Workshop on ∞-Categories, MPIM Bonn

Introduction to ∞-OPERADS (Part I)

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1. Why ∞-Operads?
2. Operads
3. Categories of operators
4. ∞-Operads and algebras
5. Examples of symmetric monoidal ∞-categories
6. Free algebras

1. Why ∞-Operads

Homotopy-coherent algebraic structures:
instead of relations (equations)
between algebraic operations, have
infinite hierarchy of (higher)
homotopies relating them

Occur in several contexts:

- spaces (loop spaces, iterated/infinite loop spaces)
- spectra (ring spectra)
- chain cx.s (e.g. A_∞ / L_∞-alg.s)
- categories (monoidal, braided mon., symmetric mon.)

Operads (in Top / Set$_\Delta$) were introduced as a framework for parametrizing such homotopy-coherence data (Boardman-Vogt / May, '70s)

But: Same drawbacks as simplicial / topological cat.-s as model for homotopy theories

- hard to define correct ht.-py theories of alg.s
- hard to define new examples of symm. mon. ht.-py theories

Slogan: ∞-Operads give a good language for working with ht.py-coherent algebraic structures (just as ∞-categories give a good language for working with ht.-py theories)

$\exists$ many ways of looking at operads $\rightsquigarrow$ many (equivalent) models for ∞-operads

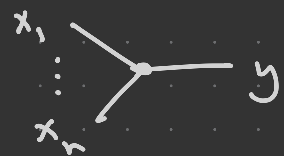
This talk: Introduce Lurie's model for ∞-operads

2. Operads

A (symmetric, coloured) **operad** (AKA symmetric multicategory) is "a category where morphisms can have many inputs (that can be permuted)".

More precisely, an operad $\mathcal{O}$ has:

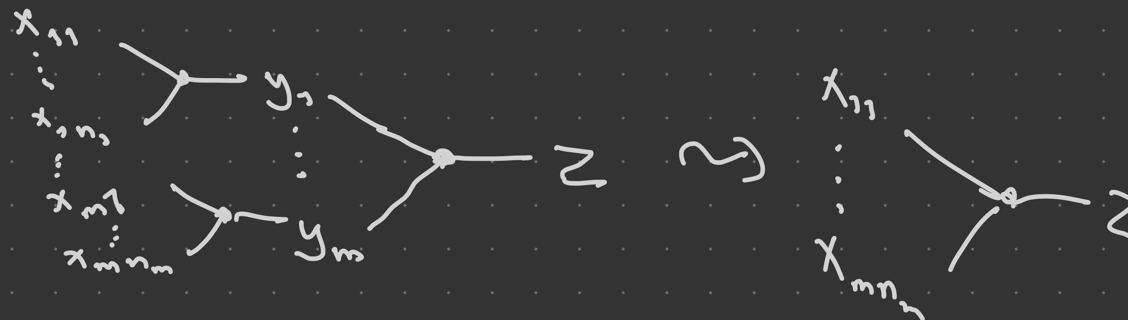
- set of objects
- $\forall$ obs $x_1, \dots, x_n, y \in \mathcal{O}$ a set $\mathcal{O}(x_1, \dots, x_n; y)$ of <u>multimorphisms</u>



($n = 0$ allowed $\bullet\!\!-\!\!-\, y$)

- $id_x \in \mathcal{O}(x; x)$
- Σ_n-action $\mathcal{O}(x_1, \dots, x_n; y) \xrightarrow[\sim]{} \mathcal{O}(x_{\sigma 1}, \dots, x_{\sigma n}; y)$

• composition



s.t. composition is unital, associative & equivariant

Notation: If $\mathcal{O}$ has one ob. o, write $\mathcal{O}(n) = \mathcal{O}(\underbrace{o,\dots,o}_{n}; o) \circlearrowleft \Sigma_n$

Examples:

- $\mathcal{V}$ symm. mon. cat. gives operad w/ $\mathcal{V}(x_1,\dots,x_n; y) = \mathcal{V}(x_1 \otimes \cdots \otimes x_n, y)$
  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ " $\mathrm{Hom}_{\mathcal{V}}(\qquad)$
- Comm has one ob., $\mathrm{Comm}(n) = * \quad \forall\, n$
- Assoc has one ob., $\mathrm{Assoc}(n) = \Sigma_n$ w/ free action
- CMod has two ob-s a, m, &
  $\mathrm{CMod}(a,\dots,a; a) = *$,
  $\mathrm{CMod}(a,\dots,a,m; m) = *$, otherwise $\varnothing$

A <u>functor</u> between operads $F: \mathcal{O} \longrightarrow \mathcal{P}$ is given by

$$x \in \mathcal{O} \longmapsto F(x) \in \mathcal{P}$$

$$f \in \mathcal{O}(x_1; \dots, x_n; y) \longmapsto F(f) \in \mathcal{P}(Fx_1; \dots, Fx_n; Fy),$$

compatibly w/ composition, identities, permutations

$\mathcal{O}$ opd., $\mathcal{V}$ symm. mon. an $\mathcal{O}$-algebra in $\mathcal{V}$ is a functor $\mathcal{O} \longrightarrow \mathcal{V}$

Examples:

- $\mathcal{C}$ symm. mon., $\mathcal{C}$-alg. in $\mathcal{V}$ = lax symmetric monoidal functor

$$\begin{array}{ccc} (id \in \mathcal{C}(x, y; x \otimes y) & \longmapsto & F(id) \in \mathcal{V}(Fx, Fy; F(x \otimes y)) \\ \wr\wr & & \shortparallel \\ \mathcal{C}(x \otimes y, x \otimes y) & & \mathcal{V}(Fx \otimes Fy, F(x \otimes y)) \end{array}$$

- Comm-alg. in $\mathcal{V}$: $* \longmapsto X \in \mathcal{V}$

  $* \in \mathrm{Comm}(n) \longmapsto X^{\otimes n} \xrightarrow{\mu_n} X$

  Trivial Σ_n-action $\longmapsto$ Σ_n-inv't.

  Compatible w/ composition $\rightsquigarrow$ μ_n = iteration of μ_2 $(n \neq 0,1)$

  - μ_2 = commutative mult.
  - $\mu_1 = \mathrm{id}$
  - μ_0 = unit for mult.

  - X is a comm. alg. in $\mathcal{V}$

- Assoc-alg.: $* \longmapsto X \in \mathcal{V}$

  operations gen. by $X^{\otimes 2} \to X$ (two of these, but related by symmetry $X^{\otimes 2} \xrightarrow{\sim} X^{\otimes 2}$)

  - X is assoc. alg. in $\mathcal{V}$

- CMod-alg in $\mathcal{V}$:

$$a \longmapsto A \in \mathcal{V} \quad \text{- } A \text{ is comm. alg.}$$

$$m \longmapsto M \in \mathcal{V}$$

with $A^{\otimes n} \otimes M \longrightarrow M$ - makes M an A-module.

3. Categories of Operators

[May-Thomason, 1978]

$\mathbb{F}_*$ = (skeleton of) finite pointed sets, ob.s $\langle n \rangle = (\{0,1,\dots,n\}, 0)$

$\mathcal{O}$ an operad

The category of operators $\mathcal{O}^{\otimes}$ has

- ob.s $(\langle n \rangle, x_1, \dots, x_n)$ $(x_i \in \mathcal{O})$
- mor. $(\langle n \rangle, x_1, \dots, x_n) \longrightarrow (\langle m \rangle, y_1, \dots, y_m) \longrightarrow (\langle k \rangle, z_1, \dots, z_k)$

  $=$ $\varphi : \langle n \rangle \longrightarrow \langle m \rangle$ in $\mathbb{F}_*$

  $+ \; \forall j = 1, \dots, m$ multimor. $(x_i)_{i \in \varphi^{-1}(j)} \longrightarrow y_j$ in $\mathcal{O}$
- comp. comes from comp. in $\mathcal{O}$.

A mor. $\varphi : \langle n \rangle \longrightarrow \langle m \rangle$ in $\mathbb{F}_*$ is

inert if φ "only sends things to base point"

- $|\varphi^{-1}(i)| = 1$ if $i \neq 0$

active if φ "doesn't send anything to base point"

- $|\varphi^{-1}(0)| = 1$

$\pi : \mathcal{O}^{\otimes} \longrightarrow \mathbb{F}_*$ satisfies

(1) For $X = (\langle n \rangle, x_1, \dots, x_n)$ & $\varphi : \langle n \rangle \to \langle m \rangle$ inert,

$\exists$ cocart. mor. $X \longrightarrow \varphi_! X$ in $\mathcal{O}^{\otimes}$,

namely $X \longrightarrow (\langle m \rangle, x'_j)$ w/ $x'_j = x_{\varphi^{-1} j}$

Aside: Every mor. in $\mathbb{F}_*$ factors (uniquely up to isomor.)
as $\xrightarrow{\text{inert}}\xrightarrow{\text{active}}$ — have a factorization system

$\bar{\varphi}: X \to Y$ in $\mathcal{O}^{\otimes}$ over φ in $\mathbb{F}_*$ is **inert** if φ inert + $\bar{\varphi}$ cocart.
— encode decomposition of X into
$(\langle 1\rangle, x_i)$

& **active** if φ is active — encode (families of) multimor.s in $\mathcal{O}$

Notation: $\rho_i : \langle n\rangle \longrightarrow \langle 1\rangle$ is inert map given by $\rho_i(j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

(2) Cocart. mor.s over ρ_i's give functors

$$\mathcal{O}^{\otimes}_{\langle n\rangle} \longrightarrow \prod_{i=1}^{n} \mathcal{O}^{\otimes}_{\langle 1\rangle}$$ — these are eq.ces

$$(\langle n\rangle, x_1, \dots, x_n) \longmapsto ((\langle 1\rangle, x_1), \dots, (\langle 1\rangle, x_n))$$

(3) For $X \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, $Y \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, $Y \to Y_i$ cocart. over ρ^i,

$$\underbrace{\mathrm{Hom}^{\varphi}_{\mathcal{O}^{\otimes}}(X,Y)}_{\text{fibre at } \varphi:\langle n\rangle\to\langle m\rangle} \longrightarrow \prod_{i=1}^{m} \mathrm{Hom}^{\rho^i\varphi}_{\mathcal{O}^{\otimes}}(X,Y_i) \quad \text{is isomor.}$$

Note:

$$\begin{tikzcd}
\rho^i\varphi : \langle n\rangle \arrow[r] \arrow[dr, "\beta_i"'] & \langle 1\rangle \\
& \langle n_i\rangle \arrow[u, "\alpha_{n_i}"']
\end{tikzcd}$$

β_i: inert

α_{n_i}: unique active to $\langle 1\rangle$ from $\langle n_i\rangle$

$X \longrightarrow X_i$ cocart. over β_i, then

$$\mathrm{Hom}^{\varphi}_{\mathcal{O}^{\otimes}}(X,Y) \cong \prod_{i=1}^{m} \underbrace{\mathrm{Hom}^{\alpha_{n_i}}_{\mathcal{O}^{\otimes}}(X_i,Y_i)}_{\text{multimor.s in } \mathcal{O}}$$

Propn.: $\pi: \mathcal{C} \longrightarrow \mathcal{F}_*$ is equivalent to cat. of operators of an operad iff π satisfies (1)–(3).

Idea of Proof: Define operad $\mathcal{O}$ by $\mathrm{ob}\,\mathcal{O} = \mathrm{ob}\,\mathcal{C}_{\langle 1\rangle}$

$$\mathcal{O}(x_1,\dots,x_n; y) = \mathrm{Hom}_{\mathcal{C}}^{\alpha_n}(X, y)$$

$$\text{w/ } X \in \mathcal{C}_{\langle n\rangle} \longleftrightarrow (x_1,\dots,x_n) \in \prod \mathcal{C}_{\langle 1\rangle}.$$

Compose via

$$\prod_{i=1}^{m} \mathcal{O}(x_{i1},\dots,x_{in_i}; y_i) \times \mathcal{O}(y_1,\dots,y_m, z)$$

$$\cong \mathrm{Hom}_{\mathcal{C}}^{\varphi}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}^{\alpha_m}(Y, z)$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{C}}^{\alpha_n}(X, z)$$

$n = \sum n_i$

$X \in \mathcal{C}_{\langle n\rangle} \longleftrightarrow (x_{11},\dots,x_{mn_m})$

$\varphi: \langle n\rangle \longrightarrow \langle m\rangle$ active

$1,\dots,n_1 \longmapsto 1$

$n_1+1,\dots,n_1+n_2 \longmapsto 2$

$\dots$

Then $\mathcal{C} \simeq \mathcal{O}^{\otimes}$ $\square$

Can say more: functors of operads $\mathcal{O} \to \mathcal{P}$

$$\longleftrightarrow \text{ functors } \begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "F"] \arrow[dr] & & \mathcal{P}^{\otimes} \arrow[dl] \\ & \mathbb{F}_* & \end{tikzcd}$$

s.t. F preserves inert mor.s

$\big(F(\langle n\rangle, x_1, \dots, x_n) = (\langle n\rangle, Fx_1, \dots, Fx_n)\big)$

Cor.: $\mathcal{O} \mapsto \mathcal{O}^{\otimes}$ gives eq.ce of cat.s $\mathrm{Opd} \xrightarrow{\sim} \mathrm{Cat}^{\mathrm{opd}}_{/\mathbb{F}_*} \subset \mathrm{Cat}_{/\mathbb{F}_*}$

= subcat. of ftr.s satisfying (1)-(3) + mor.s that preserve inerts.

Gives description of operads in terms of usual categories
– & easy to state defn. in full (w/o saying "obvious diagrams commute")

Lemma: Opd. $\mathcal{O}$ comes from symm. mon. cat. iff $\mathcal{O}^{\otimes} \to \mathbb{F}_*$ is cocart. fib. ($\exists$ cocart. mor.s over all maps in $\mathbb{F}_*$);

(strong) symm. mon. functors $\longleftrightarrow$
$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[r, "F"] \arrow[dr] & \mathcal{P}^{\otimes} \arrow[d] \\ & \mathbb{F}_* \end{tikzcd}$$
s.t. F preserves all cocart. mor.s

Idea of Proof: Cocart. mor. over $\alpha_2 : \langle 2 \rangle \to \langle 1 \rangle$ means for $x, y \in \mathcal{O}$ $\exists$ $(x,y) \longrightarrow x \otimes y$ s.t. $\mathcal{O}(x,y;z) \cong \mathcal{O}(x \otimes y; z)$

Over α_n: $\exists$ $(x_1, \dots, x_n) \longrightarrow \otimes_n(x_1, \dots, x_n)$ s.t. $\mathcal{O}(x_1, \dots, x_n; y) \cong \mathcal{O}(\otimes_n(x_1, \dots, x_n); y)$

Decompose α_n as $\langle n \rangle \to \langle n-1 \rangle \to \cdots \to \langle 1 \rangle$

$\rightsquigarrow \otimes_n(x_1, \dots, x_n) \cong x_1 \otimes (x_2 \otimes \cdots)$ as coart. mor.s compose.

Preserving coart. $\underline{(x,y) \to} x \otimes y$ means $(Fx, Fy) \to Fx \otimes Fy$, $(Fx, Fy) \to F(x \otimes y)$, $Fx \otimes Fy \xrightarrow{\wr} F(x \otimes y)$ $\square$

This arguably gives simpler defn. of symm. mon. cat. than usual one.

Can give complete defn.s w/o making arbitrary choices.

E.g. define

$\mathrm{Vect}^{\otimes} \to \mathbb{F}_*$ w/ ob.s $(\langle n \rangle, V_1, \dots, V_n)$, $V_i \in \mathrm{Vect}$,

mor. $(\langle n \rangle, V_i) \to (\langle m \rangle, W_j)$ over $\varphi : \langle n \rangle \to \langle m \rangle$

is family of multilin. maps $\prod_{i \in \varphi^{-1}j} V_i \to W_j$

— now just need to check bilinear maps are corepresentable.

Cocart. fib.s over $\mathbb{F}_*$ $\longleftrightarrow$ functors $\mathbb{F}_* \longrightarrow Cat$

Symm. mon. cat.s $\longleftrightarrow$ functors $\mathbb{F}_* \xrightarrow{F} Cat$ s.t.

$$F(\langle n \rangle) \longrightarrow \prod_{i=1}^{n} F(\langle 1 \rangle) \quad (\text{in } \rho_i\text{'s})$$

is eq.ce of cat.s

4. ∞-Operads and Algebras

[Lurie 2009]

The defn. of operads via cat.s of operators generalizes straightforwardly to ∞-cat.s:

Defn.: An **∞-operad** is a functor of ∞-cat.s $\pi : \mathcal{O} \to \mathbb{F}_*$ s.t.

(1) $\exists$ π-cocart. mor.s over inert mor.s in $\mathbb{F}_*$

[Inert mor.s in $\mathcal{O}$ = cocart. over inert
Active mor.s in $\mathcal{O}$ = lies over active } give factorization system]

(2) Cocart. mor.s over ρ_i's $\rightsquigarrow$ $\mathcal{O}_{\langle n \rangle} \longrightarrow \prod_{i=1}^{n} \mathcal{O}_{\langle 1 \rangle}$

- this is an eq.ce of ∞-cat.s

(3) $X \in \mathcal{O}_{\langle n\rangle}$, $Y \in \mathcal{O}_{\langle m\rangle}$, $Y \to Y_i$ cocart., $\varphi : \langle n\rangle \to \langle m\rangle$

$$\mathrm{Map}^{\varphi}_{\mathcal{O}}(X,Y) \xrightarrow{\sim} \prod_{i=1}^{n} \mathrm{Map}^{\rho_i\varphi}_{\mathcal{O}}(X,Y_i)$$

[If $(x_1,\dots,x_n) \in \prod_{i=1}^{n} \mathcal{O}_{\langle 1\rangle}$ corr. to $X \in \mathcal{O}_{\langle n\rangle}$ under (2), $\mathrm{Map}^{\alpha_n}_{\mathcal{O}}(X,y)$
$\longleftrightarrow$ space of multimors $(x_1,\dots,x_n) \to y$]

A <u>morphism of ∞-operads</u> is a commutative triangle of ∞-cat.s

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "F"] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \mathbb{F}_* & \end{tikzcd}$$

s.t. F preserves inert mor.s

AKA <u>$\mathcal{O}$-algebra</u> in $\mathcal{P}$

Opd_∞ = ∞-cat. of ∞-operads − subcat. of $\mathrm{Cat}_\infty/\mathbb{F}_*$

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ = ∞-cat. of $\mathcal{O}$-alg.s in $\mathcal{P}$ − full subcat. of $\mathrm{Fun}_{/\mathbb{F}_*}(\mathcal{O},\mathcal{P})$

A symmetric monoidal ∞-category $\mathcal{V}^{\otimes} \longrightarrow \mathbb{F}_*$ is an ∞-operad that is also a cocart. fibration

A symmetric monoidal functor is a functor $F: \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ over $\mathbb{F}_*$ s.t. F preserves all cocart. mor.s.

Lemma: A cocart. fib. over $\mathbb{F}_*$ is a symm. mon. ∞-cat. iff corresponding functor $\mathbb{F}_* \xrightarrow{M} \mathrm{Cat}_\infty$ is an $\mathbb{F}_*$-monoid,

AKA commutative monoid

i.e. $M(\langle n \rangle) \longrightarrow \prod_{i=1}^{n} M(\langle 1 \rangle)$ (via ρ_i's) is an eq.ce of ∞-cat.s. [I.e. condition (3) is automatic here]

[Cf. Segal's "special Γ-spaces" = $\mathbb{F}_*$-monoids in $\mathcal{S}$ = ∞-cat of spaces]

- Ordinary operads are examples of ∞-operads, so can define e.g.
    - commutative / associative alg.s
    - modules over such

  in any symmetric monoidal ∞-cat.

- Simplicial / topological operads give ∞-opd.s (define simplicial cat. of op's & take nerve)

  $\rightsquigarrow$ e.g. E_n-alg.s in symm. mon. ∞-cat.

- As far as "new objects" go, main advantage of ∞-opd.s is maybe new examples of symm. mon. ∞-cat.s, not so much new types of algebraic str.s therein

5. Examples of Symmetric Monoidal ∞-Categories

① Products

$\mathcal{C}$ an ∞-cat. w/ finite products

$\mathcal{O}$ an ∞-operad

An $\mathcal{O}$-monoid in $\mathcal{C}$ is $M : \mathcal{O} \to \mathcal{C}$ s.t.

$\forall\, X \in \mathcal{O}_{\langle n \rangle}$, $X \to X_i$ cocart. over ρ^i,

$$M(X) \xrightarrow{\sim} \prod_{i=1}^{n} M(X_i). \quad \rightsquigarrow \quad \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{O}, \mathcal{C})$$

$\exists$ symm. mon. ∞-cat. $\mathcal{C}^{\times}$ s.t. $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\times}) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$

(w/ $\otimes = \times$)

Construction:

- $\mathcal{I}(\mathbb{F}_*) \subset \mathrm{Fun}(\Delta^1, \mathbb{F}_*)$ full subcat. of inert mor.s

  (obj.s $\langle n\rangle \to \langle m\rangle$ inert $\longleftrightarrow$ $\langle n\rangle$, $S \subset \{1, \dots, n\}$)

  $ev_0, ev_1 : \mathcal{I}(\mathbb{F}_*) \longrightarrow \mathbb{F}_*$

- Define $\overline{\mathcal{C}}^\times$ by $\mathrm{Map}_{/\mathbb{F}_*}(K, \overline{\mathcal{C}}^\times) \simeq \mathrm{Map}(\underbrace{K \times_{\mathbb{F}_*}}_{ev_0} \mathcal{I}(\mathbb{F}_*), \mathcal{C})$

  obj.s of $\overline{\mathcal{C}}^\times$ over $\langle n\rangle$ $\longleftrightarrow$ functors $\underbrace{\mathcal{I}(\mathbb{F}_*)_{\langle n\rangle}}_{(\text{poset of subsets of } \{1, \dots, n\})^{op}} \to \mathcal{C}$

- $\mathcal{C}^\times \subset \overline{\mathcal{C}}^\times$ full subcat. of functors $\mathcal{I}(\mathbb{F}_*)_{\langle n\rangle} \to \mathcal{C}$

  s.t. $F(S) \simeq \prod_{i \in S} F(\{i\})$

② Coproducts

$\mathcal{C}$ ∞-cat. w/ finite coproducts

$\mathbb{F}_*^+$ = cat. of pairs $(\langle n \rangle, i)$, $i \in \{1, \dots, n\}$

mor.s $(\langle n \rangle, i) \to (\langle m \rangle, j)$ = $\varphi : \langle n \rangle \to \langle m \rangle$ in $\mathbb{F}_*$

s.t. $\varphi(i) = j$

Define $\mathcal{C}^{\amalg} \to \mathbb{F}_*$ by $\mathrm{Map}_{/\mathbb{F}_*}(K, \mathcal{C}^{\amalg}) \simeq \mathrm{Map}(K \times_{\mathbb{F}_*} \mathbb{F}_*^+, \mathcal{C})$

- ob.s over $\langle n \rangle$ = functors $(\mathbb{F}_*^+)_{\langle n \rangle} \to \mathcal{C}$ - i.e. ob.s $(x_1, \dots, x_n)$

$$(\mathbb{F}_*^+)_{\langle n \rangle} \cong \{1, \dots, n\}$$

$x_i \in \mathcal{C}$

- mor.s over $\alpha_n : \langle n \rangle \to \langle 1 \rangle$, $(x_1, \dots, x_n) \to y$

$\iff$ mor.s $x_i \to y$ $\forall i$

$\mathcal{C}^{\amalg} \to \mathrm{Fin}_*$ is always an ∞-operad,

symm. mon. iff $\mathcal{C}$ has finite coproducts

(3) Smash product of spectra — can define this "just as in Adams's book"

A prespectrum X is a seq. of pointed spaces $(X_0, X_1, \dots)$

& maps $\Sigma X_n \longrightarrow X_{n+1}$ $\rightsquigarrow$ ∞-cat. PSp

A spectrum is a prespectrum s.t. adjoint $X_n \xrightarrow{\sim} \Omega X_{n+1}$

$\rightsquigarrow$ full subcat. $Sp \subset PSp$

Can define ∞-operad $PSp^{\wedge}$ s.t. multi mor. $(X,Y) \longrightarrow Z$

is given by maps $X_n \wedge Y_m \longrightarrow Z_{n+m}$ s.t.

$$\begin{tikzcd}
\Sigma(X_n \wedge Y_m) \arrow[r] \arrow[d] & X_{n+1} \wedge Y_m \arrow[d] \\
X_n \wedge Y_{m+1} \arrow[r] & Z_{n+m+1}
\end{tikzcd}$$

commutes, etc.

Claim: If we define $Sp^\wedge \subset PSp^\wedge$ as full subcat. of ob.s corr. to lists of spectra then this is symm. mon.

④ Day convolution on presheaves [H. Heine] (other versions: Glasman, Lurie)

$\mathcal{C}^\otimes$ small symm. mon. $\rightsquigarrow$ symm. mon. str. on

$\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ given by

$$(F \otimes G)(c) = \operatorname*{colim}_{\substack{c \to x \otimes y \\ \text{in } \mathcal{C}}} F(x) \times G(y)$$

Straightening eq.ce: $\mathcal{P}(\mathcal{C}) \simeq \mathrm{RFib}(\mathcal{C}) \subset \mathrm{Cat}_{\infty/\mathcal{C}}$ full subcat. of right fib.s

$$\mathcal{P}(\mathcal{C}) \overset{\text{"}}{=} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) \simeq \mathrm{RFib}(\mathcal{C})$$

$\mathrm{RFib} \subset \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ full subcat. of right fib.s

- closed under $\times$

$ev_1 : \mathrm{RFib} \to \mathrm{Cat}_\infty$ is cart. fib. (pullback of right fib. is right fib.)

& cocart. — corresponds to LKE of presheaves

$$\begin{tikzcd} \mathrm{RFib}(\mathcal{C})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{RFib}^{\times} \arrow[d, "ev_1^{\times}"] \\ \mathrm{Fin}_* \arrow[r, "M"'] & \mathrm{Cat}_\infty^{\times} \end{tikzcd}$$

Universal property: $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}(\mathcal{C})^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{O} \times_{\mathrm{Fin}_*} \mathcal{C}^{op,\otimes}}(\mathcal{S})$

(5) ∞-Operad algebras

$\mathcal{C}^{\otimes}$ symm. mon. ∞-cat., $\mathcal{O}$ ∞-opd.

$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ has symm. mon. str. given by (pointwise) $\otimes$ in $\mathcal{C}$.

For $\mathcal{O} = \mathbb{F}_*$, this gives coproduct in $\mathrm{CAlg}(\mathcal{C}) = \mathrm{Alg}_{\mathbb{F}_*}(\mathcal{C})$

$\Rightarrow \mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{C}))$

(6) Modules

$\mathrm{Mod}(\mathcal{C}) = \mathrm{Alg}_{\mathrm{CMod}}(\mathcal{C})$, $R \in \mathrm{CAlg}(\mathcal{C})$

$$\begin{tikzcd} \mathrm{Mod}_R(\mathcal{C})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(\mathcal{C})^{\otimes} \arrow[d] \\ \mathbb{F}_* \arrow[r, "R"'] & \mathrm{CAlg}(\mathcal{C})^{\otimes} \end{tikzcd}$$

cocart. fib. if $\mathcal{C}$ has simplicial colims & $\otimes$ preserves these

7 Limits

$\mathrm{CMon}(\mathcal{C}) \longrightarrow \mathcal{C}$ preserves limits, in particular for $\mathcal{C} = \mathrm{Cat}_\infty$

$\rightsquigarrow$ given a diagram of symm. mon. ∞-cat-s & symmetric monoidal functors, the limit (in Cat_∞) gets a symm. mon. structure too

E.g. for X a derived stack/scheme

$\mathrm{QCoh}(X) = \lim\limits_{\mathrm{Spec} A \to X} \mathrm{Mod}_A$ is symm. mon.

6. Free Algebras

If $\mathcal{C}$ is a symmetric monoidal cat. compatible w/ colimits indexed by groupoids & $\mathcal{O}$ is a (small) operad w/ one ob., then the free $\mathcal{O}$-algebra on $X \in \mathcal{C}$ can be described explicitly as

$$F_{\mathcal{O}}(X) \simeq \coprod_{n=0}^{\infty} \mathcal{O}(n) \times_{\Sigma_n} X^{\otimes n}$$

Notation: $\mathcal{O}$ ∞-opd.

$\mathcal{O}^{int}$ = subcat. of $\mathcal{O}$ w/ only inert mor.s, $\mathcal{O}^{act}$ same w/ actives

$\mathcal{O}^{el} \subset \mathcal{O}^{int}$ full subcat. of ob.s over $\langle 1 \rangle$

$\shortparallel$ $\mathcal{O}^{\simeq}_{\langle 1 \rangle}$

$X \in \mathcal{O}$ $\quad \mathrm{Act}_{\mathcal{O}}(X) = \infty$-gpd. of active mor.s to X = $(\mathcal{O}^{act}_{/X})^{\simeq}$

Thm. (Lurie): $\mathcal{C}^{\otimes}$ symm. mon. ∞-cat. compatible w/ ∞-gpd.-indexed colimits

Then $U_{\mathcal{O}}: \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{O}^{el}, \mathcal{C})$ has a left adjoint $F_{\mathcal{O}}$,

s.t. $$(U_{\mathcal{O}} F_{\mathcal{O}} \Phi)(X) \simeq \operatorname*{colim}_{\substack{Y \to X \in \mathrm{Act}_{\mathcal{O}}(X) \\ Y \text{ over } \langle n \rangle}} \Phi(Y_1) \otimes \cdots \otimes \Phi(Y_n)$$
$X \in \mathcal{O}^{el}$

Idea of Proof (Chu–H.): First consider $\mathcal{C}^{\otimes} = \mathcal{S}^{\times}$

- $\mathcal{O}^{int}$ is ∞-gpd. w/ no non-trivial active mor.s

$\Rightarrow$ $\mathrm{Mon}_{\mathcal{O}^{int}}(\mathcal{S}) \subset \mathrm{Fun}(\mathcal{O}^{int}, \mathcal{S})$ is exactly full subcat. of functors RKE'd from $\mathcal{O}^{el}$

- $U_{\mathcal{O}}$ factors as $$\mathrm{Mon}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{j^*} \mathrm{Mon}_{\mathcal{O}^{int}}(\mathcal{S}) \xrightarrow[\sim]{i^*} \mathrm{Fun}(\mathcal{O}^{el}, \mathcal{S})$$

restrict along $j: \mathcal{O}^{int} \to \mathcal{O}$ restrict along $i: \mathcal{O}^{el} \to \mathcal{O}^{int}$

Enough to show LKE $j_!: \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$ takes $\mathcal{O}^{\mathrm{int}}$-monoids to $\mathcal{O}$-monoids:

$$j_!\Phi(X) \simeq \operatorname*{colim}_{Y \to X \in \mathcal{O}^{\mathrm{int}} \times_{\mathcal{O}} \mathcal{O}_{/X}} \Phi(Y) \simeq \operatorname*{colim}_{Y \to X \in \mathrm{Act}_{\mathcal{O}}(X)} \Phi(Y) \simeq \operatorname*{colim}_{(Y_i \to X_i) \in \prod \mathrm{Act}_{\mathcal{O}}(X_i)} \prod \Phi(Y_i)$$

cofinality from inert-active fact. system

$\mathrm{Act}_{\mathcal{O}}(X)$ decomposes & Φ monoid

$$\simeq \prod \operatorname*{colim}_{Y_i \to X_i \in \mathrm{Act}_{\mathcal{O}}(X_i)} \Phi(Y_i) \simeq \prod j_!\Phi(X_i).$$

$\times$ in $\mathcal{S}$ preserves colim in each variable

Next extend to Day conv. on $\mathcal{P}(\mathcal{C})$ by univ. property,
then to presentable = localization of presheaves,
(then in general can embed). $\square$

Workshop on ∞-Categories, MPIM Bonn

Introduction to ∞-OPERADS (Part II)

20/8/2020

Rune Haugseng (NTNU)

7. Variants of ∞-Operads

8. The Boardman-Vogt Tensor Product & additivity

9. Other models for ∞-operads

10. Dendroidal Segal spaces

11. Enriched ∞-Operads via Ω

12. Analytic monads

7. Variants of ∞-Operads

In some sense, ∞-operads are the universal "things that have algebras in symm. mon. ∞-cat.s".

But sometimes useful to consider alg.s for other things over $\mathbb{F}_*$ — can be hard to describe corresponding ∞-opd even if we know it exists, or sometimes more combinatorially complicated.

What did we use about $\mathbb{F}_*$ in defn. of ∞-operads/algebras?

- Inert-active fact. system
- Distinguished object $\langle 1 \rangle$

Defn.: A cartesian pattern is an ∞-cat. $\mathcal{O}$ over $\mathbb{F}_*$ (i.e.: $\mathcal{O} \to \mathbb{F}_*$)

with • factorization system
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{inert}"'] & & Y \\ & \bullet \arrow[ur, "\text{active}"'] & \end{tikzcd}$$
compatible w/ that in $\mathbb{F}_*$

$\mathcal{O}^{int} \subset \mathcal{O}$ subcat. w/ only inert mor.s

• $\mathcal{O}^{el} \subset \mathcal{O}^{int}$ full subcat. of elementary obj.s, s.t. $|E| = \langle 1 \rangle$, $E \in \mathcal{O}^{el}$

• for $X \in \mathcal{O}_{\langle n \rangle}$, $\rho_i : \langle n \rangle \to \langle 1 \rangle$, $\exists!$ inert mor. $X \xrightarrow[\rho_i^X]{} X_i$ w/ $X_i \in \mathcal{O}^{el}$!

$\mathcal{O}$ ∞-opd: inert = cocart. over inerts, act. = over act., el = all over $\langle 1 \rangle$

If $\mathcal{C}^{\otimes}$ is symm. mon., an $\mathcal{O}$-algebra in $\mathcal{C}$ is
$$\begin{tikzcd} \mathcal{O} \arrow[r, "F"] \arrow[dr] & \mathcal{C}^{\otimes} \arrow[d] \\ & \mathbb{F}_* \end{tikzcd}$$
s.t. F preserves inert mor.s.

[Cf. Lurie's "strong approximations" to ∞-opd.s]

Can define $\mathcal{O}$-∞-operads $\rightsquigarrow$ ∞-opd-s w/ $\mathbb{F}_*$ replaced by $\mathcal{O}$

$$\rightsquigarrow \quad \mathrm{Opd}_\infty^{\mathcal{O}} \quad \text{w/ adjunction} \quad \mathrm{Opd}_\infty^{\mathcal{O}} \underset{\text{pullback to } \mathcal{O}}{\overset{\mathrm{Sym}_{\mathcal{O}}}{\rightleftarrows}} \mathrm{Opd}_\infty \quad (\perp)$$

$\Rightarrow$ $\mathrm{Sym}_{\mathcal{O}}(\mathcal{O})$-alg-s in $\mathcal{C}^\otimes$ $\simeq$ $\mathcal{O}$-alg-s in $\mathcal{C}^\otimes$

Examples:

1. Δ has fact. system w/ $\varphi: [n] \to [m]$ inert if $\varphi(i) = \varphi(0) + i \ \forall i$
active if $\varphi(0) = 0, \varphi(n) = m$

$|{-}|: \Delta^{op} \to \mathbb{F}_*$, $|[n]| = \langle n \rangle$, $|\varphi|: \langle m \rangle \to \langle n \rangle$

$$i \mapsto \begin{cases} j, & \varphi(j-1) < i \le \varphi(j) \\ 0, & \text{otherwise} \end{cases}$$

$\mathrm{Sym}_\Delta \Delta = \mathrm{Assoc}$

Δ-∞-operads = non-symmetric ∞-operads

2. Φ perfect operator cat. (Barwick), $\Lambda(\Phi)$ = Leinster cat. is a cart. pattern, $\Lambda(\Phi)$-∞-opd.s = Φ-∞-opd.s

3. A generalized ∞-opd. $\mathcal{O} \to \mathbb{F}_*$ is like an ∞-opd. but $\mathcal{O}_{\langle 0 \rangle}$ is not required to be $*$ $\left(\mathcal{O}_{\langle n \rangle} \simeq \mathcal{O}_{\langle 1 \rangle} \times_{\mathcal{O}_{\langle 0 \rangle}} \cdots \times_{\mathcal{O}_{\langle 0 \rangle}} \mathcal{O}_{\langle 1 \rangle} \right)$

still gives a cart. pattern

4. $\Delta^{n,op}_1$, Θ^{op}_n

8. The Boardman-Vogt Tensor Product and Additivity

$\mu : \mathbb{F}_* \times \mathbb{F}_* \longrightarrow \mathbb{F}_*$ is the "smash product of pointed finite sets"

$$\mu(\langle n\rangle, \langle m\rangle) = \langle nm\rangle = \frac{\langle n\rangle \times \langle m\rangle}{\langle n\rangle \times \{0\} \vee \{0\} \times \langle m\rangle}$$

Makes $\mathbb{F}_* \times \mathbb{F}_*$ a cart. pattern (w/ el. $= (\langle 1\rangle, \langle 1\rangle)$)

$\mathcal{O}, \mathcal{P}$ ∞-opd.s $\quad$ $\mathcal{O} \times \mathcal{P}$ is not quite an $\mathbb{F}_* \times \mathbb{F}_*$ - ∞-opd.,

but $\mathcal{O} \times \mathcal{P} \to \mathbb{F}_* \times \mathbb{F}_* \xrightarrow[\mu]{} \mathbb{F}_*$ is a cart. pattern.

Defn.: The Boardman-Vogt tensor product is

[Lurie, Barwick] $\quad \mathcal{O} \otimes \mathcal{P} = \mathrm{Sym}(\mathcal{O} \times \mathcal{P})$

$$
\begin{tikzcd}
\mathcal{O} \otimes \mathcal{P} \arrow[r] \arrow[dr] & \mathcal{C}^{\otimes} \arrow[d] \\
& \mathbb{F}_{*}
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
\mathcal{O} \times \mathcal{P} \arrow[r] \arrow[d] & \mathcal{C}^{\otimes} \arrow[d] \\
\mathbb{F}_{*} \times \mathbb{F}_{*} \arrow[r, "\mu"'] & \mathbb{F}_{*}
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
\mathcal{O} \arrow[r] \arrow[dr] & \mathrm{ALG}_{\mathcal{P}}(\mathcal{C}^{\otimes}) \arrow[d] \\
& \mathbb{F}_{*}
\end{tikzcd}
$$

$\mathcal{C}^{\otimes}$ symm. mon., $\mathrm{ALG}_{\mathcal{P}}(\mathcal{C}^{\otimes}) = \mathrm{Alg}_{\mathcal{P}}(\mathcal{C})^{\otimes} =$ symm. mon. str. on $\mathcal{P}$-alg.s

– more generally exists as an ∞-opd.

$\rightsquigarrow$ Opd_{∞} is closed symm. mon. via Boardman-Vogt $\otimes$

$\mathbb{E}_n$ = little n-disc operad, $\mathbb{E}_n(k)$ = configuration space of k points in $\mathbb{R}^n$

$\mathbb{E}_1 \simeq$ Assoc $\simeq$ space of embedding k n-discs in a big n-disc

Dunn–Lurie Additivity Thm.: $\mathbb{E}_n \otimes \mathbb{E}_m \simeq \mathbb{E}_{n+m}$

– or $\mathrm{Assoc}^{\otimes n} \simeq \mathbb{E}_n$

$\Rightarrow$ $\mathbb{E}_n$-algebras $\simeq$ n compatible associative alg. str.s

Cor.: $\mathbb{E}_\infty := \operatorname{colim}_{n \to \infty} \mathbb{E}_n$ is equivalent to Comm.

[Proof uses that $\mathbb{E}_n(k)$ is increasingly connected as $n \to \infty$]

$\Rightarrow$ to define symm. mon. (∞, n)-cat., suffices to define compatible seq. of $\mathbb{E}_k$-mon. (∞,n)-cat.s $\forall k$

– useful e.g. for cobordism (∞,n)-cat.s.

Cor. (Baez-Dolan Stabilization Hypothesis): If $\mathcal{C}^{\otimes}$ is a symmetric mon. $(n,1)$-cat. (= mapping spaces in $\mathcal{C}$ are $(n-1)$-types)

then $\mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_{n+2}}(\mathcal{C}) \simeq \cdots \simeq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C})$

[Proof again uses connectivity of $\mathbb{E}_n(k)$]

E.g. for $n=1$ (ordinary cat.) 2 compatible associative mult.s = comm. (Eckmann-Hilton)

$n=2$: Cat is a $(2,1)$-cat.

$\mathbb{E}_1$-alg. $\longleftrightarrow$ monoidal cat.

$\mathbb{E}_2$-alg $\longleftrightarrow$ braided mon. cat.

$\mathbb{E}_3$-alg. & higher $\longleftrightarrow$ symmetric mon. cat.

9. Other models for ∞-operads

10. Dendroidal Segal Spaces

Can think of Δ as full subcat. of Cat on

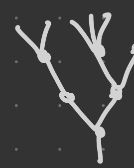
$[n] = 0 \to 1 \to \cdots \to n$ = free cat. on this graph

— encode basic shapes of compositions in a cat.

(Inert maps encode decompositions of these shapes, active encode ways to compose operations.)

Analogue for operads?

Ways of composing multimor.s $\rightsquigarrow$ trees



Want to consider full subcat. of Opd on "free operads generated by trees"

— this gives the dendroidal category Ω (Moerdijk–Weiss)

Problem: Have to define trees and this is usually annoying.

For our trees each vertex/node has one outgoing edge & some (maybe 0) incoming edges.

E = set of edges

V = set of vertices

V_* = set of vertices + one incoming edge

$$(*) \qquad \begin{array}{ccccccc} E & \xleftarrow{i} & V_* & \xrightarrow{p} & V & \xrightarrow{o} & E \\ e & \longleftarrow & (v,e) & \longmapsto & v & \longmapsto & \text{outgoing edge of } v \end{array}$$

Defn. (Kock): A **tree** consists of a diagram $(*)$ of finite sets s.t.

- o is injective
- i is injective, w/ unique elt. $R \in E$ not in image (the root)
- Define $\sigma: E \to E$ by $\sigma(R) = R$, $\sigma(e) = o p i^{-1}(e)$, then $\forall e \; \exists k$ s.t. $\sigma^k e = R$.

Given two trees, a subtree inclusion (or inert mor.) is

$$
\begin{tikzcd}
T' \arrow[d, "j"'] & = & E' \arrow[d, "j_E"'] & V'_* \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V' \arrow[r] \arrow[d, "j_V"] & E' \arrow[d, "j_E"] \\
T & = & E & V_* \arrow[l] \arrow[r] & V \arrow[r] & E
\end{tikzcd}
$$

(w/ $E' \to E$, $V' \to V$ injective - in fact automatic)

- pullback cond. says vertex in T' $\mapsto$ vertex in T w/ same number of edges

$\rightsquigarrow$ cat. Ω_{int} of trees & inert mor.s

Corolla C_n = tree w/ 1 vertex & n leaves

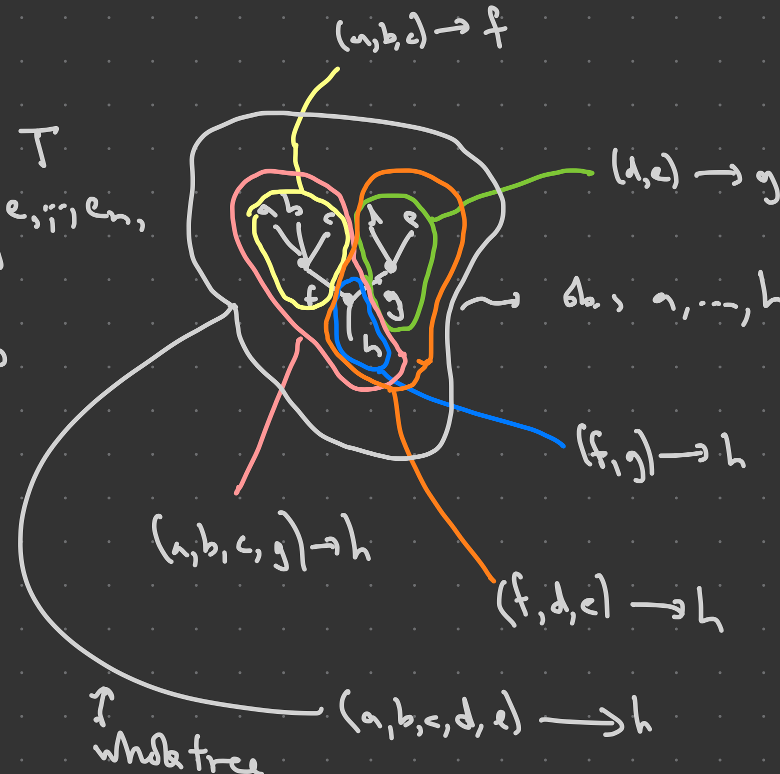
$$\underline{n} \amalg * \longleftarrow \underline{n} \longrightarrow * \hookrightarrow \underline{n} \amalg *$$

Edge η = tree w/ 0 vertices & 1 edge

$$* \longleftarrow \emptyset \longrightarrow \emptyset \longrightarrow *$$

The <u>free operad</u> $F(T)$ on a tree T has

- obs = edges of T
- multimors = subtrees of T
  $(e_1,\dots,e_n) \to e'$ w/ leaves $e_1,\dots,e_n$, root e'
- composition = gluing subtrees
- Σ_n-actions reorder leaves



Ω = full subcat. of Opd on $F(T)$

A multimor. in $F(T)$ is uniquely a composite of corollas

$\rightsquigarrow$ mor. $F(T) \xrightarrow{\varphi} F(T')$ amounts to

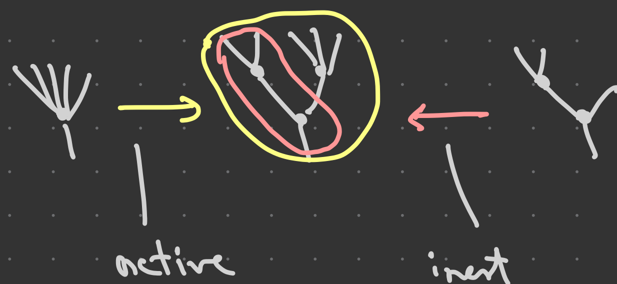
edge of $T \longmapsto$ edge of T'

corolla in $T \longmapsto$ subtree of T', same number of leaves, correct leaves & root

φ is **active** if preserves leaves & root

inert if corollas $\mapsto$ corollas (subtree inclusion)

- form a factorization system



degeneracy (active)

Recall: A Segal space is a functor (of ∞-cat.s)

$$X : \Delta^{op} \longrightarrow \mathcal{S}, \qquad X_n = X([n])$$

s.t.

$$X_n \xrightarrow{\sim} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

via the inert maps

$$[0] \longrightarrow [n] \quad (0 \mapsto i,\ i = 0, \dots, n)$$

$$[1] \longrightarrow [n] \quad \left(\begin{array}{l} 0 \mapsto i-1 \\ 1 \mapsto i, \end{array} \quad i = 1, \dots, n\right)$$

Encodes the algebraic str. of an ∞-cat.:

X_0 = space of ob.s

X_1 = space of mor.s

$s_0 : [1] \twoheadrightarrow [0] \quad \rightsquigarrow \quad X_0 \longrightarrow X_1$ - identity mor.s

$d_1, d_0 : [0] \longrightarrow [1] \quad \rightsquigarrow \quad X_1 \rightrightarrows X_0$ - source & target of mor.s

Segal cond.: $X_2 \xrightarrow{\simeq} X_1 \times_{X_0} X_1$ - space of composable mor-s

$d_1 : [1] \to [2] \rightsquigarrow X_2 \to X_1$ - composition

$$\begin{tikzcd} {[1]} \arrow[r, "d_1"] \arrow[d, "d_1"'] & {[2]} \arrow[d, "d_1"] \\ {[2]} \arrow[r, "d_2"'] & {[3]} \end{tikzcd} \rightsquigarrow \begin{tikzcd} X_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & X_2 \arrow[d] \\ X_2 \arrow[r] & X_1 \end{tikzcd}$$ - associativity up to specified ht.py

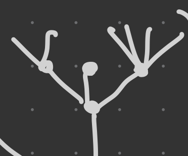
Defn.: A dendroidal Segal space (Segal presheaf on Ω) is

$$X : \Omega^{op} \longrightarrow \mathcal{S}$$

s.t.
$$X(T) \xrightarrow{\sim} \lim_{E \in (\Omega^{el}_{/T})^{op}} X(E)$$

where $\Omega^{el} \subset \Omega^{int}$ is full subcat. on C_n, η

$$\Omega^{el}_{/T} = \Omega^{el} \times_{\Omega^{int}} \Omega^{int}_{/T} \quad (\text{i.e. inert maps } C_n, \eta \longrightarrow T)$$

$X($  $) \simeq \Big(X($  $) \times X($  $) \times X($  $) \Big) \times_{X(|)^{\times 3}} X($  $)$

Encodes algebraic structure of an ∞-operad

$X(\eta)$ - sp. of ob.s

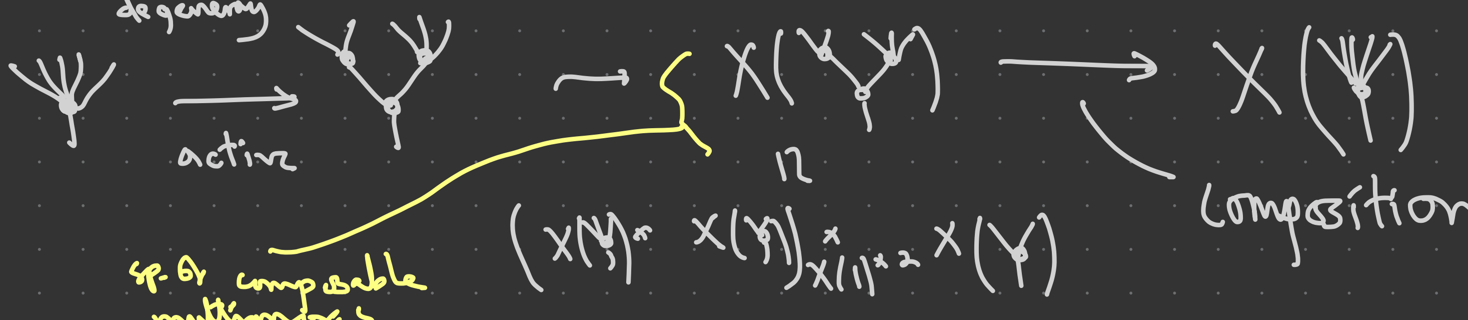
$X(C_n)$ - sp. of n-ary multimor.s

$$\begin{tikzcd} X(C_n) \arrow[r, shift left, "\text{root}"] \arrow[r, shift right, "\text{leaves}"'] & X(\eta) \end{tikzcd}$$

target ob.

n source ob.s

$C_1 \xrightarrow[\text{degeneracy}]{} \eta \longrightarrow X(\eta) \to X(C_1)$ — identities



Defn.: $F: X \to Y$ mor. of dendroidal Segal sp.-s is fully faithful if

$$\begin{tikzcd} X(C_n) \arrow[r] \arrow[d] & Y(C_n) \arrow[d] \\ X(\eta)^{\times n+1} \arrow[r] & Y(\eta)^{\times n+1} \end{tikzcd}$$

is cartesian $\forall n$ ($\Leftrightarrow$ fibres are equivalent)
 $\shortparallel$
spaces of multimor.-s

and essentially surjective if $X^{eq} \to Y^{eq}$ is surjective on π_0

$\Delta \hookrightarrow \Omega$ (incl. of trees w/ only unary vertices)

$X|_{\Delta^{op}}$ is a Segal space, X^{eq} = sp. of eq.-ces of $X|_{\Delta^{op}}$

$= \mathrm{Map}_{\mathcal{P}(\Delta)}(E_1, X)$, E_1 = nerve of universal isomor.

X is complete if $X|_\Delta$ is complete, i.e. $X(\eta) \xrightarrow{\sim} X^{eq}$

Thm. (Cisinski–Moerdijk): Complete dendroidal Segal sp.s are precisely those local w.r.t. FFES mor.s – $CSS_\Omega(\mathcal{S}) \hookrightarrow Seg_\Omega(\mathcal{S})$ has a left adj't that exhibits $CSS_\Omega(\mathcal{S})$ as localization at FFES mor.s.

$CSS_\Omega(\mathcal{S})$ is a model for ∞-opd.s

11. Enriched ∞-operads via Ω

[Chu - H.]

Defn.: For $X \in \mathcal{S}$, $\Omega_X^{op} \longrightarrow \Omega^{op}$ is left fib. corr. to $\Omega^{op} \longrightarrow \mathcal{S}$

given by RKE

$$\begin{tikzcd} \{\eta\} \arrow[r, "X"] \arrow[d] & \mathcal{S} \\ \Omega^{op} \arrow[ur] & \end{tikzcd}$$

$$T \longmapsto X^{\#\text{edges of } T}$$

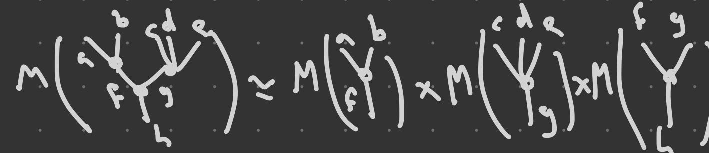
Ob. of Ω_X $=$ $(T \in \Omega,\ \overline{T}(e) \in X$ for e edge of $T)$

Mor. $\overline{T} \to \overline{T}'$ $=$ $T \xrightarrow{\varphi} T'$ in Ω

$\qquad + \ \overline{T}(e) \xrightarrow{\sim} \overline{T}'(\varphi e) \ \forall e$

An Ω_X^{op}-monoid in $\mathcal{S}$ is $M: \Omega_X^{op} \longrightarrow \mathcal{S}$ s.t.

$$M(T) \xrightarrow{\simeq} \prod_{\substack{C \text{ corollas} \\ \text{of } T}} M(C)$$



Propn.: $\mathrm{Seg}_{\Omega}(\mathcal{S})_{\underbrace{X}_{\text{sp. of ob.s } = X}} \simeq \mathrm{CMon}_{\Omega_X^{op}}(\mathcal{S}) \simeq \mathrm{Alg}_{\Omega_X^{op}}(\mathcal{S}^{\times})$

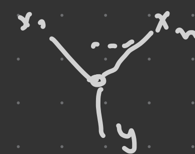
– here Ω_X^{op} is a cart. pattern via $\Omega_X^{op} \to \Omega^{op} \longrightarrow \mathbb{F}_*$

counts # corollas in trees

Defn.: $\mathcal{V}$ a symm. mon. ∞-cat. A $\mathcal{V}$-∞-operad w/ sp. of ob.s X is an Ω_X^{op}-algebra in $\mathcal{V}$ (+ completeness condition).

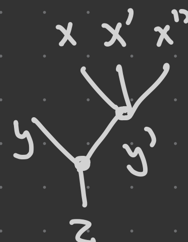
$$\begin{tikzcd} \Omega^{op}_X \arrow[r, "\mathcal{O}"] \arrow[dr] & \mathcal{V}^{\otimes} \arrow[d] \\ & \mathbb{F}_* \end{tikzcd}$$

For $x_1, \dots, x_n, y \in X$



$\longmapsto \mathcal{O}(x_1, \dots, x_n; y) \in \mathcal{V}$

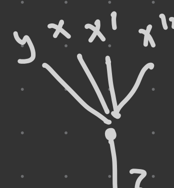
- enriched multimor.s



$\longmapsto \big(\mathcal{O}(x, x', x''; y'), \ \mathcal{O}(y, y'; z)\big)$

$\Big\downarrow$ active $\quad \sim \quad \mathcal{O}(x, x', x''; y') \otimes \mathcal{O}(y, y'; z)$

$\Big\downarrow \sim$ comp. of multimor.s



$\mathcal{O}(y, x, x', x''; z)$

Thm. (Chu–H.): Complete Ω^{or}_X-alg.s give localization at FFES mor.s.

If M is a nice symm. mon. model cat., agree w/ strict M-enriched opd.s. (E.g. M = symmetric spectra, ch. cx. / field of char. 0)

12. Analytic monads

[Gepner - H. - Kock]

A polynomial functor $S_{/I} \xrightarrow{F} S_{/J}$ is a functor of the form $t_! p_* s^*$ for some mor.s of spaces

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

[$f : X \to Y$ in S

$f^* : S_{/Y} \longrightarrow S_{/X}$ - pullbacks

$f_!$: composition $\quad f_! \dashv f^* \dashv f_*$]

Can show these are exactly the accessible functors that preserve weakly contractible limits (⟹ closed under composition)

F is analytic if p has finite discrete fibres

$\Leftrightarrow$ F preserves sifted colimits & weakly contractible limits

An **analytic monad** is a monad T on $\mathcal{S}_{/I}$ s.t.

- $T : \mathcal{S}_{/I} \to \mathcal{S}_{/I}$ is analytic
- Multiplication & unit transformations are cartesian (= naturality squares are cartesian)

$\Leftrightarrow$ monad in $(\infty,2)$-cat. of analytic functors & cart. transformations

Note: If $\mathcal{O}$ is ∞-opd. then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \longrightarrow \mathrm{Fun}(\mathcal{O}^{el}, \mathcal{S})$ is monadic right adjoint for an analytic monad

$\mathcal{O}^{el} = \mathcal{O}^{\simeq}_{\langle 1 \rangle}$

Thm. (Gepner - H. - Kock): $\mathrm{AnMnd} \simeq \mathrm{Seg}_{\Omega}(\mathcal{S})$

Idea: Can view trees as analytic endofunctors $\rightsquigarrow \Omega_{int} \to \mathrm{AnEnd}$.

Can describe free analytic monads explicitly

$\rightsquigarrow \Omega \simeq$ free analytic monads on trees $\subset \mathrm{AnMnd} \rightsquigarrow \mathrm{AnMnd} \longrightarrow \mathcal{P}(\Omega)$.

Analytic endofunctors of $\mathcal{S}$: $* \leftarrow E \xrightarrow{p} B \longrightarrow *$

finite discrete fibres

$\exists$ universal such mor.: $\coprod_n \underline{n}_{h\Sigma_n} \longrightarrow \coprod_n B\Sigma_n$ ($\leftrightarrow$ Σ_n-action on $\underline{n}$)

So $p \longleftrightarrow$ mor. $B \longrightarrow \coprod_n B\Sigma_n$ in $\mathcal{S}$

$\longleftrightarrow$ functor $\coprod_n B\Sigma_n \longrightarrow \mathcal{S}$

i.e. $X(n) \circlearrowleft \Sigma_n$ $\forall n$ – a symmetric sequence

Can show composition of analytic endofunctors

$\longleftrightarrow$ composition product of symm. seq.s,

$$(*) \quad (X \circ Y)(n) = \coprod_{k=0}^{\infty} \Big(\coprod_{i_1 + \cdots + i_k = n} (X(i_1) \times \cdots \times X(i_k)) \times_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} \Sigma_n \Big) \times_{\Sigma_k} Y(k)$$

$\Rightarrow$ ∞-opd.s w/ one ob. = analytic monads on $\mathcal{S}$ = alg.s in symm. seq.s

Cleaner description of $(*)$:

$$X(I \to J) = \prod_{j \in J} X(I_j)$$

I, J finite sets

$$(X \circ Y)(I \to K) = \underbrace{\operatorname*{colim}_{I \to J \to K}}_{\text{groupoid of such factorizations}} X(I \to J) \times Y(J \to K)$$

$\rightsquigarrow$ description of $\circ$ via $\Delta_{\mathbb{F}}$

$\Omega^{op} \to \mathcal{S}$ Segal $\longleftrightarrow$ monoids for $\Omega_X^{op} \to \mathcal{S}$ / $X \in \mathcal{S}$

$$\rightsquigarrow$$

alg.s for $\Omega_X^{\otimes} \to \mathcal{V}^{\otimes}$

$f : X \to Y$ in $\mathcal{S}$

$$\begin{tikzcd}
\Omega_X^{op} \arrow[rr] \arrow[d] & \arrow[d, phantom, "\Downarrow"] & \mathcal{V}^{\otimes} \\
\Omega_Y^{op} \arrow[urr] & &
\end{tikzcd}$$

– $X \mapsto \mathrm{Fun}(\Omega_X^{op}, \mathcal{V}^{\otimes})$

$\mathcal{P}(\underbrace{\coprod B\Sigma_n}_{\text{Day conv.}})$ – free presentable symm. mon. ∞-cat. on $*$

composition is mon. str.

$$\mathrm{Fun}^{L,\otimes}(\mathcal{P}(\coprod B\Sigma_n), \mathcal{C}) \simeq \mathrm{Fun}^{\otimes}(\coprod B\Sigma_n, \mathcal{C}) \simeq \mathcal{C}$$

$$\mathrm{Fun}^{L,\otimes}(\mathcal{P}(\coprod B\Sigma_n), \mathcal{P}(\coprod B\Sigma_n)) \simeq \mathcal{P}(\coprod B\Sigma_n)$$