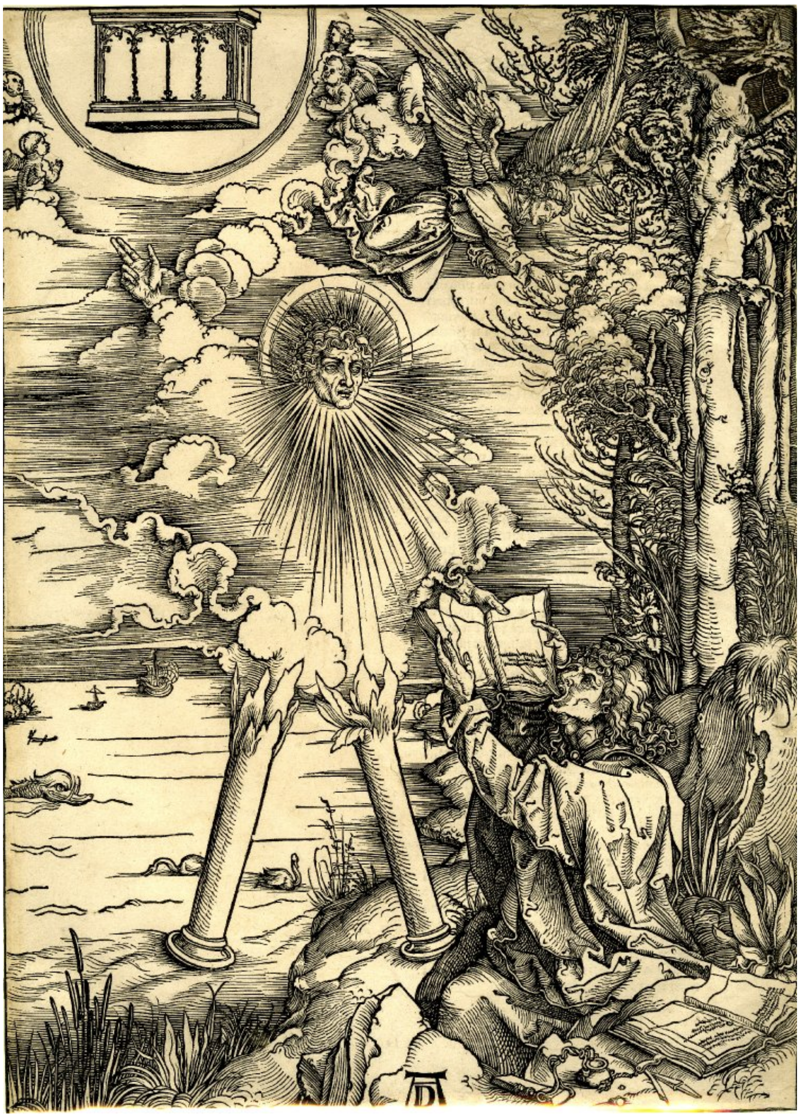
RUNE HAUGSENG

YET ANOTHER INTRODUCTION TO ∞-CATEGORIES



NTNU

Spring 2025 (last updated: April 9, 2025)

Title page illustration: artist's impression of studying higher category theory.[1]

[1]Or just possibly it might be engraving 10, "St John eating the book", from Albrecht Dürer's *Apocalypse* (1498). Source: https://commons.wikimedia.org/wiki/File:Dürer_Apocalypse_10.jpg

Contents

Preface

These are lecture notes for the course MA8403 taught at NTNU during the spring semester of 2025 (but maybe it will one day become a book). Please note that some parts of the document are currently in a pretty rough shape.

Sources

I make no claim of originality for any part of this text, except for the arrangement of the material. On the other hand, the contents have often been so substantially reorganized that it has not been possible to assign credit for most of the results as they appear. To partially make up for this, I will try to list some sources that have been particularly influential:

- Unsurprisingly, Lurie's book [Lur09], where much of the theory of ∞-categories was first developed, has influenced many parts of the text, but especially the later Chapters 9 and 10 on filtered colimits and accessible and presentable ∞-categories.
- Much of Chapter 2 is based on Martini's paper [Mar21] on the Yoneda lemma for internal ∞-categories, which in turn was heavily influenced by Cisinski's book [Cis19]. The same is true for the later discussions of left and right fibrations and cofinality.
- The approach to adjunctions via free fibrations in Section 6.3 is taken from the book of Riehl and Verity [RV22].
- The proof of the adjoint functor theorem in Section 10.2 is a combination of the proof from [Lur09] (which seems to contain a gap) and that of Nguyen, Raptis and Schrade in [NRS20].
- The proof that filtered colimits commute with finite limits in ∞-groupoids in Sections 9.8–9.9 is taken from the note [SW25] by Sattler and Wärn.
- Parts of Chapter 1 are based on my expository article on higher categories [Hau24].

That said, some parts of these notes are in a sense the outcome of more than a decade of trying to understand ∞-categories, and I have likely forgotten where I first learned about many of the ideas it contains — I would appreciate any suggestions of additional sources that should be acknowledged!

Acknowledgments

I would like to thank all the people who have taught me about ∞-categories through many, many conversations, going back to when I first heard about them in 2008. I will not try to list them all individually, but I should at least mention Clark Barwick and David Gepner, from whom I learned a lot during my PhD studies, as well as all my other past and present collaborators on ∞-categorical topics: Fernando Abellán, Shaul Barkan, Damien Calaque, Hongyi Chu, Bastiaan Cnossen, Elden Elmanto, Andrea Gagna, Owen Gwilliam, Fabian Hebestreit, Gijs Heuts, Joachim Kock, Tobias Lenz, Sil Linskens, Valerio Melani, Thomas Nikolaus, Joost Nuiten, Pavel Safronov, Claudia Scheimbauer, and Jan Steinebrunner. Many aspects of this text have been greatly influenced by discussions about their analogues for $(\infty, 2)$-categories with Fernando Abellán during his time as a postdoc in Trondheim. It is also a pleasure to thank my past and present PhD students at NTNU: Fredrik Bakke, Tallak Manum, and Louis Martini. Finally, I thank all the students who participated in the course at NTNU in spring 2025, especially for their many comments and questions during the lectures (except for the ones about set theory): Fredrik Bakke, Denis Bondarenko Bergmann, Tore Bjerkestrand Braathen, Eivind Xu Djurhuus, Vebjørn Holm-Gjerde, John Aslak Wee Kleven, Tallak Manum, Marius Verner Bach Nielsen, Trygve Poppe Oldervoll, and Mia Soheltberg.

Chapter 1

Introduction

(This introduction is currently quite rough and in particular omits any discussion of the history of higher categories; for the moment, the reader is referred to [Hau24] for some more context and references to the literature.)

1.1 What are higher categories?

Many mathematical objects have an associated notion of *morphism*, and so naturally organize themselves into *categories*. Basic examples include

- sets and functions,
- vector spaces and linear maps,
- groups and group homomorphisms.

In some cases there are also “morphisms between morphisms”, such as

- homotopies between continuous maps of topological spaces,
- chain homotopies between homomorphisms of chain complexes,
- natural transformations between functors among categories;

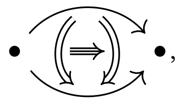
these structures give us *higher* categories. The basic concept is that of an *n-category*, which should be a structure with

- objects[1] ($\bullet$),
- morphisms (or 1-morphisms) between objects ($\bullet \to \bullet$),
- 2-morphisms between morphisms (with the same source and target), which we can depict as:

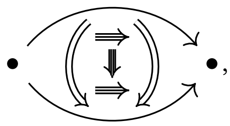
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet, \end{tikzcd}$$

[1]It is sometimes convenient to think of objects as 0-*morphisms*.

- 3-morphisms between 2-morphisms, which we can depict as:



- 4-morphisms between 3-morphisms, which we can depict as:



and so on up to n-morphisms between $(n-1)$-morphisms. We should be able to compose i-morphisms for all $0 < i \leq n$, and an i-morphism should have an identity $(i+1)$-morphism for $0 \leq i < n$; the composition should give us for all objects x, y in an n-category $\mathcal{C}$ an $(n-1)$-category $\mathcal{C}(x, y)$ of morphisms from x to y with a composition functor

$$\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$$

for any triple of objects x, y, z. In particular, a 0-category is just a set and a 1-category is an ordinary category.

Before we explain why it is not entirely straightforward to give a precise definition of an n-category, let us mention a few structures that ought to give examples of n-categories:

- Topological spaces with continuous maps, homotopies, homotopies of homotopies, etc., up to n-dimensional homotopies, should give an example of an $(n+1)$-category for any n.
- Categories with functors and natural transformations should give a 2-category.
- More generally, n-categories should form an $(n+1)$-category where the i-morphisms between $\mathcal{C}$ and $\mathcal{D}$ are functors $\mathcal{C} \times C_i \to \mathcal{D}$, with C_i being the free n-category on a single i-morphism.
- Given a commutative ring R, we should have a *Morita 2-category* of R where the objects are associative R-algbras, the 1-morphisms from A to B are A-B-bimodules, and the 2-morphisms are bimodule homomorphisms.

If we ask for the composition of i-morphisms to be *stricly* associative, so that we have *identities* $h(gf) = (hg)f$, we get the notion of *strict* n-categories. These are easy to define, but turn out to not be very useful in practice.

To see why we should not expect associativity to hold strictly, let us first point out that as n increases we have an increasingly refined notion of when two objects in an n-category are "the same" and when a morphism is "invertible":

- In a 0-category, i.e. a set, two objects x and y are the same if they are *equal* ($x = y$).
- In a 1-category $\mathcal{C}$, two objects x and y are the same if they are *isomorphic* ($x \cong y$), i.e. there exist morphisms $f : x \to y$ and $g : y \to x$ such that $gf = \mathrm{id}_x$ and $fg = \mathrm{id}_y$ in the *sets* $\mathcal{C}(x, x)$ and $\mathcal{C}(y, y)$. Here we also say that f and g are *invertible* or are *isomorphisms*.
- In a 2-category $\mathcal{C}$, two objects x and y are the same if they are *equivalent* ($x \simeq y$), i.e. there exist morphisms $f : x \to y$ and $g : y \to x$ and isomorphisms $gf \cong \mathrm{id}_x$ and $fg \cong \mathrm{id}_y$ in the *categories* $\mathcal{C}(x, x)$ and $\mathcal{C}(y, y)$. Here we also say that f and g are *invertible* or are *equivalences*.
- In general, in an n-category $\mathcal{C}$, two objects x and y are the same if they are *equivalent* ($x \simeq y$), i.e. there exist morphisms $f : x \to y$ and $g : y \to x$ and equivalences $gf \simeq \mathrm{id}_x$ and $fg \simeq \mathrm{id}_y$ in the $(n-1)$*-categories* $\mathcal{C}(x, x)$ and $\mathcal{C}(y, y)$. Here we again say that f and g are *invertible* or are *equivalences*.

A basic principle for ordinary categories is that we should never ask for two objects to be *equal*, only to be *isomorphic*. Similarly, in an n-category it is "evil" (or at least morally disreputable) to demand that two objects should be equal rather than equivalent. (For example, we should never ask for two categories to be isomorphic, we can only demand that they be equivalent.)

The strict associativity equation $h(gf) = (hg)f$ for i-morphisms in an n-category is asking for two objects in an $(n-i)$-category to be equal. We should therefore instead require a (specified) invertible $(i+1)$-morphism $h(gf) \xrightarrow{\sim} (hg)f$. However, using such $(i+1)$-morphisms there are two ways to relate compositions of 4 morphisms:

$$\begin{tikzcd}
 & & (kh)(gf) \arrow[drr] & & \\
k(h(gf)) \arrow[urr] \arrow[dr] & & & & ((kh)g)f. \\
 & k((hg)f) \arrow[rr] & & (k(hg))f \arrow[ur] &
\end{tikzcd}$$

These two composites should again be "the same", so we need an invertible $(i+2)$-morphism between them, which in turn calls for a coherence $(i+2)$-morphism relating different ways to go between compositions of 5 morphisms, and so on. This is the idea of *weak n-categories*.

As should already be plausible, it is not easy to explicitly write down a definition of weak n-categories, at least for $n > 2$. On the other hand, strict n-categories are simply not sufficient, as most real examples of n-categories are *not* strict. For example, there should be a 2-category $B(\mathsf{Vect}, \otimes)$ with a single object, where the morphisms are vector spaces, the 2-morphisms are linear maps, and the composition of 1-morphisms is given by taking tensor products. For this to be a strict 2-category we would need to have equations

$$U \otimes (V \otimes W) = (U \otimes V) \otimes W,$$

for triple tensor products, which is simply not true — instead, we have a canonical *isomorphism* between these vector spaces.

There are definitions of weak n-categories that package the coherence data directly, but in practice these have turned out to not be very usable, at least for $n > 3$. Let us also mention that every weak 2-category is equivalent to a strict 2-category, but this is false for $n > 2$.

Instead of the "bottom-up" approach to n-categories we have sketched so far, it turns out to be much easier to first use *homotopy theory* to develop a theory of "$(\infty, 1)$-categories", and then use this to define other kinds of higher categories.

Definition 1.1.1. An (n, k)-*category* is an n-category where all i-morphisms are invertible for $i > k$.

For example, an (n, n)-category is the same thing as an n-category, while an $(n, 0)$-category is an n-*groupoid*, i.e. an n-category where all i-morphisms are invertible for all i.

At least informally, we can also allow $n = \infty$ in this definition, so that we have i-morphisms for all $i > 0$.

Warning 1.1.2. In this text, and in much of the modern literature, the term ∞-*category* is an abbreviation for $(\infty, 1)$-category. This conflicts with the convention that n-categories are (n, n)-categories for finite n, but it is simply too cumbersome to keep writing out "$(\infty, 1)$-category".[2] In any case (∞, ∞)-categories (or ω-*categories*), which have non-invertible i-morphisms for all i, will not show up here.

1.2 The homotopy hypothesis

Informally, the basic idea for the homotopical approach to higher categories is that ∞-*groupoids* should capture precisely the homotopy-invariant information in topological spaces — this is the *homotopy hypothesis* of Grothendieck, which we will introduce in this section. Given this, we can then build a theory of ∞-categories on top of topological spaces (or some other model of their homotopy theory, such as simplicial sets).

We first need to recall some basic definitions from homotopy theory:

[2] On the other hand, we will not be as lazy as some young people have become these days and refer to ∞-categories as just "categories".

Definition 1.2.1. Suppose X and Y are topological spaces, and $f, g\colon X \to Y$ are continuous maps. Then a *homotopy* from f to g is a continuous map $h\colon X \times I \to Y$ (where I is the closed interval $[0,1]$) such that $h(-,0) = f$ and $h(-,1) = g$. If we instead have pointed topological spaces (X, x) and (Y, y), with f and g pointed maps, then we say that h is a *pointed homotopy* if we also have $h(x, -) = y$. We say that f and g are (pointed) *homotopic* if there exists a (pointed) homotopy between them; this is an equivalence relation on (pointed) continuous maps.

Definition 1.2.2. For a pointed topological space (X, x), we define the nth *homotopy group* $\pi_n(X, x)$ to be the set of pointed homotopy classes of maps $(S^n, *) \to (X, x)$.

Here $\pi_1(X, x)$ is the *fundamental group* of loops in X based at x, while for $n > 1$ the homotopy group $\pi_n(X, x)$ is an abelian group. We also take $\pi_0 X$ to be the (pointed) set of path components of X.

Definition 1.2.3. Let X be a topological space. The *fundamental groupoid* $\pi_{\leq 1} X$ of X has as its objects the points of X, and as its morphisms from x to y the homotopy classes of paths $I \to X$ that start at x and end at y (for homotopies that respect this condition). Composition is given by concatenation of paths (which is well-defined up to homotopy), and the identity morphisms are the constant paths; this makes $\pi_{\leq 1} X$ a category, which is indeed a groupoid since paths can be reversed.

Note also that for a point $x \in X$, the set $\pi_{\leq 1}(X)(x, x)$ is nothing but the fundamental group $\pi_1(X, x)$, so that the fundamental groupoid contains the information from $\pi_0 X$ as well as the fundamental groups at all base points of X.

The topological space X should also have a *fundamental n-groupoid* $\pi_{\leq n} X$ where

- the objects are points of X ($* \to X$),
- the morphisms are paths in X ($I \to X$),
- the 2-morphisms are homotopies between paths ($I^2 \to X$),
- …
- the n-morphisms are equivalence classes of n-dimensional homotopies $I^n \to X$.

This should also make sense for $n = \infty$, where we keep going forever (and never take equivalence classes) giving the *fundamental ∞-groupoid* $\pi_{\leq \infty} X$.

The Homotopy Hypothesis characterizes the information about X that is contained in $\pi_{\leq n} X$. To explain this more precisely, we need to introduce some more terminology:

- A continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ and homotopies $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$.

- A continuous map $f\colon X \to Y$ is a *weak homotopy equivalence* if it induces isomorphisms $\pi_0 X \xrightarrow{\sim} \pi_0 Y$ and $\pi_n(X,x) \to \pi_n(Y, f(x))$ for all $n \geq 1$ and all $x \in X$.
- Two topological spaces have the same *homotopy type* if they are in the same equivalence class under weak homotopy equivalence.

It is a theorem of Whitehead that a weak homotopy equivalence among CW-complexes is actually a homotopy equivalence. Moreover, any topological space is *weakly* homotopy equivalent to a CW-complex, so to study homotopy types of topological spaces we can either consider all topological spaces up to weak homotopy equivalence or CW-complexes up to homotopy equivalence.

Definition 1.2.4. A topological space X is an n*-type* if all homotopy groups $\pi_i(X,x)$ vanish for $i > n$, for all $x \in X$.

Given a topological space X we can construct an n-type $\tau_{\leq n}X$ by "killing" the higher homotopy groups above level n by attaching cells. We also get a map $X \to \tau_{\leq n}X$ that is isomorphism on all homotopy groups in dimension $\leq n$. Moreover, the homotopy type of $\tau_{\leq n}X$ is determined by these properties, so we call it *the n-type of X*.

Conjecture 1.2.5 (Grothendieck's Homotopy Hypothesis)**.** *There is an equivalence between (weak) n-groupoids (up to equivalence) and n-types (up to weak homotopy equivalence), such that for a topological space X the fundamental n-groupoid $\pi_{\leq n}X$ corresponds to the n-type $\tau_{\leq n}X$. For $n = \infty$, the homotopy type of X is determined by $\pi_{\leq\infty}X$.*

In low degrees, the Homotopy Hypothesis corresponds to classical results in algebraic topology:

- 0-types are weakly homotopy equivalent to sets (with discrete topology), and 0-groupoids are the same thing as sets.
- A connected 1-type is precisely an *Eilenberg–MacLane* space BG for a group G. More precisely, group are equiavlent to both pointed connected 1-types and pointed connected groupoids. This extends to an equivalence (of $(2,1)$-categories) between 1-types and groupoids.
- MacLane and Whitehead identified connected 2-types with *crossed modules*, which are also equivalent to connected strict 2-groupoids. This correspondence extends to one between general 2-types and 2-groupoids.
- It is also known that 3-types are equivalent to weak 3-groupoids.[3]

[3] This was first proved in the unpublished thesis of Leroy. See [Ber99] for a published proof by Berger.

The basic idea for the homotopical approach to higher categories is to take the Homotopy Hypothesis as given (instead of a conjecture about a hypothetical definition of weak n-groupoids), and simply *define* ∞-groupoids to be homotopy types. We can then build more complicated higher-categorical structures, such as ∞-categories, on top of these. This leads to a version of higher categories where we can avoid dealing with coherence data for composition: instead of first *choosing* specific composites and then supplying coherence data for these, we rather specify a space of all choices of compositions and then require this to be contractible. This has turned out to work much better than earlier approaches, both for developing the general theory and for constructing and working with specific examples of higher categories.

The simplest way to use homotopy theory to define a concrete notion of ∞-categories is to assume we can get away with one level of strictness (as turns out to be true) and consider categories where the morphisms form topological spaces:

Definition 1.2.6. A *topological category* (that is, a category *enriched* in topological spaces) $\mathcal{C}$ has a set of objects, and for all objects x, y a topological space $\mathcal{C}(x, y)$ of morphisms from x to y with identities $\mathrm{id}_x \in \mathcal{C}(x, x)$ and continuous composition maps

$$\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$$

for all objects x, y, z, which are (strictly) associative and unital.

While this *is* a correct definition, it has some important drawbacks. We will introduce some better-behaved models of ∞-categories below, but before that we will try to explain why ∞-categories show up in several areas of mathematics.

1.3 Localizations and ∞-categories

In an abstract sense, homotopy theory is concerned with objects that we want to consider up to some notion of equivalence that is weaker than isomorphism, such as

- topological spaces up to (weak) homotopy equivalence,
- chain complexes up to quasi-isomorphism,
- categories up to equivalence.

A *relative category* $(\mathcal{C}, W)$ consists of a category $\mathcal{C}$ with a collection W of morphisms that we think of as "weak equivalences" (formally, we can think of W as a (replete) wide subcategory of $\mathcal{C}$, so that the weak equivalences are closed under composition and contain all isomorphisms in $\mathcal{C}$). Given a relative category, we can always construct a *localization* $L\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that composition with

L identifies functors $\mathcal{C}[W^{-1}] \to \mathcal{D}$ with functors $\mathcal{C} \to \mathcal{D}$ that take morphisms in W to isomorphisms in $\mathcal{D}$. For example, topological spaces with weak homotopy equivalences localizes to the homotopy category of topological spaces, while chain complexes of R-modules with quasi-isomorphisms localizes to the derived category of R.

We can define $\mathcal{C}[W^{-1}]$ as a pushout of categories $\mathcal{C} \amalg_W W_{\mathrm{gpd}}$, where W_{gpd} is the groupoid obtained by inverting all morphisms in the subcategory W, or a bit more concretely as a category with the same objects as $\mathcal{C}$ and with morphisms from x to y given by equivalence classes of zig-zags

$$x \to \bullet \leftarrow \bullet \to \bullet \leftarrow \cdots \leftarrow y,$$

where the backward maps lie in W. In general, we have very little control over the localization $\mathcal{C}[W^{-1}]$, but in many cases we can describe it more concretely by first restricting to a class of "nice" objects and then quotienting the Hom-sets from $\mathcal{C}$ by a "homotopy" relation. For example, the homotopy category of spaces is described by CW-complexes and homotopy classes of continuous maps among them, while the derived category of R is given by taking chain complexes of projective modules and chain homotopy classes of maps.

However, it turns out that these localizations lose important homotopy-invariant information about objects of $\mathcal{C}$. For example, in topological spaces we can describe the n-sphere S^n as the pushout $D^n \amalg_{S^{n-1}} D^n$ where we glue two n-dimensional discs along their boundary S^{n-1}. On the other hand, $* \amalg_{S^{n-1}} *$ is a point, so pushouts of topological spaces are *not* homotopy-invariant. Nevertheless, it is reasonable to think of S^n as being the "homotopically correct" pushout (or "homotopy pushout") — but it is *note* the pushout in the homotopy category (which does not exist). More generally, there exists a homotopical theory of limits and colimits that we cannot see by just looking at the homotopy category.

It turns out that we can fix this problem by instead extracting from our relative category $(\mathcal{C}, W)$ an ∞-*category* that universally inverts W, as this object actually captures precisely the homotopy-invariant information from $\mathcal{C}$. The theory of ∞-categories then gives us a good language not just for working with objects of $\mathcal{C}$ up to homotopy, but also with homotopy-coherent structures (such as diagrams or algebras) built from these. This is the reason that ∞-categories have become important in several areas of mathematics, including algebraic topology, algebraic geometry, and representation theory, and in general whenever we want to work with objects up to some weak notion of equivalence.

1.4 Simplicial sets and topological spaces

The *simplex category* Δ is the category of non-empty finite ordered sets

$$[n] = \{0 < 1 < \cdots < n\}, \quad n = 0, 1, \ldots$$

A *simplicial set* is a presheaf $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$; we write $\mathsf{Set}_\Delta := \mathsf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set})$ for the category of simplicial sets.

Example 1.4.1. The *n-simplex* is the representable simplicial set

$$\Delta^n := \mathsf{Hom}_{\mathbf{\Delta}}(-, [n]).$$

The category $\mathbf{\Delta}$ is generated by

- the *face maps* $d_i \colon [n-1] \hookrightarrow [n]$ that skip $i \in [n]$
- and the *degeneracy maps* $s_i \colon [n+1] \twoheadrightarrow [n]$ that repeat $i \in [n]$,

subject to certain relations.

Definition 1.4.2. The *topological n-simplex* $|\Delta^n|$ is the topological space

$$|\Delta^n| := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \colon \sum x_i = 1, 0 \le x_i \le 1\}$$

(with the subspace topology from $\mathbb{R}^{n+1}$). For $\phi \colon [n] \to [m]$ we can define a continuous map $\phi_* \colon |\Delta^n| \to |\Delta^m|$ by

$$\phi_*(x_0, \dots, x_n)_i = \sum_{j : \phi(j) = i} x_j.$$

This gives a functor $|\Delta^\bullet| \colon \mathbf{\Delta} \to \mathsf{Top}$.

We can then define the *singular simplicial set* functor

$$\mathrm{Sing} \colon \mathsf{Top} \to \mathsf{Set}_\Delta$$

as

$$\mathrm{Sing}(X) = \mathsf{Hom}_{\mathsf{Top}}(|\Delta^\bullet|, X).$$

This has a left adjoint $|{-}| \colon \mathsf{Set}_\Delta \to \mathsf{Top}$, called the *geometric realization* functor, which is the unique colimit-preserving functor that extends $|\Delta^\bullet|$ via the Yoneda embedding. More concretely, we can define $|S|$ for a simplicial set S as the quotient of $\amalg_n S_n \times |\Delta^n|$ where we identify $(\sigma, \phi_* p)$ with $(\phi^* \sigma, p)$ for $\phi \colon [n] \to [m]$, $\sigma \in S_n$ and $p \in |\Delta^m|$. Informally, we build the topological space $|S|$ out of simplices according to the "blueprint" S.

If we say that a morphism $S \to T$ in Set_Δ is a weak equivalence if $|S| \to |T|$ is a weak homotopy equivalence[4], then the relative category consisting of Set_Δ with these weak equivalence describes the same homotopy theory as that of topological spaces;[5] for example, the counit map $|\operatorname{Sing} X| \to X$ for a topological space X is always a weak homotopy equivalence. We can also describe the weak equivalences of simplicial sets as homotopy equivalences (or describe them via homotopy groups) if we restrict to a class of nice objects, which we will introduce next.

[4] Or just a homotopy equivalence, as geometric realizations are always CW-complexes.

[5] More precisely, the adjunction $|{-}| \dashv \mathrm{Sing}$ is a *Quillen equivalence* of model categories.

Definition 1.4.3. The kth *horn* $\Lambda^n_k \subseteq \Delta^n$ is the subobject where we remove the interior and the face opposite the kth vertex. More formally,

$$(\Lambda^n_k)_i = \{\sigma\colon [i] \to [n] : \{0, \dots, k-1, k+1, \dots, n\} \not\subset \operatorname{im}(\sigma)\}.$$

Definition 1.4.4. A simplicial set S is a *Kan complex* if any horn $\Lambda^n_k \to S$ can be extended to a simplex $\Delta^n \to S$ (not necessarily uniquely).

Example 1.4.5. $\operatorname{Sing} X$ is a Kan complex for any topological space X (since $|\Lambda^n_k| \to |\Delta^n|$ is a deformation retract).

Definition 1.4.6. A *simplicial homotopy* is a map $S \times \Delta^1 \to T$.

From simplicial homotopies we get an associated notion of *homotopy equivalence* among simplicial sets.

Observation 1.4.7. Set_Δ has an internal Hom S^T for simplicial sets S and T, given by

$$(S^T)_n = \mathsf{Hom}_{\mathsf{Set}_\Delta}(T \times \Delta^n, S).$$

Fact 1.4.8. *If S is a Kan complex, then so is S^T for any simplicial set T.*

Definition 1.4.9. For a simplicial set S, we define $\pi_0 S$ to be the quotient of S_0 by the relation generated by identifying two 0-simplices if there exists a 1-simplex that connects them.

Exercise 1.1. If S is a Kan complex, then the relation defining $\pi_0 S$ is an equivalence relation.

Fact 1.4.10. *For a morphism of simplicial sets $f\colon S \to T$, we have:*

- *f is a weak equivalence if and only if for all Kan complexes K, the induced map $\pi_0 K^T \to \pi_0 K^S$ is an isomorphism.*
- *If S and T are Kan complexes, then f is a weak equivalence if and only if it is a homotopy equivalence.*

Moreover, every simplicial set S is weakly equivalent to a Kan complex (for example $\operatorname{Sing}|S|$); it follows that the homotopy category of Set_Δ can be described by taking Kan complexes and homotopy classes of maps among them. It can also be shown that weak equivalences among Kan complexes are detected on homotopy groups.

1.5 Simplicial categories

Since we can replace topological spaces by simplicial sets as a model for the homotopy theory of spaces or ∞-groupoids, it is not surprising that we can model ∞-categories by *simplicial categories*, i.e. categories enriched in Set_Δ, instead of by topological categories as we mentioned earlier. We will write Cat_Δ for the category of simplicial categories.

Exercise 1.2. Show that Cat_Δ can be described as the full subcategory of $\mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Cat})$ containing the functors whose simplical sets of objects are constant.

We can now define the right notion of weak equivalence for simplicial categories:

Definition 1.5.1. Given a simplicial category $\mathcal{C}$, we can define a *homotopy category* $h\mathcal{C}$ by taking the same objects as in $\mathcal{C}$ and setting $h\mathcal{C}(x, y) := \pi_0\mathcal{C}(x, y)$. A functor of simplicial categories $F\colon \mathcal{C} \to cD$ is a *Dwyer–Kan equivalence* if it is

- weakly fully faithful: $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is a weak equivalence of simplicial sets for all $x, y \in \mathcal{C}$;
- essentially surjective up to homotopy: $h\mathcal{C} \to h\mathcal{D}$ is essentially surjective.

A key problem with simplicial categories as a model for ∞-categories is that it is hard to access the correct ∞-groupoids or ∞-categories of functors between simplicial categories in this model. Other homotopy-invariant constructions are also hard, for example given a functor $F\colon \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is an ordinary category and $\mathcal{C}$ is a simplicial category, and we're given equivalences $F(x) \xrightarrow{\sim} G(x)$ in $\mathcal{C}$ for all $x \in \mathcal{I}$, we can't necessarily replace F by a functor that is given on objects by $x \mapsto G(x)$.

This latter problem we can remedy by considering *homotopy-coherent diagrams* of shape $\mathcal{I}$ in $\mathcal{C}$: instead of asking for F to respect composition, given $i \xrightarrow{f} j \xrightarrow{g} k$ we ask for an edge in $\mathcal{C}(Fi, Fk)$ between $F(g)F(f)$ and $F(f)$, and then given a third morphism $h\colon k \to \ell$ we ask for 2-simplices

$$\begin{tikzcd}
F(h)F(g)F(h) \arrow[r] \arrow[d] & F(h)F(gf) \arrow[d] \\
F(hg)F(f) \arrow[r] & F(hgf)
\end{tikzcd}$$

in $\mathcal{C}(i, \ell)$, and so on. We can then define a simplicial set of homotopy-coherent diagrams, which turns out to give the correct ∞-groupoid of functors. We want homotopy-coherent diagrams to be functors $\mathcal{I}_{\mathrm{coh}} \to \mathcal{C}$ from a "coherent" replacement of $\mathcal{I}$; there is also a nice way to package this data using a "nerve" functor, which we will introduce after considering a simpler version thereof.

Definition 1.5.2. We can view the ordered sets $[n]$ as categories; this gives a (fully faithful) functor $\Delta \to \mathsf{Cat}$. The *nerve* functor $N\colon \mathsf{Cat} \to \mathsf{Set}_\Delta$ is then defined by

$$\mathcal{C} \mapsto \mathsf{Hom}_{\mathsf{Cat}}([\bullet], \mathcal{C}),$$

so that

$$N\mathcal{C}_n = \{x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n\}.$$

Thus $N\mathcal{C}_n$ is the set of composable sequences of n morphisms; in particular, $N\mathcal{C}_0$ is the set of all objects of $\mathcal{C}$ and $N\mathcal{C}_1$ is the set of all morphisms — the two face maps $[0] \to [1]$ give the source and target of each morphism, and the degeneracy map $[1] \to [0]$ gives the identity morphism of each object.

Exercise 1.3. Show that $N\colon \mathsf{Cat} \to \mathsf{Set}_\Delta$ is fully faithful.

Proposition 1.5.3. *A simplicial set X is in the image of N if and only if either of the following conditions holds:*

(1) Every inner *horn $\Lambda^n_k \to X$ (where $0 < k < n$) extends to a* unique *simplex $\Delta^n \to X$.*

(2) The Segal map *$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$, which is induced by the inclusions $[1] \cong \{i-1 < i\} \hookrightarrow [n]$ and $[0] \cong \{i\} \hookrightarrow [n]$, is an isomorphism for all n.*

This is not particularly hard to prove, but we omit the details (for now). Note that a map $\Lambda^2_1 \to X$ specifies two composable edges, and the unique extension to a 2-simplex says that these edges have a unique composite. On the other hand a map $\Lambda^2_0 \to X$ specifies two edges with the same source, so in a category we should not necessarily expect an extension to Δ^2 (unless the category is a groupoid).

Exercise 1.4. Show that fillers for horns of type Λ^3_2 and Λ^3_1 give associativity for the composition of edges.

Both of the conditions in Proposition 1.5.3 can be weakened to produce good models for ∞-categories. We will introduce these below, but first we return to coherent diagrams and introduce the *coherent nerve* of simplicial categories:

Definition 1.5.4. We can explicitly describe a coherent replacement $[n]_{\mathrm{coh}}$ of $[n]$:

- the objects are $0, 1, \ldots, n$,
- the simplicial set of morphisms from i to j is $\emptyset$ if $i > j$, and otherwise it is the nerve of the partially ordered set P_{ij} of subsets of $\{i, i+1, \ldots, j\}$ that contain i and j, ordered by inclusion,
- composition if given by taking unions of subsets.

This gives a functor $[\bullet]_{\mathrm{coh}}\colon \Delta \to \mathsf{Cat}_\Delta$.

Example 1.5.5. In $[2]_{\mathrm{coh}}$ we have (as posets)

$$[2]_{\mathrm{coh}}(0,1) = \{01\}$$
$$[2]_{\mathrm{coh}}(1,2) = \{12\}$$
$$[2]_{\mathrm{coh}}(0,2) = \{02 < 012\},$$

where 012 is the composite $12 \circ 01$. Similarly, in $[3]_{\text{coh}}$ the poset of maps from 0 to 3 is

$$\begin{tikzcd} 03 \arrow[r] \arrow[d] \arrow[dr] & 013 \arrow[d] \\ 023 \arrow[r] & 0123. \end{tikzcd}$$

Definition 1.5.6. The *coherent nerve* $\mathfrak{N}\colon \mathsf{Cat}_\Delta \to \mathsf{Set}_\Delta$ is given by

$$\mathcal{C} \mapsto \mathsf{Hom}_{\mathsf{Cat}_\Delta}([\bullet]_{\text{coh}}, \mathcal{C}).$$

This has a left adjoint $\mathfrak{C}$ that extends $[\bullet]_{\text{coh}}$.

We can then describe a homotopy-coherent diagram $\mathcal{I} \to \mathcal{C}$ as a map of simplicial sets $\mathfrak{N}\mathcal{I} \to \mathfrak{N}\mathcal{C}$ where $\mathcal{I}$ and $\mathcal{C}$ are simplicial categories. (If $\mathcal{I}$ is an ordinary category, we have $\mathfrak{N}\mathcal{I} = N\mathcal{I}$.) This map indeed assigns a coherent simplex in $\mathcal{C}$ to every string of composable morphisms in $\mathcal{I}$; we can then define the coherent replacement of $\mathcal{I}$ as $\mathcal{I}_{\text{coh}} := \mathfrak{C}(\mathcal{I})$.

A simplicial category $\mathcal{C}$ is *fibrant* if $\mathcal{C}(x, y)$ is a Kan complex for all objects x, y. For $\mathcal{C}$ fibrant, the counit map $\mathfrak{C}\mathfrak{N}\mathcal{C} \to \mathcal{C}$ is always a Dwyer–Kan equivalence. All told, the coherent nerve and its adjoint gives a relationship between simplicial categories and simplicial sets that is very similar to the relation between topological spaces and simplicial sets we considered earlier. In fact, we can again describe the homotopy theory of simplicial categories very nicely using a particular class of simplicial sets.

1.6 Quasicategories

We now introduce the model for ∞-categories obtained by weaking the first condition in Proposition 1.5.3:

Definition 1.6.1. A simplicial set X is a *quasicategory* if every inner horn $\Lambda^n_k \to X$ $(0 < k < n)$ admits an extension to Δ^n (but this is *not* necessarily unique).

This condition on X says, for example, that given two composable 1-simplices, which determine a map $\Lambda^2_1 \to X$, we can compose them by choosing an extension to Δ^2, but this composite is not unique. However, one can show that the simplicial set of such composites forms a contractible Kan complex.

For $\mathcal{C}$ a fibrant simplicial category, $\mathfrak{N}\mathcal{C}$ is always a quasicategory. In fact, the adjunction $\mathfrak{C} \dashv \mathfrak{N}$ exhibits the homotopy theory of quasicategories as equivalent to that of simplicial categories, where we say that a morphism of simplicial sets $X \to Y$ is a *categorical (weak) equivalence* if $\mathfrak{C}X \to \mathfrak{C}Y$ is a Dwyer–Kan equivalence of simplicial categories.

Fact 1.6.2. *A map of quasicategories is a categorical equivalence if and only if it is "fully faithful and essentially surjective", in an appropriate sense.*

Via Proposition 1.5.3 we can think of quasicategories as a common generalization of:

- Kan complexes ("∞-groupoids"), which have fillers for *all* horns,
- and (nerves of) categories, which have *unique* fillers for *inner* horns.

In fact, the intersection of these conditions, where we demand unique fillers for all horns, characterizes precisely the simplicial sets that are nerves of groupoids.

We have seen that quasicategories give a model for ∞-categories where diagrams are *automatically* homotopy-coherent. This strongly suggests that they are a better-behaved model than simplicial categories, and this is indeed the case.

1.7 Segal spaces

We now turn to model for ∞-categories that we can obtain by weaking the second condition in Proposition 1.5.3, which said that a simplicial set X is isomorphic to the nerve of a category precisely when the Segal map

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is an isomorphism for all n.

The idea is now to replace sets by ∞-groupoids to get a similar description of ∞-categories. If we use simplicial set (Kan complexes) to model ∞-groupoids, we also have to replace the strict pullbacks by *homotopy pullbacks*. We will not go into this here, but one model for the homotopy pullback of maps $A, C \to B$ is the strict pullback

$$\begin{tikzcd} A \times^h_B C \arrow[r] \arrow[d] & B^{\Delta^1} \arrow[d] \\ A \times C \arrow[r] & B \times B; \end{tikzcd}$$

note that a point (0-simplex) in $A \times^h_B C$ then consists of a point in A, a point in C, and a path (1-simplex) between their images in B.

Informally, we then say that $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathsf{Set}_\Delta$ is a *Segal space*[6] if for all n, the simplicial set X_n is the homotopy pullback $X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$ via the Segal map.

Here we think of X_0 as the ∞-groupoid of objects in an ∞-category and X_1 as that of morphisms. The two face maps $X_1 \rightrightarrows X_0$ assign the source and target object to each morphism, while the degeneracy map $X_0 \to X_1$ gives the identity morphism for each object. We get a composition map as the composite

$$X_1 \times^h_{X_0} X_1 \xleftarrow{\sim} X_2 \to X_1$$

[6]As is conventional, here we will sometimes use *space* as a synonym for *simplicial set*.

after inverting the homotopy equivalence. The commutative square

$$\begin{tikzcd} X_3 \arrow[r, "d_2"] \arrow[d, "d_1"'] & X_2 \arrow[d, "d_1"] \\ X_2 \arrow[r, "d_1"'] & X_1 \end{tikzcd}$$

gives associativity for this composition.

For an ordinary category $\mathcal{C}$, we can view its nerve $N\mathcal{C}$ as a (discrete) Segal space. If E^1 is the "free isomorphism" with objects 0 and 1 and a unique morphism $i \to j$ for all $i, j \in \{0, 1\}$, a map $NE^1 \to X_\bullet$ then gives a coherent equivalence in X

Definition 1.7.1. A Segal space is *complete* if $X_0 \to \mathsf{Map}(NE^1, X)$ is a weak equivalence.

This condition says that the simplicial set of objects of X is the "correct" one — the paths between objects in X_0 give precisely the equivalences between these objects in $X_\bullet$.

Complete Segal spaces are another good model of ∞-categories. Moreover, if $X_\bullet$ is a Segal space, one can show that the simplicial set $(X_\bullet)_0$ of levelwise 0-simplices is a quasicategory. In fact, a key problem with quasicategories is that it is usually not possible to directly write down a simplicial set and check by hand that it has the inner horn fillers required to be a quasicategory — it is often more feasible to prove that some simplicial space is a Segal space.

1.8 What we will actually do in this text

To develop the foundations of ∞-categories rigorously you have to start with a model, such as quasicategories, and then develop analogues of basic concepts from category theory using that model. Then you have prove that constructions like homotopy (co)limits in model categories describe ∞-categorical versions in the quasicategory you get by inverting the weak equivalences. As far as ∞-categories themselves are concerned, this is essentially a "bootstrapping" process, so that in the end you know that the external notions of, say, homotopy pullbacks you use to set up the theory of quasicategories agree with the corresponding internal notion of pullbacks applied to the quasicategory of quasicategories.

On the one hand, this process takes a lot of work (which could easily fill a whole semester). However, on the other hand once it's done you can essentially forget about it: in practice we almost always want to work "model-independently" with ∞-categories, i.e. view them as themselves being objects of an ∞-category, where we can only do the fully coherent internal constructions.

There are already several books [Cis19, Lan21, RV22] that explain how to set up the foundations of quasicategories, so in this text I have attempted to take a different approach: I will try to explain directly how ∞-categories work model-independently. In my opinion there are several advantages to this approach:

- this material should hopefully be more useful to the reader who wants to learn how to use ∞-categories for applications in their own area of mathematics, rather than to work on their foundations,
- but understanding first how things work without reference to a model also often makes it much clearer what is really going on in model-specific definitions and proofs.

The disadvantage, of course, is that the start of the text will not be entirely rigorous, or at least not self-contained.

(An alternative approach to foundations, currently being worked out by Cisinski, Cnossen, Nguyen and Walde, is to instead axiomatically specify how a strict model of ∞-categories should behave. While this is not tied to any particular model, it is different from the philosophy of this document, since it still starts at a *strict* level. While this strictness is unsatisfying, it is admittedly likely that it will always be necessary to start with strict constructions to rigorously reach the platonic realm of homotopy theory from our world.)

Chapter 2

Getting started

We are going to be working with ∞-groupoids and ∞-categories as basic objects, and we will start by describing the basic operations and structures relating these. We think of this as an "external" description of the ∞-categories of (small) ∞-groupoids and ∞-categories — we emphasize that this is *informal* and should in no way be thought of as an attempt to axiomatize ∞-groupoids and ∞-categories!

2.1 Basic properties of ∞-groupoids

Fact 2.1.1. *There are objects called* ∞-groupoids.

In an ∞-groupoid we have points, paths between points, homotopies between paths, homotopies between homotopies, and so on. However, we are not allowed to distinguish between points that have a path between them, paths that have a homotopy between them. An ∞-groupoid X therefore does *not* have a well-defined "set of points", only a set $\pi_0 X$ of path components.

We also have *maps* or *morphisms* between ∞-groupoids, homotopies between morphisms, homotopies between homotopies, and so on. For maps from X to Y, these form the points, paths, homotopies, and so on in an ∞-groupoid $\mathsf{Map}(X, Y)$. We can *compose* morphisms, and for any ∞-groupoid X there is an identity morphism $\mathrm{id}_X \in \mathsf{Map}(X, X)$. Composition is unital and associative in the only way that makes sense[1]: given morphisms

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z,$$

there is a homotopy $h(gf) \simeq (hg)f$, as well as homotopies $\mathrm{id}_X \circ f \simeq f \simeq f \circ \mathrm{id}_W$.

Definition 2.1.2. A morphism $g\colon X \to Y$ is an *equivalence* if there exist a morphism $f\colon Y \to X$ and homotopies $g \circ f \simeq \mathrm{id}_Y$, $f \circ g \simeq \mathrm{id}_X$.

[1] Since we are not allowed to say that two points in an ∞-groupoid are the same, only that they are connected by a (specified) path.

Remark 2.1.3. Here we are defining "being an equivalence" as a *property*. If we wanted to define an ∞-*groupoid* of equivalences, we should take a bit more care to coherently define the *data* of an equivalence.

Fact 2.1.4. *Sets are ∞-groupoids, and if S, T are sets then $\mathsf{Map}(S,T)$ is the set $\mathsf{Hom}(S,T)$ of functions from S to T. For any ∞-groupoid X there is a canonical map $X \to \pi_0 X$ to its set of path components. If S is a set, then composition with this map gives an equivalence*

$$\mathsf{Hom}(\pi_0 X, S) \xrightarrow{\sim} \mathsf{Map}(X, S).$$

Observation 2.1.5. In particular, the one-point set $*$ is an ∞-groupoid, and for an ∞-groupoid X we have an equivalence

$$* \cong \mathsf{Hom}(\pi_0 X, *) \xrightarrow{\sim} \mathsf{Map}(X, *),$$

so that there is a unique map $X \to *$. In other words, $*$ is the terminal ∞-groupoid.

Fact 2.1.6. *The empty set $\emptyset$ is also an ∞-groupoid. For any ∞-groupoid X, the unique map $\mathsf{Map}(\emptyset, X) \to *$ is an equivalence. In other words, $\emptyset$ is the initial ∞-groupoid.*

A *commutative square*

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "h"] \\ Y \arrow[r, "k"'] & Z \end{tikzcd}$$

consists of ∞-groupoids and morphisms as shown, together with a homotopy $h \circ f \simeq k \circ g$.

Fact 2.1.7. *Given morphisms $X \to Z$ and $Y \to Z$, there exists a* pullback *square*

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd}$$

Here a point of $X \times_Z Y$ is a point in X and a point in Y together with a path between their images in Z. More generally, a map $W \to X \times_Z Y$ is determined by maps $W \to X$, $W \to Y$ and a homotopy between their composites to Z.

Warning 2.1.8. This may be the least precise part of this discussion: If you try to write down an ∞-groupoid of squares to make the universal property of pullbacks precise, you'll find that this has to be defined as itself being a pullback. This unfortunately suggests that a rigorous development of the theory has to be bootstrapped from a setting with strictly commuting squares.

Fact 2.1.9. *Pullbacks of sets are computed as usual.*

As important special cases of pullbacks, we have:

- The *fibre* $f^{-1}(b)$ at b of a map $f\colon E \to B$ is defined as the pullback

$$\begin{tikzcd} f^{-1}(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ \{b\} \arrow[r] & B. \end{tikzcd}$$

- The *product* $X \times Y$ of two ∞-groupoids X and Y is defined as the pullback

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & *. \end{tikzcd}$$

Fact 2.1.10. *For ∞-groupoids X, Y there is a canonical* evaluation map

$$\mathrm{ev}\colon \mathsf{Map}(X, Y) \times X \to Y,$$

so that for any ∞-groupoid Z the induced morphism

$$\mathsf{Map}(Z, \mathsf{Map}(X, Y)) \to \mathsf{Map}(Z \times X, \mathsf{Map}(X, Y) \times X)\mathsf{Map}(Z \times X, Y)$$

is an equivalence.

We can compose squares: Given two squares

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "b"'] & W \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "h"] \\ V \arrow[r, "c"'] & Y \arrow[r, "k"] & Z, \end{tikzcd}$$

we get a square

$$\begin{tikzcd} U \arrow[r, "fa"] \arrow[d, "b"'] & X \arrow[d, "h"] \\ V \arrow[r, "kc"'] & Z, \end{tikzcd}$$

using the given homotopies together with the coherence homotopies for associativity:

$$h(fa) \simeq (hf)a \simeq (kg)a \simeq k(ga) \simeq k(cb) \simeq (kc)b.$$

Fact 2.1.11. *The composition of two pullback squares is again a pullback square.*

Definition 2.1.12. For points $x, y \in X$, the *path space* $X(x, y)$ is the pullback

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{x\} \arrow[d] \\ \{y\} \arrow[r] & X. \end{tikzcd}$$

If the two points are the same, this gives the *loop space* $\Omega_x X := X(x, x)$. A point in the path space $X(x, y)$ is a path between x and y; in $\Omega_x X$ we thus have a canonical point, namely the constant path or the identity homotopy of the map $\{x\} \to X$. We can therefore iterate the loop space construction and obtain for each n the *n-fold loop space* $\Omega^n_x X$.

Exercise 2.1. Assuming that pushouts exist and have the expected universal property, show that $\Omega^n_x X \simeq \mathsf{Map}_*(S^n, X)$, where the n-sphere is the pushout

$$S^n := * \amalg_{S^{n-1}} *,$$

and the space of pointed maps is the pullback

$$\begin{tikzcd} \mathsf{Map}_*(S^n, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Map}(S^n, X) \arrow[d] \\ \{x\} \arrow[r] & \mathsf{Map}(*, X). \end{tikzcd}$$

Definition 2.1.13. For $x \in X$ and n, we define $\pi_n(X, x) := \pi_0 \Omega^n_x X$. These are the ($n$th) *homotopy groups* of X.

Fact 2.1.14. *$\pi_1(X, x)$ is a group, and $\pi_n(X, x)$ is an abelian group for $n > 1$.*

Fact 2.1.15. *Homotopy groups detect equivalences: a map $f : X \to Y$ is an equivalence if and only if the induced maps $\pi_0 X \to \pi_0 Y$ and*

$$\pi_n(X, x) \to \pi_n(Y, f(x))$$

are isomorphisms for all n and all $x \in \pi_0 X$.

Remark 2.1.16. If $\pi_0 X \cong \emptyset$, then this should be interpreted as implying that $\emptyset \to X$ is an equivalence.

Corollary 2.1.17. *Equivalences of ∞-groupoids satisfy the 3-for-2 property: if g and f are composable maps and two out of f, g, gf are equivalences, then so is the third.* □

Corollary 2.1.18. *A morphism $f : X \to Y$ is an equivalence if and only if $\pi_0 X \to \pi_0 Y$ is surjective and $X(x, x') \to Y(fx, fx')$ is an equivalence for all $x, x' \in X$.*

Proof. We know these conditions hold when f is an equivalence. Moreover, if f induces an *isomorphism* on π_0 then the assumption on mapping spaces implies we have isomorphisms on all homotopy groups, so that f is an equivalence by

Fact 2.1.15. It therefore suffices to show that under the given conditions the map $\pi_0 f$ is necessarily injective. But if x and x' are in different path components[2] then $X(x, x') \simeq \emptyset$ by Remark 2.1.16 and so $Y(fx, fx') \simeq \emptyset$, which implies that fx and fx' must also be in different components. □

Corollary 2.1.19. *For an ∞-groupoid X, the map $X \to \pi_0 X$ is an equivalence if and only if $X(x, x')$ is either empty or contractible for all $x, x' \in X$.*

Proof. Since we get an isomorphism on π_0, this map is an equivalence if and only if
$$X(x, x') \to \pi_0(X)([x][x'])$$
is an equivalence for all $x, x' \in X$. Here $\pi_0(X)([x][x'])$ is either a point or empty, depending on whether x and x' lie in the same path component or not. □

Corollary 2.1.20. *A non-empty ∞-groupoid X is contractible if and only if $X(x, x')$ is contractible for all $x, x' \in X$.*

Proof. If X is non-empty, then $X \to *$ is trivially surjective on π_0, so if it also gives equivalences on all path spaces it is an equivalence. □

Fact 2.1.21 (Long exact sequence of homotopy groups)**.** *For a map $f : E \to B$, a point $b \in B$ and a point $e \in f^{-1}(b)$, we have a long exact sequence of homotopy groups*
$$\cdots \to \pi_n(f^{-1}(b), e) \to \pi_n(E, e) \to \pi_n(B, b) \to \pi_{n-1}(f^{-1}(b), e) \to \cdots \to \pi_0(E) \to \pi_0(B),$$
interpreted appropriately for the group π_1 and the pointed set π_0.

Proposition 2.1.22. *A map $f : E \to B$ is an equivalence if and only if all the fibres $f^{-1}(b)$ for $b \in B$ are contractible.*

Proof. If the fibres of f are contractible, then the long exact sequence gives isomorphisms on all homotopy groups, so that f is an equivalence by Fact 2.1.15. Conversely, if f is an equivalence then the long exact sequence shows that π_0 of the fibres are points and the higher homotopy groups are 0, so they are contractible. □

Exercise 2.2. Use the 5-lemma to show that given a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B, & \end{tikzcd}$$

the morphism f is an equivalence if and only if the induced maps on fibres $p^{-1}(b) \to q^{-1}(b)$ are equivalences for all $b \in B$.

[2] Here we are somewhat informally using our intuition about "path components"!

Corollary 2.1.23. *A commutative square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & Y' \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

is a pullback if and only if for every $x \in X$, the induced map on fibres is an equivalence.

Proof. The square is a pullback if the induced map $X' \to Y' \times_Y X$ is an equivalence, which we can check on fibres over $x \in X$. But since pullbacks compose, this just gives the map on fibres $f'^{-1}(x) \to f^{-1}(g(x))$ in the original square. □

Exercise 2.3. Consider a commutative square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Show that if f is an equivalence, then the square is a pullback if and only if g is also an equivalence.

Exercise 2.4. Suppose we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X'' \arrow[d] \\ Y \arrow[r] & Y' \arrow[r] & Y''. \end{tikzcd}$$

(1) If the right and composite squares are both pullbacks, then so is the left-hand square.

(2) If $\pi_0 Y \to \pi_0 Y'$ is surjective and the left and composite squares are both pullbacks, then so is the right-hand square.

Exercise 2.5. Show that the path space $X(x, y)$ is also the fibre at $\{(x, y)\}$ of the diagonal map $X \to X \times X$.

2.2 Monomorphisms of ∞-groupoids

As a warm-up to future discussions of (full) subcategories of ∞-categories, here we will briefly discuss monomorphisms of ∞-groupoids.

Definition 2.2.1. A morphism of ∞-groupoids $f : X \to Y$ is a *monomorphism* if the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta"'] & Y \arrow[d, "\Delta"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

is a pullback.

Exercise 2.6. Show that monomorphisms are closed under base change.

Observation 2.2.2. Using Corollary 2.1.23, it follows that f is a monomorphism if and only if for all $x, x' \in X$, the induced morphism on path spaces

$$X(x, x') \to Y(fx, fx')$$

is an equivalence.

Lemma 2.2.3. *A morphism of ∞-groupoids $f : X \to Y$ is a monomorphism if and only if the fibres of f are all either empty or contractible.*

Proof. The fibre condition for pullbacks shows that f is a monomorphism if and only if the diagonal map $f^{-1}(y) \to f^{-1}(y) \times f^{-1}(y)$ is an equivalence for all $y \in Y$. We can also interpret this condition as saying that $f^{-1}(y) \to *$ is a monomorphism, or in other words that $f^{-1}(y)(p, q)$ is contractible for all $p, q \in f^{-1}(y)$. By Corollary 2.1.20 this means $f^{-1}(y)$ is either empty or contractible. □

Proposition 2.2.4. *If $X \to Y$ is a monomorphism, then $\pi_0 X \to \pi_0 Y$ is a monomorphism of sets, and the commutative square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \pi_0 X \arrow[r] & \pi_0 Y \end{tikzcd}$$

is a pullback.

Proof. It follows from the long exact sequence that we have an injection on π_0. Now we use the fibre condition for pullbacks (2.1.23) to conclude that the square is a pullback, since these are either both empty or both contractible in the two rows. □

Observation 2.2.5. Suppose $Y_0 \to Y$ is a monomorphism of ∞-groupoids. Combining the pullback from Proposition 2.2.4 with the universal property of π_0 for maps into sets from Fact 2.1.4, we see that for any ∞-groupoid Z we have a pullback square

$$\begin{tikzcd} \mathsf{Map}(Z, Y_0) \arrow[r] \arrow[d] & \mathsf{Map}(Z, Y) \arrow[d] \\ \mathsf{Hom}(\pi_0 Z, \pi_0 Y_0) \arrow[r] & \mathsf{Hom}(\pi_0 Z, \pi_0 Y). \end{tikzcd}$$

Exercise 2.7. Let X be an ∞-groupoid and consider a subset $S \subseteq \pi_0 X$. Show that if we form the pullback

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, hook] & \pi_0 X \end{tikzcd}$$

then the induced map $\pi_0 Y \to S$ is an isomorphism.

Corollary 2.2.6. *Given a subset S of $\pi_0 X$, there exists a unique monomorphism $i\colon X' \to X$ so that $\pi_0 i$ is the inclusion $S \hookrightarrow \pi_0 X$.* □

Definition 2.2.7. A morphism of ∞-groupoids $f\colon X \to Y$ is an *epimorphism* if $\pi_0 f$ is a surjection.

Observation 2.2.8. For any morphism of ∞-groupoids $f\colon X \to Y$, we can factor $\pi_0 f$ uniquely as $\pi_0 X \xrightarrow{s} S \xrightarrow{i} \pi_0 Y$ where s is surjective and i is injective. Let $P := Y \times_{\pi_0 Y} S$. Then we get a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Y \arrow[d] \\
\pi_0 X \arrow[r] & S \arrow[r] & \pi_0 Y
\end{tikzcd}$$

where the vertical maps give isomorphisms on π_0. It follows that $X \to P$ is an epimorphism, while $P \to Y$ is a monomorphism, so every map of ∞-groupoids factors in this way.

Corollary 2.2.9. *A morphism of ∞-groupoids is an equivalence if and only if it is a monomorphism and is surjective on π_0.*

Proof. It is clear that both conditions hold for an equivalence. Conversely, using Proposition 2.2.4 we see that a monomorphism that is surjective on π_0 is an isomorphism on π_0 and is pulled back from this, so it is an equivalence by Exercise 2.3. □

Exercise 2.8. Given a commutative triangle

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& B &
\end{tikzcd}$$

of ∞-groupoids, the morphism f is a monomorphism if and only if for all $b \in B$, the induced map on fibres $X_b \to Y_b$ is a monomorphism.

Lemma 2.2.10. *A commutative square of ∞-groupoids*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\
Z \arrow[r] & W,
\end{tikzcd}$$

where the vertical map are monomorphisms, is a pullback square if and only if it is one on π_0.

Proof. Consider the commutative cube

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd] & \\
& \pi_0 X \arrow[rr] \arrow[dd] & & \pi_0 Y \arrow[dd] \\
Z \arrow[rr] \arrow[dr] & & W \arrow[dr] & \\
& \pi_0 Z \arrow[rr] & & \pi_0 W,
\end{tikzcd}$$

where the top and bottom faces are pullbacks by Proposition 2.2.4. It then follows from Exercise 2.4 that the back face is a pullback if and only if the front face is a pullback, since the map $Q \to \pi_0 Q$ is surjective on π_0 for any ∞-groupoid Q. $\square$

2.3 Basic properties of ∞-categories

Fact 2.3.1. *There are objects called* ∞-categories. *There are also* functors *between ∞-categories, and given a pair of ∞-categories $\mathcal{C}, \mathcal{D}$ there is a* functor ∞-category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. *For an ∞-category $\mathcal{C}$ there is an identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$, and we can compose functors (which also induces functors on functor ∞-categories).*

Fact 2.3.2. *∞-groupoids are ∞-categories. An ∞-category $\mathcal{C}$ has an* underlying ∞-groupoid $\mathcal{C}^{\simeq}$, *obtained by "throwing away the non-invertible morphisms", with a canonical map $\mathcal{C}^{\simeq} \to \mathcal{C}$ (which is the identity if $\mathcal{C}$ is an ∞-groupoid). We write*

$$\mathsf{Map}(\mathcal{C}, \mathcal{D}) := \mathsf{Fun}(\mathcal{C}, \mathcal{D})^{\simeq};$$

then the underlying ∞-groupoid of $\mathcal{C}$ satisfies: for any ∞-groupoid X, the induced map

$$\mathsf{Map}(X, \mathcal{C}^{\simeq}) \to \mathsf{Map}(X, \mathcal{C})$$

is an equivalence.

Paths in $\mathsf{Map}(\mathcal{C}, \mathcal{D})$ are *natural equivalences*, and we can say that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a functor $G \colon \mathcal{D} \to \mathcal{C}$ and natural equivalences $GF \simeq \mathrm{id}_{\mathcal{C}}$, $FG \simeq \mathrm{id}_{\mathcal{D}}$. Composition of functors is then unital and associative up to (coherent) natural equivalences.

Fact 2.3.3. *There are pullbacks of ∞-categories, just as for ∞-groupoids, and they are preserved by* Map *and* Fun.

Fact 2.3.4. *For ∞-categories $\mathcal{C}$ and $\mathcal{D}$, there is an evaluation functor $\mathsf{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \to \mathcal{D}$, which induces an equivalence*

$$\mathsf{Fun}(\mathcal{B}, \mathsf{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathsf{Fun}(\mathcal{B} \times \mathcal{C}, \mathcal{D})$$

for all ∞-categories $\mathcal{B}$.

Fact 2.3.5. *Ordinary categories are ∞-categories. If $\mathcal{C}, \mathcal{D}$ are ordinary categories, then* $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ *is the ordinary category of functors. (And if $\mathcal{C}$ is an ordinary category then $\mathcal{C}^{\simeq}$ is its underlying groupoid.)*

Fact 2.3.6. *Any ∞-category $\mathcal{C}$ has a* homotopy category *$h\mathcal{C}$, which is an ordinary category, with a canonical functor $\mathcal{C} \to h\mathcal{C}$. If $\mathcal{D}$ is an ordinary category, then*

$$\mathsf{Fun}(h\mathcal{C}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

Fact 2.3.7. *Given functors $\mathcal{A} \to \mathcal{B}, \mathcal{A} \to \mathcal{C}$, there exists a* pushout square

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{B} \amalg_{\mathcal{A}} \mathcal{C}, \end{tikzcd}$$

so that for any ∞-category $\mathcal{D}$ the induced square

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{B} \amalg_{\mathcal{A}} \mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{B}, \mathcal{D}) \arrow[d] \\ \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \arrow[r] & \mathsf{Fun}(\mathcal{A}, \mathcal{D}) \end{tikzcd}$$

is a pullback. Pushout squares are closed under composition of squares.

Definition 2.3.8.

- Let $[n]_{\mathsf{Seg}}$ be the iterated pushout $[1] \amalg_{[0]} \cdots \amalg_{[0]} [1]$ of n copies of $[1]$ along the two inclusions of $[0]$, and define the *Segal map* to be the map determined by the inert inclusions $[1] \cong \{i-1, i\} \hookrightarrow [n]$ and $[0] \cong \{i\} \hookrightarrow [n]$.
- Let E be the iterated pushout $[0] \amalg_{[1]} [3] \amalg_{[1]} [0]$ using the inclusions $\{0 < 2\} \to [3], \{1 < 3\} \to [3]$.[3]

Fact 2.3.9. *The following functors are equivalences:*

- *The Segal maps $[n]_{\mathsf{Seg}} \to [n]$.*
- *The map $E \to *$.*
- *The map $[2] \amalg_{[1]} [2] \to [1] \times [1]$ that picks out the two composite maps $(0,0) \to (0,1) \to (1,1)$, $(0,0) \to (1,0) \to (1,1)$.*

[3] A functor $E \to \mathcal{C}$ then specifies morphisms $x \xrightarrow{f} y \xrightarrow{g} x \xrightarrow{h} y$ and identifies gf and hg with identities, so that g has a left and a right inverse. In other words, a map from E is a (coherent) equivalence in $\mathcal{C}$.

Fact 2.3.10. *Given an ∞-category $\mathcal{C}$, there exists a* localization $\|\mathcal{C}\|$ *to an ∞-groupoid, with a canonical map $\mathcal{C} \to \|\mathcal{C}\|$, so that for any ∞-groupoid X the induced map*

$$\mathsf{Map}(\|\mathcal{C}\|, X) \to \mathsf{Map}(\mathcal{C}, X)$$

is an equivalence. Moreover, $\|[1]\| \simeq *$.

Fact 2.3.11. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if the maps*

$$\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}, \quad \mathsf{Map}([1], \mathcal{C}) \to \mathsf{Map}([1], \mathcal{D})$$

are equivalences of ∞-groupoids.

Lemma 2.3.12. *The following are equivalent for an ∞-category $\mathcal{C}$:*

(1) $\mathcal{C}$ *is an ∞-groupoid.*

(2) $\mathcal{C}^{\simeq} \to \mathcal{C}$ *is an equivalence.*

(3) The map

$$\mathcal{C}^{\simeq} \cong \mathsf{Map}([0], \mathcal{C}) \to \mathsf{Map}([1], \mathcal{C})$$

induced by $[1] \to [0]$ *is an equivalence of ∞-groupoids.*

(4) The functor

$$\mathcal{C} \to \mathsf{Fun}([1], \mathcal{C})$$

induced by $[1] \to [0]$ *is an equivalence of ∞-categories.*

Proof. The first two points are equivalent since the inclusion of the underlying ∞-groupoid is assumed to be invertible when $\mathcal{C}$ is an ∞-groupoid. The third point is implied by $\mathcal{C}$ being an ∞-groupoid since $\|[1]\|$ is contractible; for the converse, we consider the commutative square

$$\begin{tikzcd}
\mathsf{Map}([0], \mathcal{C}^{\simeq}) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathsf{Map}([0], \mathcal{C}) \arrow[d] \\
\mathsf{Map}([1], \mathcal{C}^{\simeq}) \arrow[r] & \mathsf{Map}([1], \mathcal{C}).
\end{tikzcd}$$

Here the top and left maps are equivalences, so the right map is an equivalence if and only if the bottom map is, and the latter corresponds to $\mathcal{C}$ being an ∞-groupoid since $[0]$ and $[1]$ detect equivalences. Since (4) immediately implies (3), it remains to show that (3) implies (4). For this, it suffices to prove that

$$\mathsf{Map}([1], \mathcal{C}) \to \mathsf{Map}([1], \mathsf{Fun}([1], \mathcal{C})).$$

is an equivalence. Here we can identify the right-hand side as $\mathsf{Map}([1] \times [1], \mathcal{C})$ and then use the decompositions of $[1] \times [1]$ as $[2] \amalg_{[1]} [2]$ and of $[2]$ as $[1] \amalg_{[0]} [1]$ together with (3) to see that this ∞-groupoid is equivalent to $\mathcal{C}^{\simeq}$. □

Exercise 2.9. Show that if X is an ∞-groupoid, then so is $\mathsf{Fun}(\mathcal{C}, X)$ for any ∞-category $\mathcal{C}$.

Fact 2.3.13. *For an ∞-category $\mathcal{C}$, there is an* opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. *For ordinary categories this gives the usual opposite category, an satisfies*

- $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathcal{C}$,
- $(\mathcal{C}^{\mathrm{op}})^{\simeq} \simeq \mathcal{C}^{\simeq}$,
- $\mathsf{Map}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}) \simeq \mathsf{Map}(\mathcal{C}, \mathcal{D})$.

Exercise 2.10. Show that $\mathsf{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$.

2.4 Lifting properties

It will be convenient to characterize a number of important classes of functors by lifting properties, so we include a brief discussion of these.[4]

Definition 2.4.1. For morphisms $\ell \colon A \to B$ and $r \colon X \to Y$ we say that r is *right orthogonal* to ℓ (and dually that ℓ is *left orthogonal* to r) if for any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\ell"'] & X \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dotted] & Y, \end{tikzcd}$$

the space of diagonal lifts $B \to X$ is contractible. In other words, the fibres of the map

$$\mathsf{Map}(B, X) \to \mathsf{Map}(B, Y) \times_{\mathsf{Map}(A,Y)} \mathsf{Map}(A, X),$$

given by composition with ℓ and r, are contractible. This is equivalent to this map being an equivalence, or to the commutative square

$$\begin{tikzcd} \mathsf{Map}(B, X) \arrow[r, "r_*"] \arrow[d, "\ell^*"'] & \mathsf{Map}(B, Y) \arrow[d, "\ell^*"] \\ \mathsf{Map}(A, X) \arrow[r, "r_*"'] & \mathsf{Map}(A, Y) \end{tikzcd}$$

being a pullback.

Example 2.4.2. A morphism of ∞-groupoids $X \xrightarrow{f} Y$ is a monomorphism if and only if it is right orthogonal to $* \amalg * \to *$, since this amounts to having a

[4] This section can be read as either defining lifting properties among ∞-categories or inside any fixed ∞-category $\mathcal{C}$, except that we haven't quite gotten to the point where the latter makes.

pullback square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y. \end{tikzcd}$$

Lemma 2.4.3. *Epimorphisms in* Gpd_∞ *are left orthogonal to monomorphisms.*

Proof. Suppose $i\colon X_0 \to X$ is a monomorphism and $p\colon A \to B$ is an epimorphism. Then we have a commutative cube

$$\begin{tikzcd}[column sep=small] \mathsf{Map}(B, X_0) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Map}(B, X) \arrow[dr] \arrow[dd] & \\ & \mathsf{Map}(A, X_0) \arrow[rr] \arrow[dd] & & \mathsf{Map}(A, X) \arrow[dd] \\ \mathsf{Hom}(\pi_0 B, \pi_0 X_0) \arrow[rr] \arrow[dr] & & \mathsf{Hom}(\pi_0 B, \pi_0 X) \arrow[dr] & \\ & \mathsf{Hom}(\pi_0 A, \pi_0 X_0) \arrow[rr] & & \mathsf{Hom}(\pi_0 A, \pi_0 X) \end{tikzcd}$$

where the front and back faces are pullbacks. Moreover, the bottom face is a pullback since surjections are left orthogonal to injections in **Set**. Hence the top face is also a pullback, as required. □

Notation 2.4.4. We use the notation $\ell \perp r$ as an abbreviation for "ℓ is left orthogonal to r".

Exercise 2.11. Show that a map is left orthogonal to itself if and only if it is an equivalence.

Lemma 2.4.5. *Suppose* $f\colon A \to B$ *is left orthogonal to a map* $r\colon X \to Y$*. Then a map* $g\colon B \to C$ *is left orthogonal to* r *if and only if* $g \circ f$ *is so.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathsf{Map}(C, X) \arrow[r] \arrow[d] & \mathsf{Map}(B, X) \arrow[r] \arrow[d] & \mathsf{Map}(A, X) \arrow[d] \\ \mathsf{Map}(C, Y) \arrow[r] & \mathsf{Map}(B, Y) \arrow[r] & \mathsf{Map}(A, Y) \end{tikzcd}$$

where $f \perp r$ amounts to the right-hand square being a pullback. The statement then follows from the 3-for-2 property of pullbacks (Exercise 2.4). □

The next properties are proved by similar manipulations of pullbacks, and are left as an exercise for the reader.

Lemma 2.4.6. *Suppose we have a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"] & Y \arrow[l] \arrow[r] \arrow[d, "g"] & Z \arrow[d, "h"] \\ X' & Y' \arrow[l] \arrow[r] & Z', \end{tikzcd}$$

such that each of the morphisms f, g and h is left orthogonal to a morphism $r\colon U \to V$. Then the induced morphism on pushouts $X \amalg_Y Z \to X' \amalg_{Y'} Z'$ is also left orthogonal to r. □

Lemma 2.4.7. *Suppose we have a pushout square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ C \arrow[r] & B \amalg_A C \end{tikzcd}$$

where f is left orthogonal to a morphism r. Then f' is also left orthogonal to r. □

Definition 2.4.8. Recall that an object X is a *retract* of Y if there are maps $X \to Y \to X$ and a homotopy between the composite and the identity of X. Similarly, we say that a morphism f' is a *retract* of f if there is a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y' \arrow[r] & Y \arrow[r] & Y' \end{tikzcd}$$

and a homotopy between the composite and the degenerate square

$$\begin{tikzcd} X' \arrow[r, "\mathrm{id}_{X'}"] \arrow[d, "f'"'] & X' \arrow[d, "f'"] \\ Y' \arrow[r, "\mathrm{id}_{Y'}"'] & Y'. \end{tikzcd}$$

Exercise 2.12. Show that any retract of an equivalence is again an equivalence.

Lemma 2.4.9. *Suppose f' is a retract of f. If f is left orthogonal to a morphism r, then so is f'.* □

2.5 Conservative functors and mapping spaces

Definition 2.5.1. A functor $\mathcal{C} \to \mathcal{D}$ is *conservative* if it is right orthogonal to $s_0\colon [1] \to [0]$.

Exercise 2.13. The following are equivalent for a functor $F\colon \mathcal{C} \to \mathcal{D}$:

(1) F is conservative.

(2) The fibres of F are all ∞-groupoids.

(3) The commutative square

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{D}^{\simeq} \arrow[r] & \mathcal{D} \end{tikzcd}$$

is a pullback.

Remark 2.5.2. We will later see (Corollary 2.9.5) that the degeneracy map $\mathcal{C}^{\simeq} \to \mathsf{Map}([1], \mathcal{C})$ is a monomorphism of ∞-groupoids for every ∞-category $\mathcal{C}$. Using Lemma 2.2.10 it follows that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is conservative if and only if we have a pullback square of sets

$$\begin{tikzcd} \pi_0\mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \pi_0\mathcal{D}^{\simeq} \arrow[d] \\ \pi_0\mathsf{Map}([1], \mathcal{C}) \arrow[r] & \pi_0\mathsf{Map}([1], \mathcal{D}), \end{tikzcd}$$

which we can interpret as saying that a morphism in $\mathcal{C}$ has the property of being an equivalence if and only if its image in $\mathcal{D}$ is an equivalence.

Notation 2.5.3. We write $\mathsf{Ar}(\mathcal{C}) := \mathsf{Fun}([1], \mathcal{C})$.

Exercise 2.14. Use the pushout decompositions $[2] \simeq [1] \amalg_{[0]} [1]$ and $[1] \times [1] \simeq [2] \amalg_{[1]} [2]$ to show that there is a pushout square

$$\begin{tikzcd} {[1] \amalg [1]} \arrow[r, "(\mathrm{id}\times d_0 s_0)\amalg(\mathrm{id}\times d_1 s_0)"] \arrow[d, "s_0 \amalg s_0"'] & {[1] \times [1]} \arrow[d, "s_0 \times \mathrm{id}"] \\ {[0] \amalg [0]} \arrow[r, "d_0 \amalg d_1"'] & {[1].} \end{tikzcd}$$

(Informally, this says that collapsing the two vertical edges in the square $[1] \times [1]$ gives the edge $[1]$.)

Proposition 2.5.4. *$\mathcal{C} \to \mathcal{D}$ is conservative if and only if*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathsf{Ar}(\mathcal{C}) \arrow[r] & \mathsf{Ar}(\mathcal{D}) \end{tikzcd}$$

is a pullback.

Proof. Pullbacks are detected on $(-)^{\simeq}$ and $\mathsf{Map}([1], -)$. Conservativity is equivalent to the former, so it suffices to show that it implies the latter. The resulting square can be identified as

$$\begin{tikzcd}
\mathsf{Map}([1], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}([1], \mathcal{D}) \arrow[d] \\
\mathsf{Map}([1] \times [1], \mathcal{C}) \arrow[r] & \mathsf{Map}([1] \times [1], \mathcal{D}).
\end{tikzcd}$$

This is a pullback since it follows from Lemma 2.4.7, Lemma 2.4.6, and Exercise 2.14 that a conservative functor is right orthogonal to the projection $[1] \times [1] \to [1]$. □

Proposition 2.5.5. $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ *is conservative.*

Proof. Unpacking the definition, this is immediate from the pushout in Exercise 2.14. □

Definition 2.5.6. For objects $x, y \in \mathcal{C}$, the *mapping space*

$$\mathsf{Map}_{\mathcal{C}}(x, y) = \mathcal{C}(x, y)$$

is the fibre

$$\begin{tikzcd}
\mathcal{C}(x, y) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{C}) \arrow[d] \\
\{(x, y)\} \arrow[r] & \mathcal{C} \times \mathcal{C}.
\end{tikzcd}$$

Since the right vertical map is conservative, this is indeed a space.

Exercise 2.15. Use the pushout decomposition of [2] to define composition maps

$$\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z).$$

(For extra credit, use the decomposition of [3] to show this is associative up to a specified homotopy.)

2.6 Fully faithful functors

Definition 2.6.1. A functor of ∞-categories $F \colon \mathcal{C} \to \mathcal{D}$ is *fully faithful* if it is right orthogonal to $\partial[1] \to [1]$, i.e. if the commutative square

$$\begin{tikzcd}
\mathsf{Map}([1], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}([1], \mathcal{D}) \arrow[d] \\
(\mathcal{C}^{\simeq})^{\times 2} \arrow[r] & (\mathcal{D}^{\simeq})^{\times 2}
\end{tikzcd}$$

is a pullback.

Observation 2.6.2. This square is a pullback if and only if for all $x, y \in \mathcal{C}$, the map on fibres
$$\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$$
is an equivalence, which is how we normally think of full faithfulness.

Proposition 2.6.3. *$F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful if and only if the commutative square*
$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{D}) \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$
is a pullback.

Proof. Pullbacks are detected on maps from $[0]$ and $[1]$, so it suffices to show that a fully faithful map is also right orthogonal to $(\partial[1]) \times [1] \to [1] \times [1]$. Consider the composition
$$[0]^4 \to (\partial[1]) \times [1] \to [1] \times [1];$$
the first map is the coproduct of two copies of $\partial[1] \to [1]$, so using Lemma 2.4.6 and Lemma 2.4.5 we conclude that it suffices to show that $[0]^{\amalg 4} \to [1] \times [1]$ is left orthogonal to fully faithful maps. Using the decomposition $[1] \times [1] \simeq [2] \amalg_{[1]} [2]$, we can view this map as the map on pushouts in the diagram
$$\begin{tikzcd} {[0]^{\amalg 3}} \arrow[d] & {[0]^{\amalg 2}} \arrow[l] \arrow[r] \arrow[d] & {[0]^{\amalg 3}} \arrow[d] \\ {[2]} & {[1]} \arrow[l] \arrow[r] & {[2],} \end{tikzcd}$$
so using Lemma 2.4.6 it's enough to check that $[0]^{\amalg 3} \to [2]$ is left orthogonal to fully faithful maps. But this follows from the same argument applied to the diagram
$$\begin{tikzcd} {[0]^{\amalg 2}} \arrow[d] & {[0]} \arrow[l] \arrow[r] \arrow[d] & {[0]^{\amalg 2}} \arrow[d] \\ {[1]} & {[0]} \arrow[l] \arrow[r] & {[1],} \end{tikzcd}$$
using the Segal decomposition of $[2]$. □

Corollary 2.6.4. *If $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful, then so is $F_*\colon \mathsf{Fun}(\mathcal{A}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{D})$ for any ∞-category $\mathcal{A}$.*

Proof. We must show that a commutative square of the form
$$\begin{tikzcd} \mathsf{Ar}(\mathsf{Fun}(\mathcal{A}, \mathcal{C})) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathsf{Fun}(\mathcal{A}, \mathcal{C})) \arrow[d] \\ \mathsf{Fun}(\mathcal{A}, \mathcal{C})^{\times 2} \arrow[r] & \mathsf{Fun}(\mathcal{A}, \mathcal{D})^{\times 2} \end{tikzcd}$$

is a pullback. But we can rewrite this as

$$\begin{tikzcd}
\mathsf{Fun}(\mathcal{A}, \mathsf{Ar}(\mathcal{C})) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{A}, \mathsf{Ar}(\mathcal{D})) \arrow[d] \\
\mathsf{Fun}(\mathcal{A}, \mathcal{C}^{\times 2}) \arrow[r] & \mathsf{Fun}(\mathcal{A}, \mathcal{D}^{\times 2}),
\end{tikzcd}$$

and $\mathsf{Fun}(\mathcal{C}, -)$ preserves pullbacks. □

Exercise 2.16. Show that the following are equivalent for a commutative square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C}' \arrow[d] \\
\mathcal{D} \arrow[r, "G"'] & \mathcal{D}'
\end{tikzcd}$$

where F and G are fully faithful:

(1) The square is a pullback of ∞-categories.

(2) The square gives a pullback of ∞-groupoids on cores.

(3) The square gives a pullback of sets on $\pi_0(-)^{\simeq}$.

(For the last point, use Lemma 2.2.10 and Corollary 2.7.3.)

2.7 Equivalences among ∞-categories

We now want to show that a functor of ∞-categories is an equivalence if and only if it is fully faithful and essentially surjective. This will follow quite easily from the following key property of fully faithful functors:

Theorem 2.7.1 (Martini, [Mar21, Lemma 3.8.8])**.** *A fully faithful functor is right orthogonal to* $[1] \to [0]$*, i.e. fully faithful functors are conservative.*

Remark 2.7.2. In the following we will freely use properties of pushouts, such as the 3-for-2 property, that we have not actually established, though we know the dual properties for pullbacks. In fact, we can avoid assuming these as axioms since we only care about what happens when we map these pushout squares into some target ∞-category. We could therefore say that a commutative square is a "weak pushout" if it gives a pullback on $\mathsf{Map}(-, \mathcal{C})$ for any ∞-category $\mathcal{C}$; then the properties we need for weak pushouts follow from those we know for pullback squares. Since the Yoneda lemma will eventually imply that weak pushouts are pushouts, we will ignore the distinction.

Proof of Theorem 2.7.1. Recall that we defined

$$E := * \amalg_{\{0<2\}} [3] \amalg_{\{1<3\}} *,$$

and that $E \simeq *$. Our first task is to use the Segal decompositions to give an alternative characterization of E (or $*$) as a pushout. Let's first define K as the pushout

$$K := \{0 < 1 < 2\} \amalg_{\{1<2\}} \{1 < 2 < 3\}$$

of two copies of $[2]$ along a copy of $[1]$; it is easy to see that the Segal decompositions imply that the implicit map $K \to [3]$ is an equivalence. We can then consider the commutative diagram

$$\begin{tikzcd}
\{0 < 2\} \amalg \{1 < 3\} \arrow[r] \arrow[d] & K \arrow[r, "\sim"] \arrow[d] & \{[3]\} \arrow[d] \\
* \amalg * \arrow[r] & E' \arrow[r] & E,
\end{tikzcd}$$

where E' is defined as the pushout in the left square. Here the composite square is a pushout by the definition of E, so the right square is a pushout by the 3-for-2 property, which means that $E' \to E$ is an equivalence. Thus $E' \simeq *$, and it suffices to show that the composite map $\{1 < 2\} \to K \to E'$ is left orthogonal to fully faithful functors.

We have defined E' by first gluing two copies of $[2]$ and then collapsing two copies of $[1]$. We can also do this in the opposite order: if we first define H by the pushout

$$\begin{tikzcd}
\{0 < 2\} \arrow[r] \arrow[d] & * \arrow[d] \\
{[2]} \arrow[r] & H,
\end{tikzcd}$$

then we can recover E' as a pushout

$$\begin{tikzcd}
{[1]} \arrow[r, "\beta"] \arrow[d, "\alpha"'] & H \arrow[d, "(0<1<2)"] \\
H \arrow[r, "(1<2<3)"'] & E',
\end{tikzcd}$$

where the two maps from H are obtained by collapsing an edge in the two maps from $[2]$ to K; the maps α and β correspond to the edges $0 < 1$ and $1 < 2$ in $[2]$, respectively. Applying Lemma 2.4.7 and Lemma 2.4.5 we conclude that it suffices to show the maps α and β are left orthogonal to fully faithful maps; we will prove this for α, the proof for β is almost the same.

Let

$$\Lambda^2_0 := \{0 < 1\} \amalg_{\{0\}} \{0 < 2\}$$

and consider the diagram

$$\begin{tikzcd}
\{0<2\} \arrow[r] \arrow[d] & * \arrow[d] \\
\Lambda^2_0 \arrow[r] \arrow[d] & {[1]} \arrow[d, "\alpha"] \\
{[2]} \arrow[r] & H.
\end{tikzcd}$$

Here the top and composite squares are pushouts, hence so is the bottom square. It therefore suffices to show that the map $\Lambda^2_0 \to [2]$ is left orthogonal to fully faithful maps. For this we consider the square

$$\begin{tikzcd}
\{0<1\} \amalg \{2\} \arrow[r] \arrow[d] & \Lambda^2_0 \arrow[d] \\
\{0<1\} \amalg_{\{1\}} \{1<2\} \arrow[r, "\sim"] & {[2]}.
\end{tikzcd}$$

Since the bottom map is an equivalence, it follows from Lemma 2.4.5 that it's enough to show the top and left maps are left orthogonal to fully faithful functors. But these are both pushouts of $\partial[1] \to [1]$, so this holds by Lemma 2.4.7. $\square$

Corollary 2.7.3. *If $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful, then its underlying morphism of ∞-groupoids $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a monomorphism of spaces.*

Proof. By definition F is right orthogonal to $[0] \amalg [0] \to [1]$, and also to $[1] \to [0]$ by Theorem 2.7.1. It is then also right orthogonal to the composite $[0] \amalg [0] \to [0]$, as required, by Lemma 2.4.5. $\square$

Definition 2.7.4. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if the induced morphism $\pi_0\mathcal{C}^{\simeq} \to \pi_0\mathcal{D}^{\simeq}$ is surjective.

Observation 2.7.5. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ gives an equivalence on cores. Then F is fully faithful if and only if $\mathsf{Map}([1], \mathcal{C}) \to \mathsf{Map}([1], \mathcal{D})$ is an equivalence. Since equivalences are detected on maps from $[0]$ and $[1]$ (Fact 2.3.11), this means that F is an equivalence if and only if it is fully faithful and $F^{\simeq}$ is an equivalence of ∞-groupoids.

Corollary 2.7.6. *A functor of ∞-categories is an equivalence if and only if it is fully faithful and essentially surjective.*

Proof. By Observation 2.7.5, a functor of ∞-categories is an equivalence if and only if it is fully faithful and induces an equivalence on cores. By Corollary 2.7.3, any fully faithful functor is a monomorphism on cores, so it suffices to observe that a monomorphism of ∞-groupoids is an equivalence if and only if it is surjective on π_0 by Corollary 2.2.9. $\square$

2.8 Full subcategories

Fact 2.8.1. *Given an ∞-category $\mathcal{C}$ and a monomorphism of ∞-groupoids $i\colon X \hookrightarrow \mathcal{C}^\simeq$, there exists a functor of ∞-categories $\bar{\imath}\colon i^*\mathcal{C} \to \mathcal{C}$ such that $\bar{\imath}^\simeq \simeq i$, and for any ∞-category $\mathcal{D}$ the commutative square*

$$\begin{tikzcd}
\mathsf{Map}(\mathcal{D}, i^*\mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{D}, \mathcal{C}) \arrow[d] \\
\mathsf{Map}(\mathcal{D}^\simeq, X) \arrow[r] & \mathsf{Map}(\mathcal{D}^\simeq, \mathcal{C}^\simeq)
\end{tikzcd}$$

is a pullback. Taking $\mathcal{D} = [1]$ we see that $\bar{\imath}$ is fully faithful.

Lemma 2.8.2. *Suppose $j\colon \mathcal{C}' \to \mathcal{C}$ is a fully faithful functor. Then $j \simeq \overline{j^\simeq}$.*

Proof. Let $i := j^\simeq$. Then we get a factorization of j as

$$\mathcal{C}' \xrightarrow{f} i^*\mathcal{C} \xrightarrow{\bar{\imath}} \mathcal{C}$$

such that $f^\simeq \simeq \mathrm{id}_{\mathcal{C}'^\simeq}$. Moreover, f is fully faithful since $\bar{\imath}$ and j are so (by the dual of Lemma 2.4.5), hence f is fully faithful and essentially surjective, and so an equivalence. □

Lemma 2.8.3. *Essentially surjective functors are left orthogonal to fully faithful ones.*

Proof. Suppose $i\colon \mathcal{C}' \to \mathcal{C}$ is fully faithful and $p\colon \mathcal{A} \to \mathcal{B}$ is essentially surjective. Then we have a commutative cube

$$\begin{tikzcd}
\mathsf{Map}(\mathcal{B}, \mathcal{C}') \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Map}(\mathcal{B}, \mathcal{C}) \arrow[dr] \arrow[dd] & \\
& \mathsf{Map}(\mathcal{A}, \mathcal{C}') \arrow[rr, crossing over] & & \mathsf{Map}(\mathcal{A}, \mathcal{C}) \arrow[dd] \\
\mathsf{Map}(\mathcal{B}^\simeq, \mathcal{C}'^\simeq) \arrow[rr] \arrow[dr] & & \mathsf{Map}(\mathcal{B}^\simeq, \mathcal{C}^\simeq) \arrow[dr] & \\
& \mathsf{Map}(\mathcal{A}^\simeq, \mathcal{C}'^\simeq) \arrow[rr] \arrow[from=uu, crossing over] & & \mathsf{Map}(\mathcal{A}^\simeq, \mathcal{C}^\simeq)
\end{tikzcd}$$

where the front and back faces are pullbacks by Fact 2.8.1 and Lemma 2.8.2, while the bottom face is a pullback by Lemma 2.4.3. Hence the top face is also a pullback, as required. □

Observation 2.8.4. Since monomorphisms of ∞-groupoids with target X are uniquely determined by subsets of $\pi_0(X)$ by Corollary 2.2.6, it follows that any subset S of $\pi_0(\mathcal{C}^\simeq)$ determines a unique fully faithful functor $\mathcal{C}' \hookrightarrow \mathcal{C}$. We refer to $\mathcal{C}'$ as the *full subcategory spanned by* the objects in S.

Observation 2.8.5. Consider an arbitrary functor $F\colon \mathcal{C} \to \mathcal{D}$. Then by Observation 2.2.8 we can factor $F^{\simeq}$ as

$$\mathcal{C}^{\simeq} \xrightarrow{s} X \xrightarrow{i} \mathcal{D}^{\simeq}$$

where s is surjective on π_0 and i is a monomorphism. It follows that F factors through the full subcategory $i^*\mathcal{D}$ as

$$\mathcal{C} \xrightarrow{G} i^*\mathcal{D} \hookrightarrow \mathcal{D},$$

where $G^{\simeq} \simeq s$. Thus any functor has a canonical factorization as an essentially surjective functor followed by a fully faithful one.

Lemma 2.8.6. *Suppose $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is fully faithful. Then for any ∞-category $\mathcal{D}$, the fully faithful functor $\mathsf{Fun}(\mathcal{D}, \mathcal{C}_0) \hookrightarrow \mathsf{Fun}(\mathcal{D}, \mathcal{C})$ presents the full subcategory spanned by the objects $F\colon \mathcal{D} \to \mathcal{C}$ such that $F(d) \in \mathcal{C}_0$ for all $[d] \in \pi_0(\mathcal{D}^{\simeq})$.*

Proof. It follows from Lemma 2.8.2 and Observation 2.2.5 that we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Map}(\mathcal{D}, \mathcal{C}_0) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{D}, \mathcal{C}) \arrow[d] \\
\mathsf{Map}(\mathcal{D}^{\simeq}, \mathcal{C}_0^{\simeq}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{D}^{\simeq}, \mathcal{C}^{\simeq}) \arrow[d] \\
\mathsf{Hom}(\pi_0(\mathcal{D}^{\simeq}), \pi_0(\mathcal{C}_0^{\simeq})) \arrow[r] & \mathsf{Hom}(\pi_0(\mathcal{D}^{\simeq}), \pi_0(\mathcal{C}^{\simeq}))
\end{tikzcd}$$

where both squares are pullbacks. □

2.9 Equivalences in an ∞-category

Proposition 2.9.1. *For any ∞-category $\mathcal{C}$, the degeneracy functor $\mathcal{C} \to \mathsf{Ar}(\mathcal{C})$ is fully faithful.*

Proof. Unpacking the orthogonality condition, we once again get $\mathsf{Map}(-, \mathcal{C})$ applied to the pushout square from Exercise 2.14. □

Notation 2.9.2. It is convenient to introduce the notation $\mathsf{Ar}_{\mathrm{eq}}(\mathcal{C})$ for the full subcategory of $\mathsf{Ar}(\mathcal{C})$ that is the image of the degeneracy functor — this is the full subcategory of invertible morphisms, since we can also think of it as the image of $\mathsf{Fun}(E, \mathcal{C}) \to \mathsf{Ar}(\mathcal{C})$.

Observation 2.9.3. We have a commutative diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "i"] \arrow[ddl, bend right, "="'] \arrow[ddr, bend left, "="] & \\ & \mathrm{Ar}_{\mathrm{eq}}(\mathcal{C}) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathcal{C} & & \mathcal{C}, \end{tikzcd}$$

where i is an equivalence. It follows that both s and t are also equivalences, and the composite

$$\mathcal{C} \xrightarrow{t^{-1}} \mathrm{Ar}_{\mathrm{eq}}(\mathcal{C}) \xrightarrow{s} \mathcal{C}$$

is equivalent to the identity of $\mathcal{C}$.

Notation 2.9.4. It will sometimes be convenient to write

$$\mathcal{C}[n] := \mathrm{Map}([n], \mathcal{C}).$$

Corollary 2.9.5. *The degeneracy map $\mathcal{C}[0] \to \mathcal{C}[1]$ is a monomorphism of ∞-groupoids for any ∞-category $\mathcal{C}$, with image the equivalences in $\mathcal{C}$.* □

Exercise 2.17 (⋆). The fact that the degeneracy map is a monomorphism for any ∞-category suggests that we should have a pushout square

$$\begin{tikzcd} {[1] \amalg [1]} \arrow[r] \arrow[d] & {[0] \amalg [0]} \arrow[d] \\ {[1]} \arrow[r] & {[0].} \end{tikzcd}$$

Give a direct proof of this. (You will have to use that $E \simeq *$, as we did in the proof of Theorem 2.7.1 — indeed, for an arbitrary simplicial ∞-groupoid X it is not the case that the degeneracy $X_0 \to X_1$ is necessarily a monomorphism.)

Lemma 2.9.6. *Suppose $f\colon x \to y$ is a morphism in an ∞-category $\mathcal{C}$. Then the following are equivalent:*

(1) f is an equivalence.

(2) For all $c \in \mathcal{C}$, the morphism $f_*\mathcal{C}(c, x) \to \mathcal{C}(c, y)$ is an equivalence of ∞-groupoids.

(3) For all $c \in \mathcal{C}$, the morphism $f^*\mathcal{C}(y, c) \to \mathcal{C}(x, c)$ is an equivalence of ∞-groupoids.

Proof. We prove that the first and second conditions are equivalent; the equivalence of the first and third conditions is proved similarly. First note that if f is an equivalence with inverse g, then for any $c \in \mathcal{C}$, the morphism g_* gives an inverse to f_*, so that f_* is an equivalence. Conversely, if the second condition holds then composition with f in particular gives equivalences

$$\mathcal{C}(y,x) \xrightarrow{\sim} \mathcal{C}(y,y), \quad \mathcal{C}(x,x) \xrightarrow{\sim} \mathcal{C}(x,y).$$

The fibre of the first map at id_y is then contractible, so there exists $g\colon y \to x$ and an equivalence $gf \simeq \mathrm{id}_y$. The fibre of the second map at f is then contractible, and contains both id_x and fg, so we must also have an equivalence $fg \simeq \mathrm{id}_x$. We can then use this data to extend f, viewed as a map $[1] \to \mathcal{C}$, to a functor $E \to \mathcal{C}$, so that f is an equivalence in the sense of being in the image of the degeneracy map $\mathcal{C}^{\simeq} \to \mathcal{C}[1]$. □

Corollary 2.9.7. *An ∞-category $\mathcal{C}$ is an ∞-groupoid if and only if every morphism in $\mathcal{C}$ is an equivalence.*

Proof. Since the degeneracy $\mathcal{C}[0] \hookrightarrow \mathcal{C}[1]$ is a monomorphism, it is an equivalence if and only if it is also surjective on π_0, which means precisely that every morphism in $\mathcal{C}$ is an equivalence. □

Proposition 2.9.8. *For ∞-categories $\mathcal{C}, \mathcal{D}$, a morphism in $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ is an equivalence if and only if its component at every $[c] \in \pi_0\mathcal{C}^{\simeq}$ is an equivalence in $\mathcal{D}$.*

Proof. We can identify the degeneracy $\mathsf{Fun}(\mathcal{C},\mathcal{D}) \to \mathsf{Ar}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$ with the functor $\mathsf{Fun}(\mathcal{C},\mathcal{D}) \to \mathsf{Fun}(\mathcal{C},\mathsf{Ar}(\mathcal{D}))$ given by composing with the degeneracy of $\mathcal{D}$. The image of this fully faithful functor admits the desired description by Lemma 2.8.6. □

Exercise 2.18. Show that if $\mathcal{C}' \to \mathcal{C}$ is essentially surjective, then

$$\mathsf{Fun}(\mathcal{C},\mathcal{D}) \to \mathsf{Fun}(\mathcal{C}',\mathcal{D})$$

is conservative.

2.10 Monomorphisms of ∞-categories

Definition 2.10.1. A functor of ∞-categories $\mathcal{C} \to \mathcal{D}$ is a *monomorphism* if the commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathcal{C}\times\mathcal{C} \arrow[r] & \mathcal{D}\times\mathcal{D} \end{tikzcd}$$

is a pullback.

Exercise 2.19. Show that the following are equivalent for $F\colon \mathcal{C} \to \mathcal{D}$:

(1) F is a monomorphism.

(2) $F([i])\colon \mathcal{C}([i]) \to \mathcal{C}([i])$ is a monomorphism of ∞-groupoids for $i = 0, 1$.

(3) F is right orthogonal to $[i] \amalg [i] \to [i]$ for $i = 0, 1$.

Exercise 2.20. Use Exercise 2.8 to show that $F \colon \mathcal{C} \to \mathcal{D}$ is a monomorphism if and only if $F^{\simeq} \colon \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ and $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ are monomorphisms for all $x, y \in \mathcal{C}$.

Observation 2.10.2. Since $\mathsf{Fun}(\mathcal{A}, -)$ preserves pullbacks, if $\mathcal{C} \to \mathcal{D}$ is a monomorphism, then so is $\mathsf{Fun}(\mathcal{A}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{D})$.

Observation 2.10.3. We have a commutative diagram

$$\begin{tikzcd}
{[0] \amalg [0]} \arrow[r] \arrow[d] & {[1] \amalg [1]} \arrow[r] \arrow[d] & {[0] \amalg [0]} \arrow[d] \\
{[0]} \arrow[r] & {[1]} \arrow[r] & {[0],}
\end{tikzcd}$$

where the horizontal maps in the left square are given by taking the same object of $[1]$ three times. The horizontal composites are both identities, so this shows that the fold map of $[0]$ is a retract of that for $[1]$. Applying Lemma 2.4.9, this means that a functor of ∞-categories is a monomorphism if and only if it is right orthogonal to $[1] \amalg [1] \to [1]$.

Lemma 2.10.4. *Monomorphisms are conservative, i.e. if $F \colon \mathcal{C} \to \mathcal{D}$ is a monomorphism, then we have a pullback square*

$$\begin{tikzcd}
\mathcal{C}[0] \arrow[r, hook] \arrow[d] & \mathcal{D}[0] \arrow[d] \\
\mathcal{C}[1] \arrow[r, hook] & \mathcal{D}[1].
\end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd}
{[1]} \arrow[r] \arrow[d] & {[0]} \arrow[d] \\
{[1] \amalg [1]} \arrow[r] \arrow[d] & {[0] \amalg [1]} \arrow[d] \\
{[1]} \arrow[r] & {[0].}
\end{tikzcd}$$

Here the top and composite squares are pushouts, hence so is the bottom square, so a monomorphism is right orthogonal to the map $[0] \amalg [1] \to [0]$ by Lemma 2.4.7. But $[1] \to [0]$ is a retract of this, hence a monomorphism is right orthogonal to it by Lemma 2.4.9. $\square$

Lemma 2.10.5. *Fully faithful functors are monomorphisms.*

Proof. Suppose F is fully faithful. We saw in Corollary 2.7.3 that F is then right orthogonal to $[0] \amalg [0] \to [0]$, and it remains to show that it is also right orthogonal to $[1] \amalg [1] \to [1]$. For this we consider the commutative square

$$\begin{tikzcd} {[0]^{\amalg 4}} \arrow[r] \arrow[d] & {[1] \amalg [1]} \arrow[d] \\ {[0]^{\amalg 2}} \arrow[r] & {[1].} \end{tikzcd}$$

Here we see that F is right orthogonal to the left vertical map since this is a coproduct of two copies of the fold map of $[0]$, and right orthogonal to the top horizontal map for the same reason. It follows by Lemma 2.4.5 that F is also right orthogonal to the right vertical map. □

2.11 Subcategories

Fact 2.11.1. *Let $\mathcal{C}$ be an ∞-category and $i_1 \colon X_1 \hookrightarrow \mathsf{Map}([1], \mathcal{C})$ a monomorphism of ∞-groupoids. Define X_0 by the pullback*

$$\begin{tikzcd} X_0 \arrow[r, hook, "i_0"] \arrow[d] & \mathcal{C}^{\simeq} \arrow[d, "s_0"] \\ X_1 \arrow[r, hook, "i_1"'] & \mathsf{Map}([1], \mathcal{C}) \end{tikzcd}$$

and suppose the monomorphisms i_0, i_1 satisfy the following properties[5]*:*

- *the source and target of the morphisms in X_1 lie in X_0, i.e. the composites*

  $$X_1 \hookrightarrow \mathsf{Map}([1], \mathcal{C}) \xrightarrow{d_i} \mathcal{C}^{\simeq}$$

  factor through X_0

- *the morphisms in X_1 are closed under composition, i.e. the composite*

  $$X_1 \times_{X_0} X_1 \hookrightarrow \mathsf{Map}([1], \mathcal{C}) \times_{\mathcal{C}^{\simeq}} \mathsf{Map}([1], \mathcal{C}) \simeq \mathsf{Map}([2], \mathcal{C}) \xrightarrow{d_1} \mathsf{Map}([1], \mathcal{C})$$

  factors through X_1.

Then there exists a functor of ∞-categories $\bar{\imath} \colon i^*\mathcal{C} \to \mathcal{C}$ such that $\mathsf{Map}([1], \bar{\imath}) \simeq i_1$, and for any ∞-category $\mathcal{D}$, the commutative square

$$\begin{tikzcd} \mathsf{Map}(\mathcal{D}, i^*\mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{D}, \mathcal{C}) \arrow[d] \\ \mathsf{Map}(\mathcal{D}([1]), X_1) \arrow[r] & \mathsf{Map}(\mathcal{D}([1]), \mathcal{C}([1])) \end{tikzcd}$$

is a pullback. We refer to $i^*\mathcal{C}$ as the subcategory *determined by the morphisms in X_1.*

[5] Note that all properties can be checked on π_0, since this detects factorizations through monomorphisms.

Warning 2.11.2. Our subcategories are always *replete*, that is they contain all equivalences among their objects. (Indeed, weaker notions of subcategory are not invariant under equivalence of ∞-categories, and so do not really make sense except in some model of (∞-)categories — note that this is already true for ordinary categories.)

Observation 2.11.3. Taking $\mathcal{D} = [0]$ in this square, we get a pullback square

$$\begin{tikzcd} (i^*\mathcal{C})^{\simeq} \arrow[r] \arrow[d] & \mathcal{C}^{\simeq} \arrow[d] \\ X_1 \arrow[r] & \mathcal{C}([1]), \end{tikzcd}$$

so that $(i^*\mathcal{C})^{\simeq}$ is X_0. In particular, $\bar{\imath}$ is a monomorphism.

Lemma 2.11.4. *Given an ∞-category $\mathcal{C}$ and a monomorphism $i_0 \colon X_0 \to \mathcal{C}^{\simeq}$, we can define X_1 by the pullback*

$$\begin{tikzcd} X_1 \arrow[r, "i_1"] \arrow[d] & \mathcal{C}([1]) \arrow[d] \\ X_0 \times X_0 \arrow[r, "i_0 \times i_0"] & \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}. \end{tikzcd}$$

Then the resulting monomorphism $\bar{\imath} \colon i^*\mathcal{C} \to \mathcal{C}$ is the fully faithful functor associated to the inclusion i_0 in Fact 2.11.1.

Proof. First consider the commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & \mathcal{C}^{\simeq} \arrow[d] \\ X_1 \arrow[r, "i_1"] \arrow[d] & \mathcal{C}[1] \arrow[d] \\ X_0 \times X_0 \arrow[r] & \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}. \end{tikzcd}$$

Here the bottom and composite squares are pullbacks (since i_0 is a monomorphism), hence so is the top square. This shows that $\bar{\imath}$ is given by i_0 on cores. Moreover, since $\bar{\imath}$ is given by i_1 on $\mathsf{Map}([1], -)$, the bottom square then shows that $\bar{\imath}$ is fully faithful. Since full subcategories are uniquely determined by their objects, the functor $\bar{\imath}$ must necessarily be the fully faithful functor corresponding to the monomorphism i_0. □

Lemma 2.11.5. *Suppose $F \colon \mathcal{C}' \to \mathcal{C}$ is a monomorphism, and let*

$$i_1 := F([1]) \colon \mathcal{C}'[1] \to \mathcal{C}[1].$$

Then $F \simeq \bar{\imath}$.

Proof. We get a factorization of F as $\mathcal{C}' \xrightarrow{F'} i^*\mathcal{C} \xrightarrow{\bar{i}} \mathcal{C}$ such that $F'[1] \simeq \mathrm{id}_{\mathcal{C}'[1]}$. Moreover, F' is a monomorphism, so $F'[0]$ is also an equivalence by Lemma 2.10.4. Thus F' is an equivalence. □

Observation 2.11.6. It follows that a monomorphism of ∞-categories $j\colon \mathcal{C}' \to \mathcal{C}$ is uniquely determined by a subset S of $\pi_0\mathcal{C}[1]$ such that S is closed under composition and contains the identity morphisms on the source and target of each of its elements. We refer to $\mathcal{C}'$ as the *subcategory* of $\mathcal{C}$ generated by the morphisms in S.

Example 2.11.7. It follows from Corollary 2.9.5 that the canonical map $\mathcal{C}^{\simeq} \to \mathcal{C}$ is always a monomorphism, and gives the subcategory of $\mathcal{C}$ that contains only the equivalences.

Lemma 2.11.8. *Suppose $j\colon \mathcal{C}' \to \mathcal{C}$ is the inclusion of the subcategory generated by $S \subseteq \pi_0\mathcal{C}[1]$. Then $j_*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{C}') \to \mathsf{Fun}(\mathcal{D}, \mathcal{C})$ exhibits $\mathsf{Fun}(\mathcal{D}, \mathcal{C}')$ as the subcategory generated by the natural transformations whose components at all objects in $\mathcal{D}$ lie in S.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathsf{Map}(\mathcal{D} \times [1], \mathcal{C}') \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{D} \times [1], \mathcal{C}) \arrow[d] \\
\mathsf{Map}(\mathcal{D}[1] \times [1][1], \mathcal{C}'[1]) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{D}[1] \times [1][1], \mathcal{C}[1]) \arrow[d] \\
\mathsf{Hom}(\pi_0(\mathcal{D}[1]) \times [1][1], S) \arrow[r] & \mathsf{Hom}(\pi_0(\mathcal{D}[1]) \times [1][1], \pi_0(\mathcal{C}[1]))
\end{tikzcd}$$

where both squares are pullbacks. This says precisely that a natural transformation factors through $\mathcal{C}'$ precisely when its components lie in S and its source and target send all morphisms in $\mathcal{D}$ to S. □

Chapter 3

Fibrations

3.1 Why fibrations?

One of the key differences between ordinary category theory and ∞-category theory is that we can't "just write down" a functor between ∞-categories: this requires specifying an infinite number of coherences, and in practice this is usually impossible (also when we work in a model like quasicategories). The theory of *fibrations*, which for ordinary categories goes back to work of Grothendieck in the 60s, therefore plays a far more prominent role in the theory of ∞-categories than for ordinary categories: This theory provides an identification, the *straightening* equivalence, between functors from an ∞-category $\mathcal{B}$ to the ∞-categories of (small) ∞-groupoids and ∞-categories with certain classes of functors to $\mathcal{B}$ (the aforementioned fibrations), and defining and manipulating such fibrations often gives us the only way to construct important functors among ∞-categories.

The simplest case of this phenomenon is the identification between the overcategory $\mathsf{Set}_{/S}$ for a set S and the functor category $\mathsf{Fun}(S, \mathsf{Set})$, given by the functors

$$\begin{aligned} (E \to S) \quad &\mapsto \quad (s \in S \mapsto E_s), \\ (F\colon S \to \mathsf{Set}) &\mapsto (\coprod_{s \in S} F(s) \to S). \end{aligned}$$

If we write Gpd_∞ for the ∞-category of (small) ∞-groupoids, this extends to an equivalence

$$\mathsf{Gpd}_{\infty/X} \simeq \mathsf{Fun}(X, \mathsf{Gpd}_\infty),$$

under which a functor from X corresponds to its colimit.

More generally, we will see that the ∞-categories of functors $\mathcal{B} \to \mathsf{Gpd}_\infty$ and $\mathcal{B}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ can be identified with certain full subcategories of $\mathsf{Cat}_{\infty/\mathcal{B}}$, whose objects are the *left* and *right* fibrations over $\mathcal{B}$.

If Cat_∞ is the ∞-category of (small) ∞-categories, we will similarly identify $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)$ and $\mathsf{Fun}(\mathcal{B}^{\mathrm{op}}, \mathsf{Cat}_\infty)$ with certain (non-full) subcategories of $\mathsf{Cat}_{\infty/\mathcal{B}}$, whose objects are the *cocartesian* and *cartesian* fibrations over $\mathcal{B}$.

By combining these constructions we will, for example, be able to construct the mapping ∞-groupoid functor

$$\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Gpd}_\infty$$

out of the restriction $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$.

3.2 Left and right fibrations

Definition 3.2.1. A functor of ∞-categories $p\colon \mathcal{E} \to \mathcal{B}$ is a *left fibration* if it is right orthogonal to the inclusion $\{0\} \to [1]$, and a *right fibration* if it is orthogonal to $\{1\} \to [1]$. Equivalently, p is a left or right fibration if the corresponding commutative square

$$\begin{tikzcd}
\mathsf{Map}([1],\mathcal{E}) \arrow[r] \arrow[d] & \mathsf{Map}([1],\mathcal{B}) \arrow[d] \\
\mathcal{E}^{\simeq} \arrow[r] & \mathcal{B}^{\simeq}
\end{tikzcd}$$

is a pullback.

Exercise 3.1. Every left/right fibration is conservative.

Observation 3.2.2. A functor p is a left fibration if and only if p^{op} is a right fibration.

Lemma 3.2.3. *If $\mathcal{B}$ is an ∞-groupoid, then a functor $p\colon \mathcal{E} \to \mathcal{B}$ is a left/right fibration if and only if $\mathcal{E}$ is an ∞-groupoid. In particular, any morphism of ∞-groupoids is both a left and a right fibration.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}[0] \arrow[r] \arrow[d] \arrow[dd, bend right, "="'] & \mathcal{B}[0] \arrow[d, "\sim"] \arrow[dd, bend left, "="] \\
\mathcal{E}[1] \arrow[r] \arrow[d] & \mathcal{B}[1] \arrow[d, "\sim"] \\
\mathcal{E}[0] \arrow[r] & \mathcal{B}[0]
\end{tikzcd}$$

Here the top right arrow is an equivalence since $\mathcal{B}$ is an ∞-groupoid, and so the bottom right arrow is also an equivalence since the right vertical composite is an identity. The bottom square is therefore a pullback if and only if the bottom left arrow is an equivalence (Exercise 2.3). The left vertical composite is also an identity, so this holds if and only if the top left arrow is an equivalence, which holds precisely when $\mathcal{E}$ is an ∞-groupoid. $\square$

Proposition 3.2.4. *p is a left/right fibration if and only if the commutative square*

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{E}) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{B}) \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{B} \end{tikzcd}$$

is a pullback.

Proof. We consider the case of left fibrations, without loss of generality. The given condition clearly implies that p is a left fibration by passing to cores, so it remains to show that if p is a left fibration then we get a pullback upon applying $\mathsf{Map}([1], -)$ to the square. Here we get the square

$$\begin{tikzcd} \mathsf{Map}([1] \times [1], \mathcal{E}) \arrow[r] \arrow[d] & \mathsf{Map}([1] \times [1], \mathcal{B}) \arrow[d] \\ \mathsf{Map}([1], \mathcal{E}) \arrow[r] & \mathsf{Map}([1], \mathcal{B}) \end{tikzcd}$$

where the horizontal maps are given by restriction along $\{0\} \times [1] \hookrightarrow [1] \times [1]$, so it suffices to check that this is left orthogonal to all left fibrations. Using the pushout decomposition of $[1] \times [1]$, we can identify this as the map on pushouts in the diagram

$$\begin{tikzcd} \{0 < 1\} \arrow[d] & \{0\} \arrow[l] \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ {[2]} & {[1]} \arrow[l] \arrow[r] & {[2],} \end{tikzcd}$$

so by Lemma 2.4.6 it suffices to check that the maps $\{0 \to 1\} \to [2]$ and $\{0\} \to [2]$ are left orthogonal to left fibrations. For the former this follows from decomposing it as the map on pushouts in

$$\begin{tikzcd} {[1]} \arrow[d] & {[0]} \arrow[l] \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ {[1]} & {[0]} \arrow[l] \arrow[r] & {[1],} \end{tikzcd}$$

and applying Lemma 2.4.6 again, while the latter is the composition

$$\{0\} \hookrightarrow \{0 < 1\} \to [2],$$

so this follows from Lemma 2.4.5. □

Corollary 3.2.5. *If $p \colon \mathcal{E} \to \mathcal{B}$ is a left/right fibration, then so is $p_* \colon \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{B})$ for any ∞-category $\mathcal{C}$.*

Proof. We must show that a commutative square of the form

$$\begin{tikzcd} \mathsf{Ar}(\mathsf{Fun}(\mathcal{C},\mathcal{E})) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathsf{Fun}(\mathcal{C},\mathcal{B})) \arrow[d] \\ \mathsf{Fun}(\mathcal{C},\mathcal{E}) \arrow[r] & \mathsf{Fun}(\mathcal{C},\mathcal{B}) \end{tikzcd}$$

is a pullback. But we can rewrite this as

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{C},\mathsf{Ar}(\mathcal{E})) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{C},\mathsf{Ar}(\mathcal{B})) \arrow[d] \\ \mathsf{Fun}(\mathcal{C},\mathcal{E}) \arrow[r] & \mathsf{Fun}(\mathcal{C},\mathcal{B}), \end{tikzcd}$$

and $\mathsf{Fun}(\mathcal{C},-)$ preserves pullbacks. □

Proposition 3.2.6. *The following are equivalent for a left (or right) fibration $p\colon \mathcal{E} \to \mathcal{B}$:*

(1) p is an equivalence.

(2) The map on cores $p^{\simeq}\colon \mathcal{E}^{\simeq} \to \mathcal{B}^{\simeq}$ is an equivalence of ∞-groupoids.

(3) The fibre $\mathcal{E}_b$ is contractible for every $b \in \mathcal{B}$.

Proof. By assumption we have a pullback square

$$\begin{tikzcd} \mathsf{Map}([1],\mathcal{E}) \arrow[r] \arrow[d] & \mathsf{Map}([1],\mathcal{B}) \arrow[d] \\ \mathcal{E}^{\simeq} \arrow[r] & \mathcal{B}^{\simeq}, \end{tikzcd}$$

so if p gives an equivalence on cores it also gives an equivalence on $\mathsf{Map}([1],-)$ and so is an equivalence of ∞-categories (Fact 2.3.11). Moreover, we have a pullback square

$$\begin{tikzcd} \mathcal{E}^{\simeq} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{B}^{\simeq} \arrow[r] & \mathcal{B} \end{tikzcd}$$

by combining Exercise 2.13 and Exercise 3.1. Therefore $\mathcal{E}_b$ is also the fibre of $p^{\simeq}$ at b, so if these ∞-groupoids are all contractible then $p^{\simeq}$ is an equivalence by Proposition 2.1.22. □

Observation 3.2.7. Given a commutative triangle

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

where p and q are both left (or both right) fibrations, then so is F — this follows from the dual of Lemma 2.4.5.

Corollary 3.2.8. *Suppose given a commutative triangle*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

where p and q are both left (or both right) fibrations. Then the following are equivalent:

(1) F is an equivalence.

(2) The underlying map of ∞-groupoids $\mathcal{E}^{\simeq} \to \mathcal{F}^{\simeq}$ is an equivalence.

(3) The functor on fibres $\mathcal{E}_b \to \mathcal{F}_b$ is an equivalence for all $b \in \mathcal{B}$.

Proof. The functor F is a left (or right) fibration by Observation 3.2.7, so the equivalence of the first two conditions follows from Observation 3.2.7. Since the fibres of p and q are the same as those of $p^{\simeq}$ and $q^{\simeq}$, respectively, we can apply Exercise 2.2 to the triangle

$$\begin{tikzcd} \mathcal{E}^{\simeq} \arrow[rr] \arrow[dr, "p"'] & & \mathcal{F}^{\simeq} \arrow[dl, "q"] \\ & \mathcal{B}^{\simeq} & \end{tikzcd}$$

to see that the last two conditions are equivalent. □

Exercise 3.2. Show that a commutative square

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"'] & \mathcal{F} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

where p and q are both left (or both right) fibrations is a pullback if and only if the induced map on fibres $\mathcal{E}_b \to \mathcal{F}_{F(b)}$ is an equivalence of ∞-groupoids for all $b \in \mathcal{B}$.

3.3 Slices and cones

In this section we define over- and undercategories of an ∞-category, and show that these give left and right fibrations.

Definition 3.3.1. For an object $c \in \mathcal{C}$ we define the *overcategory* $\mathcal{C}_{/c}$ as the pullback

$$\begin{tikzcd} \mathcal{C}_{/c} \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{C}) \arrow[d, "\mathrm{ev}_1"] \\ \{c\} \arrow[r] & \mathcal{C}; \end{tikzcd}$$

the functor ev_0 pulls back to a functor $\mathcal{C}_{/c} \to \mathcal{C}$. Dually, we define the *undercategory* $\mathcal{C}_{c/}$ by pulling back ev_0; then ev_1 gives a functor $\mathcal{C}_{c/} \to \mathcal{C}$. (We refer to over- and undercategories jointly as *slice* ∞-categories.)

Observation 3.3.2. Suppose X is an ∞-groupoid. Then the slice $X_{/x}$ is contractible for any $x \in X$, since in the pullback square

$$\begin{tikzcd} X_{/x} \arrow[r] \arrow[d] & \mathsf{Ar}(X) \arrow[d, "\mathrm{ev}_1"] \\ \{x\} \arrow[r] & X, \end{tikzcd}$$

the right vertical arrow is an equivalence (since it is a one-sided inverse of the degeneracy $X \xrightarrow{\sim} \mathsf{Ar}(X)$). Similarly, $X_{x/} \simeq *$.

We can describe maps to slice ∞-categories in terms of *cones*, in the following sense:

Definition 3.3.3. For an ∞-category $\mathcal{K}$, the *left cone* $\mathcal{K}^{\triangleleft}$ and *right cone* $\mathcal{K}^{\triangleright}$ on $\mathcal{K}$ are defined by the pushouts

$$\mathcal{K}^{\triangleleft} := \{-\infty\} \amalg_{\mathcal{K}\times\{0\}} \mathcal{K} \times [1],$$

$$\mathcal{K}^{\triangleright} := \mathcal{K} \times [1] \amalg_{\mathcal{K}\times\{1\}} \{\infty\}.$$

The *cone point* is the object $\pm\infty$.

Remark 3.3.4. It can be shown (e.g. using quasicategories) that cores and mapping spaces in the cone $\mathcal{K}^{\triangleleft}$ are given by

$$(\mathcal{K}^{\triangleleft})^{\simeq} \simeq \{-\infty\} \amalg \mathcal{K}^{\simeq},$$

$$\mathcal{K}^{\triangleleft}(x,y) \simeq \begin{cases} \emptyset, & x \in \mathcal{K}, y \simeq -\infty, \\ *, & x \simeq -\infty, \\ \mathcal{K}(x,y), & x, y \in \mathcal{K}. \end{cases}$$

Lemma 3.3.5. *For ∞-categories $\mathcal{K}$ and $\mathcal{C}$ and an object x in $\mathcal{C}$, there are natural equivalences*

$$\mathsf{Fun}(\mathcal{K}, \mathcal{C}_{/x}) \simeq \mathsf{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \times_{\mathsf{Fun}(\{\infty\},\mathcal{C})} \{x\},$$

$$\mathsf{Fun}(\mathcal{K}, \mathcal{C}_{x/}) \simeq \mathsf{Fun}(\mathcal{K}^{\triangleleft}, \mathcal{C}) \times_{\mathsf{Fun}(\{-\infty\},\mathcal{C})} \{x\}.$$

Proof. We prove the first case. By definition of $\mathcal{C}_{/x}$ and $\mathcal{K}^{\triangleright}$, we have pullback squares

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{K}, \mathcal{C}_{/x}) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Fun}(\mathcal{K} \times [1], \mathcal{C}) \arrow[d] \\ \{x\} \arrow[r] & \mathsf{Fun}(\mathcal{K} \times \{1\}, \mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Fun}(\mathcal{K} \times [1], \mathcal{C}) \arrow[d] \\ \mathsf{Fun}(\{\infty\}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\mathcal{K} \times \{1\}, \mathcal{C}). \end{tikzcd}$$

The factorization $\{x\} \to \mathsf{Fun}(\{\infty\}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{K} \times \{1\}, \mathcal{C})$ in the bottom row therefore induces a commutative diagram

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{K}, \mathcal{C}_{/x}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K} \times [1], \mathcal{C}) \arrow[d] \\ \{x\} \arrow[r] & \mathsf{Fun}(\{\infty\}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\mathcal{K} \times \{1\}, \mathcal{C}), \end{tikzcd}$$

where the left square is a pullback by the 3-for-2 property, as required. □

Observation 3.3.6. The pushout decompositions of $[1] \times [1]$ and $[2]$ imply that we have a pushout

$$\begin{tikzcd} {[1] \times \{1\}} \arrow[r] \arrow[d] & {[1] \times [1]} \arrow[d] \\ {[0]} \arrow[r] & {[2],} \end{tikzcd}$$

so that $[1]^{\triangleright} \simeq [2]$ with the inclusion of $[1]$ corresponding to d_2; similarly, $[1]^{\triangleleft} \simeq [2]$ (with the inclusion of $[1]$ given by d_0).

Proposition 3.3.7. *We have pullback squares*

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}_{/c}) \arrow[r] \arrow[d, "t"] & \mathsf{Fun}([2], \mathcal{C}) \arrow[d] \\ \mathcal{C}_{/c} \arrow[r] & \mathsf{Fun}(\{1 < 2\}, \mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Ar}(\mathcal{C}_{c/}) \arrow[r] \arrow[d, "s"] & \mathsf{Fun}([2], \mathcal{C}) \arrow[d] \\ \mathcal{C}_{c/} \arrow[r] & \mathsf{Fun}(\{0 < 1\}, \mathcal{C}). \end{tikzcd}$$

Proof. We prove the first case. Combining Observation 3.3.6 with Lemma 3.3.5, we get a pullback square

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}_{/c}) \arrow[r] \arrow[d] & \mathsf{Fun}([2], \mathcal{C}) \arrow[d] \\ \{c\} \arrow[r] & \mathsf{Fun}(\{2\}, \mathcal{C}). \end{tikzcd}$$

Using that the equivalence from 3.3.5 is natural in $\mathcal{K}$, we get a factorization of this square as

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}_{/c}) \arrow[r] \arrow[d, "t"] & \mathsf{Fun}([2], \mathcal{C}) \arrow[d] \\ \mathcal{C}_{/c} \arrow[r] \arrow[d] & \mathsf{Fun}(\{1 < 2\}, \mathcal{C}) \arrow[d] \\ \{c\} \arrow[r] & \mathsf{Fun}(\{2\}, \mathcal{C}) \end{tikzcd}$$

where also the bottom square is a pullback. The top square is then a pullback by the 3-for-2 property, as required. □

Exercise 3.3. Show that more generally, we have equivalences

$$[n]^{\triangleleft} \simeq [n+1] \simeq [n]^{\triangleright}.$$

Corollary 3.3.8. *For $c \in \mathcal{C}$, $\mathcal{C}_{/c} \to \mathcal{C}$ is a right fibration and $\mathcal{C}_{c/} \to \mathcal{C}$ is a left fibration.*

Proof. We prove the first case — the second follows similarly, or by observing that $(\mathcal{C}_{c/})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{/c}$ and using Observation 3.2.2. Informally, we want to say that a morphism in $\mathcal{C}_{/c}$ is a commutative triangle

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "p"'] & & y \arrow[dl, "q"] \\ & c, & \end{tikzcd}$$

and the forgetful functor to $\mathcal{C}$ takes this to the horizontal morphism f; the condition for being a right fibration says that given $x \xrightarrow{f} y$ and the object $y \xrightarrow{q} c$ in $\mathcal{C}_{/c}$, there is a unique morphism in $\mathcal{C}_{/c}$ lifting f with target q, which follows from the uniqueness of the composite p.

Using Proposition 3.3.7, we have a commutative diagram

$$\begin{tikzcd} \mathsf{Map}([1], \mathcal{C}_{/c}) \arrow[r] \arrow[d] & \mathsf{Map}([2], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Map}(\{0 < 1\}, \mathcal{C}) \arrow[d] \\ (\mathcal{C}_{/c})^{\simeq} \arrow[r] & \mathsf{Map}(\{1 < 2\}, \mathcal{C}) \arrow[r] & \mathsf{Map}(\{1\}, \mathcal{C}) \end{tikzcd}$$

where the left square is a pullback. The right square is also a pullback by the Segal decomposition of $[2]$. The composite square is therefore a pullback, which says precisely that $\mathcal{C}_{/c} \to \mathcal{C}$ is a right fibration. □

Corollary 3.3.9. *For a morphism $f\colon x \to y$ in an ∞-category $\mathcal{C}$, we have equivalences*

$$(\mathcal{C}_{/y})_{/f} \xrightarrow{\sim} \mathcal{C}_{/x},$$

$$(\mathcal{C}_{x/})_{f/} \xrightarrow{\sim} \mathcal{C}_{y/},$$

under which the forgetful functors to $\mathcal{C}_{/y}$ *and* $\mathcal{C}_{x/}$ *correspond to the functor given by composition with* f.

Proof. We prove the first case. Here we have a commutative diagram

$$\begin{tikzcd}
(\mathcal{C}_{/y})_{/f} \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{C}_{/y}) \arrow[r] \arrow[d, "t"] & \mathsf{Fun}([2], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}(\{0 < 1\}, \mathcal{C}) \arrow[d] \\
\{f\} \arrow[r] & \mathcal{C}_{/y} \arrow[r] & \mathsf{Fun}(\{1 < 2\}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\{1\}, \mathcal{C}),
\end{tikzcd}$$

where all three squares are pullbacks: the left square by definition, the middle square by Proposition 3.3.7, and the right square by the Segal decomposition of $[2]$. It follows that the composite square is a pullback, but the bottom horizontal composite picks out $x \in \mathcal{C}$, so this pullback is also $\mathcal{C}_{/x}$, as required. □

Finally, we note that we can reformulate the definition of left fibrations in terms of slices, which will sometimes be useful:

Lemma 3.3.10. *A functor* $p\colon \mathcal{E} \to \mathcal{B}$ *is a left fibration if and only if for every* $x \in \mathcal{E}$, *the induced functor* $\mathcal{E}_{x/} \to \mathcal{B}_{p(x)/}$ *is an equivalence.*

Proof. If p is a left fibration, then by Proposition 3.2.4, the commutative square

$$\begin{tikzcd}
\mathsf{Ar}(\mathcal{E}) \arrow[r] \arrow[d, "\mathrm{ev}_0"] & \mathsf{Ar}(\mathcal{B}) \arrow[d, "\mathrm{ev}_0"] \\
\mathcal{E} \arrow[r] & \mathcal{B}
\end{tikzcd}$$

is a pullback. The map on fibres over $x \in \mathcal{E}$ is precisely $\mathcal{E}_{x/} \to \mathcal{B}_{p(x)/}$, which is therefore an equivalence. Conversely, if these functors are equivalences, then the maps on fibres in the square

$$\begin{tikzcd}
\mathcal{E}[1] \arrow[r] \arrow[d, "\mathrm{ev}_0"] & \mathcal{B}[1] \arrow[d, "\mathrm{ev}_0"] \\
\mathcal{E}^{\simeq} \arrow[r] & \mathcal{B}^{\simeq}
\end{tikzcd}$$

are equivalences of ∞-groupoids, so this is a pullback, i.e. p is a left fibration. □

3.4 Straightening for left and right fibrations

Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a left fibration. Then p is supposed to determine a functor from $\mathcal{B}$ to ∞-groupoids; let's see how the basic data of such a functor follows from the definition of left fibrations:

- First consider the commutative diagram

$$\begin{tikzcd}
\mathcal{E}[0] \arrow[d] & \mathcal{E}[1] \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & \mathcal{E}[0] \arrow[d] \\
\mathcal{B}[0] & \mathcal{B}[1] \arrow[l, "s"] \arrow[r, "t"'] & \mathcal{B}[0],
\end{tikzcd}$$

  where the left square is a pullback since p is a left fibration. On fibres over a morphism $f\colon b \to b'$ in $\mathcal{E}$, we get

$$\mathcal{E}_b \xleftarrow{\sim} \mathcal{E}[1]_f \to \mathcal{E}_{b'},$$

  which (inverting the equivalence) we can think of as a functor $f_!\colon \mathcal{E}_b \to \mathcal{E}_{b'}$.

- If we consider the identity map id_b, we have a commutative diagram

$$\begin{tikzcd}
& \mathcal{E}[0] \arrow[dl, "="'] \arrow[d] \arrow[dr, "="] & \\
\mathcal{E}[0] & \mathcal{E}[1] \arrow[l, "s"] \arrow[r, "t"'] & \mathcal{E}[0].
\end{tikzcd}$$

  On fibres, we get

$$\begin{tikzcd}
& \mathcal{E}_b \arrow[dl, "="'] \arrow[d] \arrow[dr, "="] & \\
\mathcal{E}_b & \mathcal{E}[1]_{\mathrm{id}_b} \arrow[l, "\sim"] \arrow[r] & \mathcal{E}_b,
\end{tikzcd}$$

  which shows that $(\mathrm{id}_b)_!$ is (homotopic to) the identity of $\mathcal{E}_b$.

- For two composable maps $f\colon b \to b'$, $g\colon b' \to b''$ we can consider fibres in the diagram

$$\begin{tikzcd}
& & \mathcal{E}[1] \arrow[ddll, bend right, "s"'] \arrow[ddrr, bend left, "t"] & & \\
& & \mathcal{E}[2] \arrow[u, "d_1"] \arrow[dl] \arrow[dr] & & \\
& \mathcal{E}[1] \arrow[dl, "s"'] \arrow[dr, "t"] & & \mathcal{E}[1] \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\mathcal{E}[0] & & \mathcal{E}[0] & & \mathcal{E}[0],
\end{tikzcd}$$

where the middle square is a pullback. This produces the diagram

$$\begin{tikzcd}
 & & \mathcal{E}[1]_{gf} \arrow[ddll, bend right, "\sim"'] \arrow[ddrr, bend left] & & \\
 & & \mathcal{E}[2]_{g,f} \arrow[u, "\sim"] \arrow[dl, "\sim"'] \arrow[dr] & & \\
 & \mathcal{E}[1]_f \arrow[dl, "\sim"'] \arrow[dr] & & \mathcal{E}[1]_g \arrow[dl, "\sim"'] \arrow[dr] & \\
\mathcal{E}_b & & \mathcal{E}_{b'} & & \mathcal{E}_{b''},
\end{tikzcd}$$

which gives a homotopy

$$(gf)_! \simeq g_! f_!.$$

Exercise 3.4. Check that dually, a right fibration $\mathcal{E} \to \mathcal{B}$ gives the basic data for a contravariant functor from $\mathcal{B}^{\mathrm{op}}$ to ∞-groupoids.

Fact 3.4.1. *There is a (large) ∞-category Cat_∞ of (small) ∞-categories; we write Gpd_∞ for the full subcategory of this spanned by the ∞-groupoids.*

Definition 3.4.2. For an ∞-category $\mathcal{B}$, let $\mathsf{LFib}(\mathcal{B})$ and $\mathsf{RFib}(\mathcal{B})$ denote the full subcategories of $\mathsf{Cat}_{\infty/\mathcal{B}}$ on the left and right fibrations, respectively.

Theorem 3.4.3 (Lurie)**.** *For an ∞-category $\mathcal{B}$, there is an equivalence*

$$\mathrm{Str}^{\mathrm{L}}_{\mathcal{B}} \colon \mathsf{LFib}(\mathcal{B}) \xrightarrow{\sim} \mathsf{Fun}(\mathcal{B}, \mathsf{Gpd}_\infty),$$

called the straightening *equivalence for left fibrations. This is moreover natural in $\mathcal{B}$ with respect to precomposition of functors and pullback of fibrations.*

Here the naturality means that if a fibration $p \colon \mathcal{E} \to \mathcal{B}$ straightens to a functor $F \colon \mathcal{B} \to \mathsf{Gpd}_\infty$, and we have a pullback square

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, "\phi"'] & \mathcal{B},
\end{tikzcd}$$

then the left fibration q straightens to $F \circ \phi$.

Observation 3.4.4. Taking opposite ∞-categories gives an equivalence $(-)^{\mathrm{op}} \colon \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$. This induces an equivalence on slices

$$\mathsf{Cat}_{\infty/\mathcal{B}} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_{\infty/\mathcal{B}^{\mathrm{op}}},$$

which restricts to an equivalence

$$\mathsf{RFib}(\mathcal{B}) \simeq \mathsf{LFib}(\mathcal{B}^{\mathrm{op}}).$$

Combining this with the straightening equivalence from Theorem 3.4.3, we get a straightening equivalence

$$\mathrm{Str}^{\mathrm{R}}_{\mathcal{B}} \colon \mathsf{RFib}(\mathcal{B}) \xrightarrow{(-)^{\mathrm{op}}} \mathsf{LFib}(\mathcal{B}^{\mathrm{op}}) \xrightarrow{\mathrm{Str}^{\mathrm{L}}_{\mathcal{B}^{\mathrm{op}}}} \mathsf{Fun}(\mathcal{B}^{\mathrm{op}}, \mathsf{Gpd}_{\infty})$$

for right fibrations.

Remark 3.4.5. We are going to treat the straightening theorem as a black box, and will not go into the proof (which almost certainly needs to be done in some model to avoid circularity). However, we'll try to say a few words about the proof in the simplest case, where the base is $[1]$. Here the straightening of a left fibration $\mathcal{E} \to [1]$ is (as we saw above) the functor

$$\mathcal{E}_0 \xleftarrow{\sim} \mathsf{Map}_{/[1]}([1]\mathcal{E}) \to \mathcal{E}_1$$

given by restricting a section to the two objects of $[1]$. There is a formal dual to this construction, which takes a span

$$A \leftarrow X \to B$$

of ∞-groupoids to the pushout $A \amalg_{X\times\{0\}} (X \times [1]) \amalg_{X\times\{1\}} B$. To see that this gives an inverse to straightening, we need to understand (a special case of) this pushout. This is in fact possible (in a model), but I'm not sure what the best "axioms" are to understand this in a model-independent setting.

3.5 Cartesian and cocartesian fibrations

We now want to introduce (co)cartesian fibration, which can be thought of as generalizing left and right fibrations by allowing the fibres to be ∞-categories instead of ∞-groupoids. These will not be defined using orthogonality[1], but via the existence of sufficiently many *(co)cartesian morphisms*, and we start by introducing this notion.

Exercise 3.5. Given a functor $p \colon \mathcal{E} \to \mathcal{B}$ and a morphism $f \colon x \to y$ in $\mathcal{E}$, show that for an object $z \in \mathcal{E}$ we get commutative squares

$$\begin{tikzcd}
\mathcal{E}(y,z) \arrow[r, "f^*"] \arrow[d] & \mathcal{E}(x,z) \arrow[d] \\
\mathcal{B}(py,pz) \arrow[r, "p(f)^*"'] & \mathcal{B}(px,pz)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}(z,x) \arrow[r, "f_*"] \arrow[d] & \mathcal{E}(z,y) \arrow[d] \\
\mathcal{B}(pz,px) \arrow[r, "p(f)_*"'] & \mathcal{B}(pz,py)
\end{tikzcd}$$

[1]This is not possible using just ∞-categories, but can be done in the ∞-category of *marked* ∞-categories, meaning ∞-categories with a chosen collection of morphisms.

by taking appropriate fibres in a cube with top face

$$\begin{tikzcd} \mathcal{E}[2] \arrow[r, "d_1"] \arrow[d] & \mathcal{E}[1] \arrow[d] \\ \mathcal{B}[2] \arrow[r, "d_1"'] & \mathcal{B}[1]. \end{tikzcd}$$

Similarly, we get

$$\begin{tikzcd} \mathcal{E}_{y/} \arrow[r, "f^*"] \arrow[d] & \mathcal{E}_{x/} \arrow[d] \\ \mathcal{B}_{py/} \arrow[r, "p(f)^*"'] & \mathcal{B}_{px/} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_{/x} \arrow[r, "f_*"] \arrow[d] & \mathcal{E}_{/y} \arrow[d] \\ \mathcal{B}_{/px} \arrow[r, "p(f)_*"'] & \mathcal{B}_{/py} \end{tikzcd}$$

by taking fibres in similar cubes of ∞-categories.

Definition 3.5.1. Given a functor $p\colon \mathcal{E} \to \mathcal{B}$, we say a morphism $f\colon x \to y$ in $\mathcal{E}$ is *p-cocartesian* if for every object $z \in \mathcal{E}$, the commutative square

$$\begin{tikzcd} \mathcal{E}(y,z) \arrow[r, "f^*"] \arrow[d] & \mathcal{E}(x,z) \arrow[d] \\ \mathcal{B}(py,pz) \arrow[r, "p(f)^*"'] & \mathcal{B}(px,pz) \end{tikzcd}$$

is a pullback. Dually, we say that f is *p-cartesian* if all squares

$$\begin{tikzcd} \mathcal{E}(z,x) \arrow[r, "f_*"] \arrow[d] & \mathcal{E}(z,y) \arrow[d] \\ \mathcal{B}(pz,px) \arrow[r, "p(f)_*"'] & \mathcal{B}(pz,py) \end{tikzcd}$$

are pullbacks.

Remark 3.5.2. Informally, a morphism $f\colon x \to y$ is p-cocartesian if given $g\colon x \to z$ and a factorization

$$\begin{tikzcd} p(x) \arrow[rr, "p(g)"] \arrow[dr, "p(f)"'] & & p(z) \arrow[dl, "\phi"] \\ & p(y), & \end{tikzcd}$$

there exists a unique lift of this triangle to a factorization

$$\begin{tikzcd} x \arrow[rr, "g"] \arrow[dr, "f"'] & & z \arrow[dl] \\ & y, & \end{tikzcd}$$

of g through f.

Exercise 3.6. Show that a morphism f is p-cocartesian if and only if the commutative square

$$\begin{tikzcd} \mathcal{E}_{y/} \arrow[r, "f^*"] \arrow[d] & \mathcal{E}_{x/} \arrow[d] \\ \mathcal{B}_{py/} \arrow[r, "p(f)^*"'] & \mathcal{B}_{px/} \end{tikzcd}$$

is a pullback.

Lemma 3.5.3. *If $f\colon x \to y$ is p-cocartesian, then a morphism $g\colon y \to z$ is p-cocartesian if and only if $g \circ f$ is p-cocartesian.*

Proof. For $w \in \mathcal{E}$, we consider the commutative diagram

$$\begin{tikzcd} \mathcal{E}(z,w) \arrow[r, "g^*"] \arrow[d] & \mathcal{E}(y,w) \arrow[r, "f^*"] \arrow[d] & \mathcal{E}(x,w) \arrow[d] \\ \mathcal{B}(pz,pw) \arrow[r, "p(g)^*"'] & \mathcal{B}(py,pw) \arrow[r, "p(f)^*"'] & \mathcal{B}(px,pw). \end{tikzcd}$$

Here the right square is a pullback, so the left square is a pullback if and only if the composite square is so. □

Definition 3.5.4. Given a morphism $f\colon a \to b$ in $\mathcal{B}$ and an object $x \in \mathcal{E}$ with $p(x) \simeq a$, a *p-cocartesian lift* of f to x is a p-cocartesian morphism $\bar{f}\colon x \to y$ such that $p(\bar{f}) \simeq f$. More precisely, it is a lift in the commutative square

$$\begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ {[1]} \arrow[r, "f"'] \arrow[ur, dashed, "\bar{f}"] & \mathcal{B} \end{tikzcd}$$

such that $\bar{f}$ is p-cocartesian. Dually, we define p-cartesian lifts of f to y with $p(y) \simeq b$.

Lemma 3.5.5. *Suppose $f\colon a \to b$ is an equivalence in $\mathcal{B}$. Then a commutative diagram*

$$\begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ {[1]} \arrow[r, "f"'] \arrow[ur, "\bar{f}"] & \mathcal{B} \end{tikzcd}$$

gives a p-cocartesian lift of f if and only if $\bar{f}$ is an equivalence.

Proof. Since f is an equivalence, the bottom horizontal map in the commutative square

$$\begin{tikzcd} \mathcal{E}(y,z) \arrow[r, "\bar{f}^*"] \arrow[d] & \mathcal{E}(x,z) \arrow[d] \\ \mathcal{B}(py,pz) \arrow[r, "f^*"'] & \mathcal{B}(px,pz) \end{tikzcd}$$

is an equivalence. The square is therefore a pullback if and only if the top horizontal morphism is an equivalence by Exercise 2.3. This holds for all $z \in \mathcal{E}$ if and only if $\bar{f}$ is an equivalence in $\mathcal{E}$ by Lemma 2.9.6. $\square$

Lemma 3.5.6. *Let $\mathcal{E}[1]_{\mathrm{coct}}$ denote the subspace of $\mathcal{E}[1]$ spanned by the cocartesian morphisms. Then the morphism*

$$\mathcal{E}[1]_{\mathrm{coct}} \to \mathcal{B}[1] \times_{\mathcal{B}^{\simeq}} \mathcal{E}^{\simeq}$$

induced by restricting along $\{0\} \hookrightarrow [1]$ is a monomorphism, with image the pairs $(a \to b, x, p(x) \simeq a)$ for which a cocartesian lift exists.

Proof. We must show that if there exists a cocartesian lift $\bar{f} \colon x \to y$ of $f \colon a \to b$ with source x, then the fibre over $(f, x, p(x) \simeq a)$ is contractible. We may identify this fibre as the subspace of $(\mathcal{E}_{x/})^{\simeq} \times_{(\mathcal{B}_{a/})^{\simeq}} \{f\}$ spanned by the cocartesian lifts of f. By Exercise 3.6, composition with $\bar{f}$ identifies the larger space with the pullback

$$(\mathcal{E}_{y/})^{\simeq} \times_{(\mathcal{B}_{b/})^{\simeq}} \{\mathrm{id}_b\} \simeq (\mathcal{E}_{b,y/})^{\simeq}.$$

It follows from Lemma 3.5.3 that under this equivalence the subspace of cocartesian lifts of f corresponds to that of cocartesian lifts of id_b, which by Lemma 3.5.5 is the subspace $(\mathcal{E}_b^{\simeq})_{y/}$ of equivalences out of y. This completes the proof, since the latter ∞-groupoid is contractible by Observation 3.3.2. $\square$

Definition 3.5.7. For a functor $p \colon \mathcal{E} \to \mathcal{B}$, we say that $\mathcal{E}$ *has p-cocartesian lifts* of a class S of morphisms in $\mathcal{B}$ if for any morphism $f \colon a \to b$ in S and $x \in \mathcal{E}$ with $p(x) \simeq a$, there exists a p-cocartesian lift of f to x. Dually, we can ask for $\mathcal{E}$ to *have p-cartesian lifts* of S. We say that p is a *(co)cartesian fibration* if $\mathcal{E}$ has p-(co)cartesian lifts of all morphisms in $\mathcal{B}$.

Observation 3.5.8. p is a cocartesian fibration if and only if the monomorphism

$$\mathcal{E}[1]_{\mathrm{coct}} \to \mathcal{B}[1] \times_{\mathcal{B}^{\simeq}} \mathcal{E}^{\simeq}$$

is an equivalence.

We next want to characterize left fibrations as a particular class of cocartesian fibrations.

Proposition 3.5.9. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a left fibration. Then every morphism in $\mathcal{E}$ is p-cocartesian. Dually, if p is a right fibration, then every morphism in $\mathcal{E}$ is p-cartesian.*

Proof. We prove the first statement. If p is a left fibration, then $\mathcal{E}_{x/} \to \mathcal{B}_{px/}$ is an equivalence by Lemma 3.3.10. For any morphism $f\colon x \to y$, the vertical morphisms in the commutative square

$$\begin{tikzcd} \mathcal{E}_{y/} \arrow[r, "f^*"] \arrow[d, "\sim"'] & \mathcal{E}_{x/} \arrow[d, "\sim"] \\ \mathcal{B}_{py/} \arrow[r, "p(f)^*"'] & \mathcal{B}_{px/} \end{tikzcd}$$

are therefore equivalences. This means the square is a pullback by Exercise 2.3, so f is p-cocartesian by Exercise 3.6. □

Corollary 3.5.10. *Left fibrations are cocartesian fibrations, and right fibrations are cartesian fibrations.*

Corollary 3.5.11. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. Then the following are equivalent:*

(1) p is a left fibration.

(2) Every morphism in $\mathcal{E}$ is p-cocartesian.

(3) p is conservative.

(4) The fibres of p are ∞-groupoids.

Proof. We have already seen that (1) implies (2). Conversely, if every morphism in $\mathcal{E}$ is p-cocartesian, then Observation 3.5.8 implies that p is a left fibration. By Lemma 3.5.5 we know that a morphism is p-cocartesian over an equivalence if and only if it is itself an equivalence, so (2) implies (3). Conversely, since p is a cocartesian fibration, any morphism in $\mathcal{E}$ factors (uniquely) as a cocartesian morphism followed by a morphism over an equivalence; if (3) holds the latter is an equivalence. But an equivalence is p-cocartesian and p-cocartesian morphisms compose by Lemma 3.5.3, so this implies (2). We also know that (3) is equivalent to (4) from Exercise 2.13. □

Definition 3.5.12. A *morphism of cocartesian fibrations* is a commutative square

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "p"'] & \mathcal{E}' \arrow[d, "p'"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{B}' \end{tikzcd}$$

where p and p' are cocartesian fibrations and F takes p-cocartesian morphisms to p'-cocartesian ones. For $G = \mathrm{id}_{\mathcal{B}}$ we say this is a *morphism of cocartesian fibrations over* $\mathcal{B}$.

Observation 3.5.13. Given $p\colon \mathcal{E} \to \mathcal{B}$ and a p-cocartesian lift $\overline{f}\colon x \to y$ of $f\colon a \to b$, we have in particular for $z \in \mathcal{E}_b$ a pullback square

$$\begin{tikzcd}
\mathcal{E}(y,z) \arrow[r, "f^*"] \arrow[d] & \mathcal{E}(x,z) \arrow[d] \\
\mathcal{B}(b,b) \arrow[r, "p(f)^*"'] & \mathcal{B}(a,b).
\end{tikzcd}$$

Taking the fibre at id_b this gives an equivalence

$$\mathcal{E}_b(y,z) \xrightarrow{\sim} \mathcal{E}(x,z)_f.$$

Proposition 3.5.14. *If*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\
& \mathcal{B} &
\end{tikzcd}$$

is a morphism of cocartesian fibrations over $\mathcal{B}$*, then* F *is an equivalence if and only if* $F_b\colon \mathcal{E}_b \to \mathcal{F}_b$ *is an equivalence for all* $b \in \mathcal{B}$.

Proof. It is clear that if F is an equivalence then it gives an equivalence on fibres, so it remains to see that the latter condition implies that F is an equivalence. Since $(-)^{\simeq}$ preserves pullbacks, we know that the morphisms on fibres in the commutative triangle

$$\begin{tikzcd}
\mathcal{E}^{\simeq} \arrow[rr] \arrow[dr] & & \mathcal{F}^{\simeq} \arrow[dl] \\
& \mathcal{B}^{\simeq} &
\end{tikzcd}$$

are equivalences of ∞-categories, so $F^{\simeq}$ is an equivalence by Exercise 2.2. By Observation 2.7.5 it then suffices to check that F is fully faithful. (Alternatively, we can observe that F is essentially surjective, since every object must lie in some fibre, and use Corollary 2.7.6.)

We will prove that F is fully faithful by seeing that we get an equivalence on fibres in the triangle

$$\begin{tikzcd}
\mathcal{E}(x,y) \arrow[rr] \arrow[dr] & & \mathcal{F}(Fx,Fy) \arrow[dl] \\
& \mathcal{B}(a,b) &
\end{tikzcd}$$

for $x, y \in \mathcal{E}$ lying over $a, b \in \mathcal{B}$. For $f: a \to b$, we choose a p-cocartesian lift $\bar{f}: x \to x'$ of f at x; then by assumption $F(\bar{f})$ is a q-cocartesian lift of f at $F(x)$. In the commutative diagram

$$
\begin{tikzcd}
\mathcal{E}(x', y) \arrow[r, "\bar{f}^*"] \arrow[d] & \mathcal{E}(x, y) \arrow[d] \\
\mathcal{F}(Fx', Fy) \arrow[r, "F(\bar{f})^*"] \arrow[d] & \mathcal{F}(Fx, Fy) \arrow[d] \\
\mathcal{B}(b, b) \arrow[r, "f^*"] & \mathcal{B}(a, b),
\end{tikzcd}
$$

the bottom and composite squares are therefore pullbacks, hence so is the top square. Taking fibres at id_b we therefore get a commutative square

$$
\begin{tikzcd}
\mathcal{E}_b(x', y) \arrow[r, "\sim"] \arrow[d] & \mathcal{E}(x, y)_f \arrow[d] \\
\mathcal{F}_b(Fx', Fy) \arrow[r, "\sim"'] & \mathcal{F}(Fx, Fy)_f .
\end{tikzcd}
$$

Here the left vertical map is also an equivalence by assumption, hence so is the right vertical map, as required. $\square$

Proposition 3.5.15. *Suppose $p: \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. Given a pullback square*

$$
\begin{tikzcd}
\mathcal{E}' \arrow[r, "G"] \arrow[d, "q"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{A} \arrow[r, "F"'] & \mathcal{B},
\end{tikzcd}
$$

the functor q is also a cocartesian fibration, and a morphism in $\mathcal{E}'$ is q-cocartesian if and only if its image in $\mathcal{E}$ is p-cocartesian.

Proof. Given a morphism $\bar{f}: x \to x'$ in $\mathcal{E}'$ lifting $f: a \to b$ in $\mathcal{A}$ and an object y over c, we have a commutative cube

$$
\begin{tikzcd}[row sep=small, column sep=small]
\mathcal{E}'(x', y) \arrow[rr] \arrow[dr, "\bar{f}^*"] \arrow[dd] & & \mathcal{E}(Gx', Gy) \arrow[dr, "G(\bar{f})^*"] \arrow[dd] & \\
& \mathcal{E}'(x, y) \arrow[rr, crossing over] & & \mathcal{E}(Gx, Gy) \arrow[dd] \\
\mathcal{A}(b, c) \arrow[rr] \arrow[dr, "f^*"] & & \mathcal{B}(Fb, Fc) \arrow[dr, "F(f)^*"] & \\
& \mathcal{A}(a, c) \arrow[rr] \arrow[from=uu, crossing over] & & \mathcal{B}(Fa, Fc)
\end{tikzcd}
$$

Here the back and front faces are pullbacks since $\mathcal{E}'$ is a pullback. If $G(\bar{f})$ is p-cocartesian, the right face is also a pullback, which implies that the left face is a pullback by the 3-for-2 property. Thus $\bar{f}$ is q-cocartesian if $G(\bar{f})$ is p-cocartesian. Moreover, given $f\colon a \to b$ in $\mathcal{A}$ and x over a in $\mathcal{E}'$, there exists a p-cocartesian lift of $F(f)$ at $G(x)$; this data then determines a morphism of $\mathcal{E}'$ that is a q-cocartesian lift of f at x. Since q-cocartesian lifts are unique when they exist by Lemma 3.5.6, it follows that all q-cocartesian morphisms must map to p-cocartesian morphisms under G. □

Example 3.5.16. For any ∞-category $\mathcal{A}$ the unique functor $\mathcal{A} \to [0]$ is both a cartesian and a cocartesian fibration; the (co)cartesian morphisms are the equivalences in $\mathcal{A}$. It follows that for any ∞-category $\mathcal{B}$, the projection $p\colon \mathcal{A}\times\mathcal{B} \to \mathcal{B}$ is both a cartesian and a cocartesian fibration, and a morphism is p-(co)cartesian precisely when its image in $\mathcal{A}$ is an equivalence.

Exercise 3.7. Generalize Proposition 3.5.15 to show that for a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"'] & \mathcal{F} \arrow[d, "q"] & \mathcal{E}' \arrow[l] \arrow[d, "p'"] \\ \mathcal{B} \arrow[r] & \mathcal{C} & \mathcal{B}' \arrow[l] \end{tikzcd}$$

where both squares are morphisms of cocartesian fibrations, then the induced map on pullbacks

$$q\colon \mathcal{E} \times_{\mathcal{F}} \mathcal{E}' \to \mathcal{B} \times_{\mathcal{C}} \mathcal{B}'$$

is a cocartesian fibration, and a morphism is q-cocartesian if and only if its images in $\mathcal{E}$ and $\mathcal{E}'$ are p- and p'-cocartesian.

Exercise 3.8. Use Proposition 3.5.14 and Proposition 3.5.15 to show that a morphism of cocartesian fibrations

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "p"'] & \mathcal{E}' \arrow[d, "p'"] \\ \mathcal{B} \arrow[r, "G"'] & \mathcal{B}' \end{tikzcd}$$

is a pullback square if and only if the induced functor on fibres $\mathcal{E}_b \to \mathcal{E}'_{G(b)}$ is an equivalence for all $b \in \mathcal{B}$.

3.6 Arrow ∞-categories and fibrations

In this section we will identify two important families of cocartesian fibrations, and give a simple characterization of cocartesian fibrations. The starting point for this is the following description of mapping spaces in $\mathsf{Ar}(\mathcal{C})$:

Proposition 3.6.1. *Given morphisms $f\colon x \to y$ and $g\colon x' \to y'$ in an ∞-category $\mathcal{C}$, the mapping space $\mathsf{Ar}(\mathcal{C})(f,g)$ is the pullback*

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C})(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(x,x') \arrow[d, "g_*"] \\ \mathcal{C}(y,y') \arrow[r, "f^*"'] & \mathcal{C}(x,y'). \end{tikzcd}$$

Proof. We want to identify the fibre at (f, g) of the map

$$\mathsf{Map}([1] \times [1], \mathcal{C}) \to \mathcal{C}[1]^{\times 2}.$$

We can do this in two steps, by first identifying the map on fibres in the triangle

$$\begin{tikzcd}
\mathsf{Map}([1] \times [1], \mathcal{C}) \arrow[rr] \arrow[dr] & & \mathcal{C}[1]^{\times 2} \arrow[dl] \\
& \mathcal{C}[0]^{\times 4} &
\end{tikzcd}$$

at the objects (x, y, x', y'). Using the decompositions of $[1] \times [1]$ and $[2]$ as usual, we can identify this as

$$(\mathcal{C}(x, x') \times \mathcal{C}(x', y')) \times_{\mathcal{C}(x,y')} (\mathcal{C}(x, y) \times \mathcal{C}(y, y')) \to \mathcal{C}(x, y) \times \mathcal{C}(x', y'),$$

with both maps to $\mathcal{C}(x, y')$ in the pullback given by composition. Now taking the fibre of this at (f, g) produces the required description, since pullbacks commute. $\square$

Exercise 3.9. Extends this description to show that for a morphism $\alpha \colon f \to g$ in $\mathsf{Ar}(\mathcal{C})$, given by a commutative square

$$\begin{tikzcd}
x \arrow[r, "e"] \arrow[d, "f"'] & z \arrow[d, "g"] \\
y \arrow[r, "h"'] & w,
\end{tikzcd}$$

precomposition with α fits in a commutative cube (where the left and right faces are pullbacks)

$$\begin{tikzcd}
\mathsf{Ar}(\mathcal{C})(g, q) \arrow[rr, "\alpha^*"] \arrow[dr] \arrow[dd] & & \mathsf{Ar}(\mathcal{C})(f, q) \arrow[dr] \arrow[dd] & \\
& \mathcal{C}(z, u) \arrow[rr, "e^*"] \arrow[dd] & & \mathcal{C}(x, u) \arrow[dd] \\
\mathcal{C}(w, v) \arrow[rr] \arrow[dr] & & \mathcal{C}(y, v) \arrow[dr] & \\
& \mathcal{C}(z, v) \arrow[rr, "e^*"] & & \mathcal{C}(x, v)
\end{tikzcd}$$

for $q \colon u \to v$.

Proposition 3.6.2. *For any ∞-category $\mathcal{C}$, the functor*

$$\mathrm{ev}_1 \colon \mathsf{Ar}(\mathcal{C}) \to \mathcal{C}$$

is a cocartesian fibration; a morphism

$$\begin{tikzcd}
x \arrow[r, "e"] \arrow[d, "f"'] & z \arrow[d, "g"] \\
y \arrow[r, "h"'] & w
\end{tikzcd}$$

is ev_1*-cocartesian if and only if* e *is an equivalence, i.e. the morphism is taken to an equivalence by* ev_0*. Dually,* $\mathrm{ev}_0 \colon \mathsf{Ar}(\mathcal{C}) \to \mathcal{C}$ *is a cartesian fibration; a morphism as above is* ev_0*-cartesian if and only if its image under* ev_1 *is an equivalence.*

Proof. We prove the first claim; the second is proved similarly, or by taking op. For a square of the given form with e an equivalence, viewed as a morphism $\alpha \colon f \to g$ in $\mathsf{Ar}(\mathcal{C})$, composition with α gives by Exercise 3.9 a commutative cube

$$\begin{tikzcd}
\mathsf{Ar}(\mathcal{C})(g,q) \arrow[rr, "\alpha^*"] \arrow[dr] \arrow[dd] & & \mathsf{Ar}(\mathcal{C})(f,q) \arrow[dr] \arrow[dd] & \\
& \mathcal{C}(z,u) \arrow[rr, "e^*"] \arrow[dd] & & \mathcal{C}(x,u) \arrow[dd] \\
\mathcal{C}(w,v) \arrow[rr] \arrow[dr] & & \mathcal{C}(y,v) \arrow[dr] & \\
& \mathcal{C}(z,v) \arrow[rr, "e^*"] & & \mathcal{C}(x,v),
\end{tikzcd}$$

where the left and right faces are pullbacks by Proposition 3.6.1. Since e is an equivalence, the horizontal morphisms in the front face are equivalences, so that this is also a pullback by Exercise 2.3. Then the back face is also a pullback by the 3-for-2 property, which says precisely that α is ev_1-cocartesian.

By the uniqueness of cocartesian lifts it only remains to show that given an object $x \xrightarrow{f} y$ of $\mathsf{Ar}(\mathcal{C})$ and a morphism $y \xrightarrow{g} z$ in $\mathcal{C}$, there exists a cocartesian lift of the required form, which we can take to be the degenerate square

$$\begin{tikzcd}
x \arrow[r, "="] \arrow[d, "f"'] & x \arrow[d, "gf"] \\
y \arrow[r, "g"'] & z,
\end{tikzcd}$$

viewed as a morphism in $\mathsf{Ar}(\mathcal{C})$ over g with source f. $\square$

Lemma 3.6.3. *Given a functor* $p \colon \mathcal{E} \to \mathcal{B}$*, the following are equivalent for a morphism* $\bar{f} \colon x \to y$ *in* $\mathcal{E}$ *over* $f \colon a \to b$ *in* $\mathcal{B}$*:*

(1) The morphism $\bar{f}$ *is* p*-cocartesian.*

(2) For every morphism $\bar{g} \colon z \to w$ *in* $\mathcal{E}$ *over* $g \colon c \to d$ *in* $\mathcal{B}$*, the commutative square of mapping* ∞*-groupoids*

$$\begin{tikzcd}
\mathsf{Ar}(\mathcal{E})(\bar{f},\bar{g}) \arrow[r] \arrow[d] & \mathcal{E}(x,z) \arrow[d] \\
\mathsf{Ar}(\mathcal{B})(f,g) \arrow[r] & \mathcal{B}(a,c)
\end{tikzcd}$$

is a pullback.

(3) The preceding square is a pullback whenever $\bar{g}$ is id_z for some $z \in \mathcal{E}$.

Proof. Since the description of mapping ∞-groupoids in Proposition 3.6.1 is natural with respect to the functor p, we get a commutative cube

$$\begin{tikzcd}
\mathsf{Ar}(\mathcal{E})(\bar{f},\bar{g}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{E}(x,z) \arrow[dr, "\bar{g}_*"] \arrow[dd] & \\
& \mathcal{E}(y,w) \arrow[rr, "\bar{f}^*" near start] \arrow[dd] & & \mathcal{E}(x,w) \arrow[dd] \\
\mathsf{Ar}(\mathcal{B})(f,g) \arrow[rr] \arrow[dr] & & \mathcal{B}(a,c) \arrow[dr, "g_*"] & \\
& \mathcal{B}(b,d) \arrow[rr, "f^*"] & & \mathcal{B}(a,d)
\end{tikzcd}$$

where the top and bottom faces are pullbacks. If $\bar{f}$ is p-cocartesian, then the front face is also a pullback, hence so is the back face by the 3-for-2 property. Thus (1) implies (2). Since (3) is a special case of (2), it remains to check that $\bar{f}$ is p-cocartesian if (2) holds. When $\bar{g} = \mathrm{id}_z$, our cube takes the form

$$\begin{tikzcd}
\mathsf{Ar}(\mathcal{E})(\bar{f},\mathrm{id}_z) \arrow[rr] \arrow[dr, "\sim"] \arrow[dd] & & \mathcal{E}(x,z) \arrow[dr, "="] \arrow[dd] & \\
& \mathcal{E}(y,z) \arrow[rr, "\bar{f}^*" near start] \arrow[dd] & & \mathcal{E}(x,z) \arrow[dd] \\
\mathsf{Ar}(\mathcal{B})(f,\mathrm{id}_c) \arrow[rr] \arrow[dr, "\sim"] & & \mathcal{B}(a,c) \arrow[dr, "="] & \\
& \mathcal{B}(b,c) \arrow[rr, "f^*"] & & \mathcal{B}(a,c),
\end{tikzcd}$$

where the morphisms from the back to the front are equivalences since the top and bottom faces are pullbacks. It follows that the back face is a pullback if and only if the front face is so, and the former property for all $z \in \mathcal{E}$ says precisely that $\bar{f}$ is p-cocartesian. □

Proposition 3.6.4. *For a functor $p\colon \mathcal{E} \to \mathcal{B}$ and a monomorphism $S \hookrightarrow \mathcal{E}[1]$, let $\mathsf{Ar}_S(\mathcal{E}) \subseteq \mathsf{Ar}(\mathcal{E})$ denote the full subcategory spanned by morphisms in S. Then the following are equivalent:*

(1) p is a cocartesian fibration and S is the ∞-groupoid of p-cocartesian morphisms.

(2) The commutative square

$$\begin{tikzcd}
\mathsf{Ar}_S(\mathcal{E}) \arrow[r, "\mathrm{ev}_0"] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
\mathsf{Ar}(\mathcal{B}) \arrow[r, "\mathrm{ev}_0"'] & \mathcal{B}
\end{tikzcd}$$

is a pullback.

Proof. Let q denote the functor

$$\mathsf{Ar}_S(\mathcal{E}) \to \mathsf{Ar}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{E}$$

corresponding to the square above, so that (2) is equivalent to q being an equivalence.

First suppose (1) holds; then q is an equivalence on cores by Observation 3.5.8, so it suffices to show that it is fully faithful. This follows from Lemma 3.6.3, since for p-cocartesian morphisms $\bar{f}: x \to y$ and $\bar{g}: z \to w$ over $f: a \to b$ and $g: c \to d$, we have

$$\mathsf{Ar}_S(\mathcal{E})(\bar{f}, \bar{g}) \simeq \mathsf{Ar}(\mathcal{E})(\bar{f}, \bar{g}) \xrightarrow{\sim} \mathsf{Ar}(\mathcal{B})(f, g) \times_{\mathcal{B}(a,c)} \mathcal{E}(x, z).$$

Now we prove the converse. Assuming (2), we first note that on cores we have a pullback

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathcal{E}^{\simeq} \arrow[d] \\ \mathcal{B}[1] \arrow[r] & \mathcal{B}^{\simeq}, \end{tikzcd}$$

so that given $x \in \mathcal{E}$ and $f: p(x) \to b$, there exists a (unique) lift $\bar{f}: x \to y$ in S such that $p(\bar{f}) \simeq f$. By uniqueness of p-cocartesian lifts, it therefore suffices to show that a morphism $s: x \to x'$ in S must be p-cocartesian.

The pullback above also implies that the degeneracy $\mathcal{E}^{\simeq} \to \mathcal{E}[1]$ factors through S, so that S contains all equivalences. We can thus conclude that condition (3) in Lemma 3.6.3 holds for any $\bar{f}$ in S and $z \in \mathcal{E}$, so that $\bar{f}$ is indeed p-cocartesian. □

Corollary 3.6.5. *Suppose $p: \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. Then for any ∞-category $\mathcal{K}$, the functor*

$$p_*: \mathsf{Fun}(\mathcal{K}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{K}, \mathcal{B}),$$

given by composition with p, is also a cocartesian fibration. A natural transformation $\phi: \mathcal{K} \times [1] \to \mathcal{E}$ is p_*-cocartesian if and only if its components $\phi_a: [1] \to \mathcal{E}$ are p-cocartesian for all $a \in \mathcal{K}$.

Proof. Let $S \subseteq \mathsf{Map}([1], \mathsf{Fun}(\mathcal{K}, \mathcal{E}))$ be the sub-∞-groupoid of natural transformations all of whose components are p-cocartesian. Then under the equivalence $\mathsf{Fun}(\mathcal{K}, \mathsf{Ar}(\mathcal{E})) \simeq \mathsf{Ar}(\mathsf{Fun}(\mathcal{K}, \mathcal{E}))$, Lemma 2.8.6 identifies $\mathsf{Ar}_S(\mathsf{Fun}(\mathcal{K}, \mathcal{E}))$ with $\mathsf{Fun}(\mathcal{K}, \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}))$. Since p is a cocartesian fibration, this is equivalent to

$$\mathsf{Fun}(\mathcal{K}, \mathsf{Ar}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{E}) \simeq \mathsf{Ar}(\mathsf{Fun}(\mathcal{K}, \mathcal{B})) \times_{\mathsf{Fun}(\mathcal{K}, \mathcal{B})} \mathsf{Fun}(\mathcal{K}, \mathcal{E}).$$

But then Proposition 3.6.4 implies that p_* is a cocartesian fibration and S is precisely the ∞-groupoid of p-cocartesian morphisms. □

Observation 3.6.6. Given a functor $p\colon \mathcal{E} \to \mathcal{B}$, let $\mathsf{Ar}_{\mathrm{coct}}(\mathcal{E})$ denote the full subcategory of $\mathsf{Ar}(\mathcal{E})$ spanned by the p-cocartesian morphisms. Then Proposition 3.6.4 implies that p is a cocartesian fibration if and only if

$$\mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \to \mathsf{Ar}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{E}$$

is an equivalence.

Observation 3.6.7. Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. Inverting the equivalence from Proposition 3.6.4, we get a functor

$$\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \xrightarrow{\sim} \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \xrightarrow{\mathrm{ev}_1} \mathcal{E},$$

which takes an object $(x, f\colon p(x) \to b)$ in the source to the target $f_!x$ of a cocartesian lift of f with source x; we call this the *cocartesian transport functor*.

Taking the fibre of this map at an object $b \in \mathcal{B}$, we get a functor

$$\mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{/b} \to \mathcal{E}_b$$

and restricting this to the fibre of the source over $b' \in \mathcal{B}$, we get a functor

$$\mathcal{E}_{b'} \times \mathcal{B}(b', b) \to \mathcal{E}_b$$

or

$$\mathcal{B}(b', b) \to \mathsf{Map}(\mathcal{E}_{b'}, \mathcal{E}_b).$$

The *straightening theorem*, which we will discuss in the next section, says that this assignment is part of a functor $\mathcal{B} \to \mathsf{Cat}_\infty$ determined by p.

Example 3.6.8. Consider the cocartesian fibration $\mathrm{ev}_1\colon \mathsf{Ar}(\mathcal{C}) \to \mathcal{C}$. Then $\mathsf{Ar}_{\mathrm{coct}}(\mathsf{Ar}(\mathcal{C}))$ is the full subcategory of $\mathsf{Fun}([1] \times [1], \mathcal{C})$ spanned by commutative squares

$$\begin{tikzcd} x \arrow[r] \arrow[d, "\sim"'] & y \arrow[d] \\ z \arrow[r] & w \end{tikzcd}$$

where the left vertical map is an equivalence. We therefore get an equivalence $\mathsf{Ar}_{\mathrm{coct}}(\mathsf{Ar}(\mathcal{C})) \simeq \mathsf{Fun}([2], \mathcal{C})$, under which the equivalence

$$\mathsf{Ar}_{\mathrm{coct}}(\mathsf{Ar}(\mathcal{C})) \xrightarrow{\sim} \mathsf{Ar}(\mathcal{C}) \times_{\mathcal{C}} \mathsf{Ar}(\mathcal{C})$$

is that given by the pushout $[2] \simeq [1] \amalg_{[0]} [1]$; the cocartesian transport functor from Observation 3.6.7 is then just the functor

$$\mathsf{Ar}(\mathcal{C}) \times_{\mathcal{C}} \mathsf{Ar}(\mathcal{C}) \xrightarrow{\sim} \mathsf{Fun}([2], \mathcal{C}) \to \mathsf{Ar}(\mathcal{C})$$

given by composition.

3.7 Straightening for (co)cartesian fibrations

We can now see how a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$ determines the basic data of a functor $\mathcal{B} \to \mathsf{Cat}_\infty$:

- We have a commutative diagram

$$
\begin{tikzcd}
\mathcal{E} \arrow[d] & \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \arrow[l, "s"'] \arrow[r, "t"] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{B} & \mathsf{Ar}(\mathcal{B}) \arrow[l, "s"] \arrow[r, "t"'] & \mathcal{B},
\end{tikzcd}
$$

  where the left square is a pullback by Proposition 3.6.4. On fibres over $f\colon b \to b'$ in $\mathsf{Ar}(cB)$, we therefore get

$$\mathcal{E}_b \xleftarrow{\sim} \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \to \mathcal{E}_{b'},$$

  which (inverting the equivalence) we think of as a functor $f_!\colon \mathcal{E}_b \to \mathcal{E}_{b'}$.

- For the identity map id_b we have a commutative diagram

$$
\begin{tikzcd}
& \mathcal{E} \arrow[dl, "="'] \arrow[d] \arrow[dr, "="] & \\
\mathcal{E} & \mathsf{Ar}(\mathcal{E}) \arrow[l, "s"] \arrow[r, "t"'] & \mathcal{E},
\end{tikzcd}
$$

  so on fibres we get

$$
\begin{tikzcd}
& \mathcal{E}_b \arrow[dl, "="'] \arrow[d] \arrow[dr, "="] & \\
\mathcal{E}_b & \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E})_{\mathrm{id}_b} \arrow[l, "\sim"] \arrow[r] & \mathcal{E}_b,
\end{tikzcd}
$$

  which shows that $(\mathrm{id}_b)_! \simeq \mathrm{id}_{\mathcal{E}_b}$.

- For two composable maps $f\colon b \to b'$, $g\colon b' \to b''$ we can consider fibres in the diagram

$$
\begin{tikzcd}
& & \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \arrow[ddll, bend right, "s"'] \arrow[ddrr, bend left, "t"] & & \\
& & \mathsf{Fun}_{\mathrm{coct}}([2], \mathcal{E}) \arrow[u, "d_1"] \arrow[dl] \arrow[dr] & & \\
& \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \arrow[dl, "s"] \arrow[dr, "t"] & & \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\mathcal{E} & & \mathcal{E} & & \mathcal{E},
\end{tikzcd}
$$

where $\mathsf{Fun}_{\mathrm{coct}}([2], \mathcal{E})$ denotes the full subcategory of $\mathsf{Fun}([2], \mathcal{E})$ on functors that take all maps to p-cocartesian morphisms in $\mathcal{E}$, so that the middle square is a pullback. This produces the diagram

$$\begin{tikzcd}
 & & \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E})_{gf} \arrow[lld, bend right, "\sim"'] \arrow[rrd, bend left] & & \\
 & & \mathsf{Fun}_{\mathrm{coct}}([2], \mathcal{E})_{g,f} \arrow[u, "\sim"] \arrow[ld, "\sim"'] \arrow[rd] & & \\
 & \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E})_f \arrow[ld, "\sim"'] \arrow[rd] & & \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E})_g \arrow[ld, "\sim"'] \arrow[rd] & \\
\mathcal{E}_b & & \mathcal{E}_{b'} & & \mathcal{E}_{b''},
\end{tikzcd}$$

which gives a homotopy

$$(gf)_! \simeq g_! f_!.$$

Definition 3.7.1. For an ∞-category $\mathcal{B}$, let $\mathsf{Cart}(\mathcal{B})$ and $\mathsf{Cocart}(\mathcal{B})$ denote the subcategories of $\mathsf{Cat}_{\infty/\mathcal{B}}$ whose objects are the (co)cartesian fibrations and whose morphisms are the morphisms of (co)cartesian fibrations over $\mathcal{B}$ (i.e. the functors over $\mathcal{B}$ that preserve (co)cartesian morphisms).

Theorem 3.7.2 (Lurie). *For an ∞-category $\mathcal{B}$, there is an equivalence*

$$\mathrm{Str}_{\mathcal{B}}^{\mathrm{coct}} \colon \mathsf{Cocart}(\mathcal{B}) \xrightarrow{\sim} \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty),$$

called the straightening *equivalence for cocartesian fibrations. This is moreover natural in $\mathcal{B}$ with respect to precomposition of functors and pullback of fibrations.*

Here the naturality means that if a fibration $p \colon \mathcal{E} \to \mathcal{B}$ straightens to a functor $F \colon \mathcal{B} \to \mathsf{Cat}_\infty$, and we have a pullback square

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r, "\phi"'] & \mathcal{B},
\end{tikzcd}$$

then the cocartesian fibration q straightens to $F \circ \phi$.

Observation 3.7.3. Taking opposite ∞-categories induces an equivalence

$$\mathsf{Cart}(\mathcal{B}) \simeq \mathsf{Cocart}(\mathcal{B}^{\mathrm{op}}).$$

We can combine this with the straightening equivalence from Theorem 3.7.2 to get a straightening equivalence for cartesian fibrations. For a cartesian fibration $p \colon \mathcal{E} \to \mathcal{B}$ we would like this to produce a contravariant functor that takes

$b \in \mathcal{B}$ to $\mathcal{E}_b$ (and not $\mathcal{E}_b^{\mathrm{op}}$ — in particular, straightening over $[0]$ should give the identity of Cat_∞). We therefore define this as the composite

$$\mathrm{Str}^{\mathrm{cart}}_{\mathcal{B}} \colon \mathsf{Cart}(\mathcal{B}) \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cocart}(\mathcal{B}^{\mathrm{op}}) \xrightarrow{\mathrm{Str}^{\mathrm{coct}}_{\mathcal{B}^{\mathrm{op}}}} \mathsf{Fun}(\mathcal{B}^{\mathrm{op}}, \mathsf{Cat}_\infty) \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Fun}(\mathcal{B}^{\mathrm{op}}, \mathsf{Cat}_\infty).$$

Example 3.7.4. The projection $\mathcal{A} \times \mathcal{B} \to \mathcal{A}$ is a cocartesian fibration (Example 3.5.16). Since pullback of fibrations corresponds to composition of functors, this straightens to the composite

$$\mathcal{B} \to * \xrightarrow{\mathcal{A}} \mathsf{Cat}_\infty,$$

i.e. the constant functor with value $\mathcal{B}$.

We will treat straightening as a black box, but we will need to look inside a little bit in order to prove that the Yoneda embedding is fully faithful. More precisely, we will need the following:

Fact 3.7.5. *Suppose $p \colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration, and let $F \colon \mathcal{B} \to \mathsf{Cat}_\infty$ be its straightening. Then for objects $x, y \in \mathcal{B}$, the morphism of mapping spaces*

$$\mathcal{B}(x, y) \to \mathsf{Map}(F(x), F(y)) \simeq \mathsf{Map}(\mathcal{E}_x, \mathcal{E}_y)$$

determined by F is equivalent to the map constructed in Observation 3.6.7 from the cocartesian transport functor $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \xleftarrow{\sim} \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \xrightarrow{\mathrm{ev}_1} \mathcal{E}$.

Chapter 4

Representability and the Yoneda embedding

4.1 Terminal objects and representable fibrations

Our main goal in this section is to characterize those right fibrations over $\mathcal{B}$ that are of the form $\mathcal{B}_{/x} \to \mathcal{B}$ for some object x. We start by introducing initial and terminal objects in an ∞-category:

Definition 4.1.1. An object $x \in \mathcal{C}$ is *initial* if $\mathcal{C}(x, c)$ is contractible for all $c \in \mathcal{C}$, and *terminal* if $\mathcal{C}(c, x)$ is contractible for all c.

Observation 4.1.2. If $\mathcal{C}_{\mathrm{init}}$ denotes the full subcategory of $\mathcal{C}$ spanned by the initial objects, then all mapping ∞-groupoids in $\mathcal{C}_{\mathrm{init}}$ are contractible. It therefore follows from Corollary 2.1.20 that $\mathcal{C}_{\mathrm{init}}$ is either empty or contractible. In other words, initial objects are unique if they exist.

Observation 4.1.3. The object x is initial if and only if $\mathcal{C}_{x/} \to \mathcal{C}$ is an equivalence, and terminal if and only if $\mathcal{C}_{/x} \to \mathcal{C}$ is an equivalence.

Definition 4.1.4. A right fibration $p\colon \mathcal{E} \to \mathcal{B}$ is *representable* if $\mathcal{E}$ has a terminal object, while a left fibration p is *corepresentable* if $\mathcal{E}$ has an initial object. We also say that p is *represented by* $b \in \mathcal{B}$ if $\mathcal{E}$ has a terminal object that lies over b; we may also say an object $x \in \mathcal{E}$ *exhibits* p *as represented by* b if x is terminal and $p(x) \simeq b$.

Definition 4.1.5. We say a presheaf $\phi\colon \mathcal{B}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ is *representable* if the corresponding right fibration is representable, and that a copresheaf $\mathcal{B} \to \mathsf{Gpd}_\infty$ is *corepresentable* if the corresponding left fibration is corepresentable. We will similarly say that ϕ is represented by $b \in \mathcal{B}$ and that a point $x \in \phi(b)$ exhibits ϕ as represented by b if the corresponding statement hold for the associated right fibration.

Observation 4.1.6. Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a right fibration. Then for any object $x \in \mathcal{E}$ we have a commutative square

$$\begin{tikzcd} \mathcal{E}_{/x} \arrow[r, "\sim"] \arrow[d] & \mathcal{B}_{/b} \arrow[d] \\ \mathcal{E} \arrow[r, "p"] & \mathcal{B}, \end{tikzcd}$$

where the top horizontal morphism is an equivalence by Lemma 3.3.10. Inverting this equivalence, we see that x determines a morphism $s_x\colon \mathcal{B}_{/b} \to \mathcal{E}$ over $\mathcal{B}$, which takes id_b to x.

Proposition 4.1.7. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a right fibration. Then an object $x \in \mathcal{E}$ exhibits p as represented by $b \simeq p(x)$ if and only if $s_x\colon \mathcal{B}_{/b} \to \mathcal{E}$ is an equivalence.*

Proof. By definition, s_x is the composite $\mathcal{B}_{/b} \xrightarrow{\sim} \mathcal{E}_{/x} \to \mathcal{E}$, so the composite is an equivalence if and only if $\mathcal{E}_{/x} \to \mathcal{E}$ is an equivalence, i.e. x is a terminal object. □

Observation 4.1.8. In terms of presheaves, this says that given a point $x \in \phi(b)$ for a presheaf $\phi\colon \mathcal{B}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$, there is a canonical natural transformation $\mathcal{B}(-, b) \to \phi$ that takes id_b to x, and x exhibits ϕ as represented by b precisely if this is an equivalence.

Proposition 4.1.9. id_x *is initial in $\mathcal{C}_{x/}$ and terminal in $\mathcal{C}_{/x}$. In particular, $\mathcal{C}_{/x} \to \mathcal{C}$ is a representable right fibration and $\mathcal{C}_{x/} \to \mathcal{C}$ is a corepresentable left fibration.*

Proof. We prove the first case by showing that the forgetful functor $p\colon (\mathcal{C}_{x/})_{\mathrm{id}_x/} \to \mathcal{C}_{x/}$ is an equivalence. Using Proposition 3.3.7 and the naturality in $\mathcal{K}$, of Lemma 3.3.5 we can identify the forgetful functor $(\mathcal{C}_{x/})_{f/} \to \mathcal{C}_{x/}$ with the map on fibres over f in the square

$$\begin{tikzcd} \mathsf{Fun}([2], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}(\{0 < 2\}, \mathcal{C}) \arrow[d] \\ \mathsf{Fun}(\{0 < 1\}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\{0\}, \mathcal{C}). \end{tikzcd}$$

From the Segal decomposition of $[2]$ we get a pushout square

$$\begin{tikzcd} \{0 < 1\} \arrow[r] \arrow[d] & {[2]} \arrow[d] \\ \{0\} \arrow[r] & {[1]}, \end{tikzcd}$$

and so we have a commutative diagram

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}) \arrow[r] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}([2], \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}(\{0 < 2\}, \mathcal{C}) \arrow[d] \\ \mathcal{C} \arrow[r] & \mathsf{Fun}(\{0 < 1\}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\{0\}, \mathcal{C}) \end{tikzcd}$$

where the left square is a pullback and both horizontal composites are identities. Taking fibres at $x \in \mathcal{C}$, we get

$$\mathcal{C}_{x/} \xrightarrow{\sim} (\mathcal{C}_{x/})_{\mathrm{id}_x/} \xrightarrow{p} \mathcal{C}_{x/}$$

where the first map is an equivalence and the composite is the identity. It follows that our map p is an equivalence, as required. □

Combining Proposition 4.1.7 and Proposition 4.1.9, we get:

Corollary 4.1.10. *A right fibration over $\mathcal{B}$ is representable if and only if it is equivalent to $\mathcal{B}_{/b} \to \mathcal{B}$ for some $b \in \mathcal{B}$, while a left fibration is corepresentable if and only if it is equivalent to $\mathcal{B}_{b/} \to \mathcal{B}$ for some $b \in \mathcal{B}$.* □

4.2 A weak Yoneda lemma

Our goal in this section is to prove a weak form of the Yoneda lemma, namely that for a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ and a *fixed* object $x \in \mathcal{C}$, the ∞-groupoid of natural transformations $\mathcal{C}(-,x) \to F$ is equivalent to $F(x)$. This will be a consequence of the following property of initial and terminal objects:

Proposition 4.2.1. *Suppose $\mathcal{I}$ has an initial object x. Then any left fibration $p\colon \mathcal{E} \to \mathcal{B}$ is right orthogonal to the inclusion $\{x\} \to \mathcal{I}$. Dually, any right fibration is right orthogonal to the inclusion of a terminal object.*

Proof. We prove the case of a left fibration p. By Corollary 3.2.5, $p_*\colon \mathsf{Fun}(\mathcal{K},\mathcal{E}) \to \mathsf{Fun}(\mathcal{K},\mathcal{B})$ is a left fibration for any ∞-category $\mathcal{K}$, which implies that p is right orthogonal to $\{0\} \times \mathcal{K} \to [1] \times \mathcal{K}$. Now we have by definition a pushout

$$\begin{tikzcd}
\{0\} \times \mathcal{K} \arrow[r] \arrow[d] & {[1] \times \mathcal{K}} \arrow[d] \\
\{-\infty\} \arrow[r] & \mathcal{K}^{\triangleleft},
\end{tikzcd}$$

so p is also right orthogonal to the inclusion of the cone point in $\mathcal{K}^{\triangleleft}$ by Lemma 2.4.7. By Lemma 2.4.5, it then follows that p is right orthogonal to $\mathcal{L}^{\triangleleft} \to \mathcal{K}^{\triangleleft}$ for any functor $\mathcal{L} \to \mathcal{K}$.

In particular, p is right orthogonal to $\{x\}^{\triangleleft} \to \mathcal{I}^{\triangleleft}$. Since x is an initial object, the forgetful functor $\mathcal{I}_{x/} \to \mathcal{I}$ is an equivalence, so it have an inverse section $\mathcal{I} \to \mathcal{I}_{x/}$. Using Lemma 3.3.5, this corresponds to a functor $\phi\colon \mathcal{I}^{\triangleleft} \to \mathcal{I}$ that takes $-\infty$ to x and restricts to the identity on $\mathcal{I}$. But this means we have a retract diagram

$$\begin{tikzcd}
\{x\} \arrow[r] \arrow[d] & \{x\}^{\triangleleft} \arrow[r] \arrow[d] & \{x\} \arrow[d] \\
\mathcal{I} \arrow[r] & \mathcal{I}^{\triangleleft} \arrow[r, "\phi"'] & \mathcal{I},
\end{tikzcd}$$

so that p is indeed right orthogonal to $\{x\} \to \mathcal{I}$ by Lemma 2.4.9. □

Corollary 4.2.2. *Suppose $p\colon \mathcal{E} \to \mathcal{C}$ is a right fibration; then for $x \in \mathcal{C}$, the morphism*

$$\mathsf{Map}_{/\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}) \to \mathsf{Map}_{/\mathcal{C}}(\{x\}, \mathcal{E}) \simeq \mathcal{E}_x$$

is an equivalence. Dually, if p is a left fibration, the map

$$\mathsf{Map}_{/\mathcal{C}}(\mathcal{C}_{x/}, \mathcal{E}) \to \mathsf{Map}_{/\mathcal{C}}(\{x\}, \mathcal{E}) \simeq \mathcal{E}_x$$

is an equivalence.

Proof. We prove the first case. By Proposition 4.2.1, p is right orthogonal to the inclusion of the initial object $\{\mathrm{id}_x\} \to \mathcal{C}_{/x}$, so we have a pullback square

$$\begin{tikzcd}
\mathsf{Map}(\mathcal{C}_{/x}, \mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Map}(\mathcal{C}_{/x}, \mathcal{C}) \arrow[d] \\
\mathsf{Map}(\{x\}, \mathcal{E}) \arrow[r] & \mathsf{Map}(\{x\}, \mathcal{C}).
\end{tikzcd}$$

Taking vertical fibres over the projection $\mathcal{C}_{/x} \to \mathcal{C}$ we get the equivalence we want. □

Remark 4.2.3. An alternative approach to the proof Corollary 4.2.2 is to use that $\mathcal{C}_{/x} \to \mathcal{C}$ is the free cartesian fibration on $\{x\} \to \mathcal{C}$, as we will see below in §6.1.

Remark 4.2.4. If p is a left fibration, then for $f\colon \mathcal{I} \to \mathcal{B}$ and $e \in \mathcal{E}$ over $f(x)$ we have an induced functor $\mathcal{I}_{x/} \to \mathcal{B}_{fx/} \xrightarrow{s_e} \mathcal{E}$, which lifts f if x is initial. It is thus not completely mysterious that we get an inverse to the restrictions in Corollary 4.2.2.

Notation 4.2.5. We write $\mathsf{PSh}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Gpd}_\infty)$ for the ∞-category of *presheaves* on $\mathcal{C}$. Note that straightening then gives an equivalence $\mathsf{PSh}(\mathcal{C}) \simeq \mathsf{RFib}(\mathcal{C})$.

Observation 4.2.6. Under straightening, Corollary 4.2.2 gives a weak form of the Yoneda lemma: for a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$, we get an equivalence

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathcal{C}(-,x), F) \simeq F(x),$$

since the functor $\mathcal{C}(-,x)$ is (by definition) the straightening of the right fibration $\mathcal{C}_{/x} \to \mathcal{C}$.

Exercise 4.1. Show that, for a right fibration $p\colon \mathcal{E} \to \mathcal{B}$, the maps $s_x\colon \mathcal{B}_{/p(x)} \to \mathcal{E}$ for $x \in \mathcal{E}$ are natural in x, and use this to produce a section of the map

$$\mathsf{Map}_{/\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{E}) \to \mathsf{Map}_{/\mathcal{C}}(\{x\}, \mathcal{E})$$

(This is then necessarily its inverse by Corollary 4.2.2.)

4.3 Bifibrations

A *bifibration* over $\mathcal{A} \times \mathcal{B}$ is supposed to encode a functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Gpd}_\infty$. Here we introduce this class of functors and explain how this claim follows from the straightening results we have discussed above. We apply this to construct the mapping ∞-groupoid functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Gpd}_\infty$.

Notation 4.3.1. Given $p\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ we write $p_\mathcal{A}, p_\mathcal{B}$ for the compositions of p with the projections to $\mathcal{A}$ and $\mathcal{B}$.

Definition 4.3.2. A functor $p\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ is a *bifibration* if

- $p_\mathcal{A}$ is a cartesian fibration,
- $p_\mathcal{B}$ is a cocartesian fibration,
- a morphism in $\mathcal{E}$ is $p_\mathcal{A}$-cartesian if and only if its image under $p_\mathcal{B}$ is an equivalence,
- a morphism in $\mathcal{E}$ is $p_\mathcal{B}$-cocartesian if and only if its image under $p_\mathcal{A}$ is an equivalence.

Warning 4.3.3. In parts of the category theory literature the term "bifibration" is used for a functor that is both a cartesian and a cocartesian fibration. Our terminology follows *Higher Topos Theory*; the 1-categorical analogue of what we call bifibrations might elsewhere be called "two-sided discrete fibrations".

Observation 4.3.4. Given a commutative triangle

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\phi"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{A} \times \mathcal{B} & \end{tikzcd}$$

where both p and q are bifibrations, the functor ϕ must preserve $p_\mathcal{A}$-cartesian and $p_\mathcal{B}$-cocartesian morphisms, since it necessarily preserves the classes that map to equivalences in $\mathcal{A}$ or $\mathcal{B}$.

Corollary 4.3.5. *Suppose $p\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ is a bifibration. Then:*

(1) The functor $p_a\colon \mathcal{E}_a \to \mathcal{B}$ on fibres over $a \in \mathcal{A}$ is a left fibration for all $a \in \mathcal{A}$.

(2) The functor $p_b\colon \mathcal{E}_b \to \mathcal{A}$ on fibres over $b \in \mathcal{B}$ is a right fibration for all $b \in \mathcal{B}$.

Proof. We prove the first case. For $a \in \mathcal{A}$, we have a pullback square

$$\begin{tikzcd} \mathcal{E}_a \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ \{a\} \times \mathcal{B} \arrow[r] & \mathcal{A} \times \mathcal{B} \end{tikzcd}$$

of cocartesian fibrations over $\mathcal{B}$ along functors that preserve cocartesian morphisms (since the cocartesian morphisms in $\mathcal{A} \times \mathcal{B}$ over $\mathcal{B}$ are those that map to equivalences in $\mathcal{A}$). Hence Exercise 3.7 implies that $p_a \colon \mathcal{E}_a \to \mathcal{B}$ is a cocartesian fibration, and the cocartesian morphisms are those that map to $p_{\mathcal{B}}$-cocartesian morphisms in $\mathcal{E}$. But since p is a bifibration, *every* morphism in $\mathcal{E}_a$ maps to a $p_{\mathcal{B}}$-cocartesian morphism, since their images all lie over id_a in $\mathcal{A}$. Thus every morphism in $\mathcal{E}_a$ is p_a-cocartesian, so that p_a is a left fibration by Corollary 3.5.11. □

Remark 4.3.6. In fact, we can characterize bifibrations as *precisely* the functors $p \colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ such that either of the following equivalent conditions hold:

(1) $p_{\mathcal{A}}$ is a cartesian fibration, $p_{\mathcal{A}}$-cartesian morphisms lie over equivalences in $\mathcal{B}$, and $p_a \colon \mathcal{E}_a \to \mathcal{B}$ is a left fibration for all $a \in \mathcal{A}$.

(2) $p_{\mathcal{B}}$ is a cocartesian fibration, $p_{\mathcal{B}}$-cocartesian morphisms lie over equivalences in $\mathcal{A}$, and $p_b \colon \mathcal{E}_b \to \mathcal{B}$ is a right fibration for all $b \in \mathcal{B}$.

See [HHLN23a, 2.3.3 and 2.3.13] for a proof.

Construction 4.3.7. Given a bifibration $p \colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$, we can consider the commutative triangle

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "p"] \arrow[dr, "p_{\mathcal{B}}"'] & & \mathcal{A} \times \mathcal{B} \arrow[dl] \\ & \mathcal{B} & \end{tikzcd}$$

as a morphism of cocartesian fibrations over $\mathcal{B}$ (since by Example 3.5.16 a cocartesian morphism in $\mathcal{A} \times \mathcal{B}$ over $\mathcal{B}$ is precisely one whose component in $\mathcal{A}$ is an equivalence). We can therefore straighten this to a natural transformation

$$\mathcal{B} \times [1] \to \mathsf{Cat}_\infty,$$

Here the target is the constant functor with value $\mathcal{A}$, so this is equivalently a functor

$$\mathcal{B} \to \mathsf{Cat}_{\infty/\mathcal{A}}.$$

For $b \in \mathcal{B}$, the value of this functor is $\mathcal{E}_a \to \mathcal{A}$, which is a right fibration. This functor therefore takes values in the full subcategory $\mathsf{RFib}(\mathcal{A})$. Now we can use straightening for right fibrations over $\mathcal{A}$ to get a functor

$$\mathcal{B} \to \mathsf{RFib}(\mathcal{A}) \simeq \mathsf{Fun}(\mathcal{A}^{\mathrm{op}}, \mathsf{Gpd}_\infty),$$

or $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Gpd}_\infty$.

Remark 4.3.8. Of course, we could also have done this straightening in the other order: first constructed a functor $\mathcal{A}^{\mathrm{op}} \to \mathsf{LFib}(\mathcal{B})$ and then straightened the fibrewise left fibrations to a functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Gpd}_\infty$. It is not immediately obvious that the two constructions produce the same functor, but in fact it can be shown that there is a *unique* straightening equivalence for bifibrations (as well as for (co)cartesian fibrations and their two-variable version); see [HHLN23b, Appendix A] for a proof.

Example 4.3.9. The functor $p\colon \mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ given by restriction along $\{0, 1\} \to [1]$ is a bifibration. Indeed, this is precisely the content of Proposition 3.6.2. Applying Construction 4.3.7, we obtain a functor

$$\mathsf{y}_{\mathcal{C}}\colon \mathcal{C} \to \mathsf{RFib}(\mathcal{C}) \xrightarrow{\sim} \mathsf{PSh}(\mathcal{C}),$$

which we call the *Yoneda embedding*. We can also equivalently view this as the *mapping ∞-groupoid functor*

$$\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Gpd}_\infty.$$

Lemma 4.3.10. *Suppose $p\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ is a bifibration. For functors $F\colon \mathcal{A}' \to \mathcal{A}$ and $G\colon \mathcal{B}' \to \mathcal{B}$, we define $(F,G)^*\mathcal{E}$ by the pullback*

$$\begin{tikzcd}
(F,G)^*\mathcal{E} \arrow[r] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{A}' \times \mathcal{B}' \arrow[r, "F\times G"'] & \mathcal{A} \times \mathcal{B}.
\end{tikzcd}$$

Then $(F,G)^\times\mathcal{E} \to \mathcal{A}' \times \mathcal{B}'$ is also a bifibration.

Proof. From the pullback square defining $(F,G)^*\mathcal{E}$ we get a pullback of cartesian fibrations

$$\begin{tikzcd}
(F,G)^*\mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F^*\mathcal{E} \arrow[d, "p"] \\
\mathcal{A}' \times \mathcal{B}' \arrow[r, "F\times G"'] & \mathcal{A}' \times \mathcal{B}
\end{tikzcd}$$

over $\mathcal{A}'$, so that (Exercise 3.7) $p'_{\mathcal{A}'}$ is a cartesian fibration, with a $p'_{\mathcal{A}'}$-cartesian morphism being one that maps to an equivalence in $\mathcal{B}'$ and a p-cartesian morphism in $\mathcal{E}$; since p is a bifibration this reduces to the morphisms that map to equivalences in $\mathcal{B}'$. We similarly get a pullback of cocartesian fibrations over $\mathcal{B}'$, and together these show that p' is a bifibration. □

4.4 The Yoneda embedding

Our goal in this section is to show that the Yoneda embedding $\mathsf{y}_{\mathcal{C}}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{C})$ is fully faithful for any ∞-category $\mathcal{C}$. More precisely, we will show that the functor

$$j\colon \mathcal{C} \to \mathsf{RFib}(\mathcal{C}),$$

obtained by unstraightening $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ in the second variable, is fully faithful, with image the full subcategory $\mathsf{RFib}_{\mathrm{rep}}(\mathcal{C}) \subseteq \mathsf{RFib}(\mathcal{C})$ of representable presheaves. We will then use this to prove a stronger version of the Yoneda lemma, namely an identification of the presheaf

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}_{\mathcal{C}}(-), F)$$

with F, for a fixed $F \in \mathsf{PSh}(\mathcal{C})$.

Proposition 4.4.1. *The functor $j: \mathcal{C} \to \mathsf{RFib}(\mathcal{C})$ is fully faithful with image the representable right fibrations.*

Proof. We first show that j is fully faithful, i.e. that for $x, y \in \mathcal{C}$ the induced morphism

$$\mathcal{C}(x,y) \to \mathsf{Map}_{/\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{C}_{/y}) \tag{4.1}$$

is an equivalence. Using Corollary 4.2.2, it suffices to show that the composite

$$\mathcal{C}(x,y) \to \mathsf{Map}_{/\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{C}_{/y}) \xrightarrow{\sim} \mathsf{Map}_{\mathcal{C}}(\{x\}, \mathcal{C}_{/y}) \simeq (\mathcal{C}_{/y})_x \simeq \mathcal{C}(x,y)$$

is an equivalence.

The functor j was obtained from $\mathrm{ev}_1 : \mathsf{Ar}(\mathcal{C}) \to \mathcal{C}$ by cocartesian straightening, so by Fact 3.7.5 we can describe (4.1) via the cocartesian transport functor from Observation 3.6.7. For ev_1 this was identified in Example 3.6.8, so we conclude that (4.1) is adjoint to the map

$$\begin{tikzcd} \mathcal{C}_{/x} \times \mathcal{C}(x,y) \arrow[rr] \arrow[dr] & & \mathcal{C}_{/y} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

given by composition. Restricting this to $\{\mathrm{id}_x\}$ we get that our composite map

$$\mathcal{C}(x,y) \to (\mathcal{C}_{/y})_x$$

is given by composition with id_x, and so is the identity.

To identify the image of j, we observe that on the one hand, $j(x) = (\mathcal{C}_{/x} \to \mathcal{C})$ is representable for every $x \in \mathcal{C}$, and on the other hand Proposition 4.1.7 shows that every representable right fibration is equivalent to one in the image of j. □

Suppose $F: \mathcal{C}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ is a presheaf with right fibration $\mathcal{F} \to \mathcal{C}$. We already proved a weak form of the Yoneda lemma, namely an equivalence

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}_{\mathcal{C}}(x), F) \simeq \mathsf{Map}_{/\mathcal{C}}(\mathcal{C}_{/x}, \mathcal{F}) \simeq \mathcal{F}_x \simeq F(x)$$

for a fixed x. We will now strengthen this to an identification of the presheaf

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}_{\mathcal{C}}(-), F)$$

with F. Since composition of functors corresponds under straightening to pullbacks of fibrations, we can identify the right fibration for this presheaf as the pullback

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{PSh}(\mathcal{C})_{/F} \arrow[d] \\ \mathcal{C} \arrow[r, "\mathsf{y}"'] & \mathsf{PSh}(\mathcal{C}) \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{RFib}(\mathcal{C})_{/p} \arrow[d] \\ \mathcal{C} \arrow[r, "j"'] & \mathsf{RFib}(\mathcal{C}), \end{tikzcd}$$

where $p \colon \mathcal{E} \to \mathcal{C}$ is the right fibration for F. Our goal is therefore to find a pullback square of the form

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{RFib}(\mathcal{C})_{/p} \arrow[d] \\ \mathcal{C} \arrow[r, "j"'] & \mathsf{RFib}(\mathcal{C}). \end{tikzcd}$$

Observation 4.4.2. If $p \colon \mathcal{E} \to \mathcal{C}$ is a right fibration, then the equivalence

$$(\mathsf{Cat}_{\infty/\mathcal{C}})_{/p} \simeq \mathsf{Cat}_{\infty/\mathcal{E}}$$

of Corollary 3.3.9 (under which the forgetful functor to $\mathsf{Cat}_{\infty/\mathcal{C}}$ corresponds to composition with p) restricts to an equivalence

$$\mathsf{RFib}(\mathcal{C})_{/p} \simeq \mathsf{RFib}(\mathcal{E}).$$

Indeed, it follows from (the dual of) Lemma 2.4.5 that a functor $q \colon \mathcal{X} \to \mathcal{E}$ is a right fibration if and only if pq is a right fibration.

Observation 4.4.3. For an arbitrary functor $p \colon \mathcal{E} \to \mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{E}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Ar}(\mathcal{C}) \arrow[dr] \arrow[dd] & \\ & \mathcal{E}^{\times 2} \arrow[rr] & & \mathcal{C}^{\times 2} \\ \mathcal{E} \arrow[rr] & & \mathcal{C}, & \end{tikzcd}$$

where the top square consists of morphisms of cocartesian fibrations. This straightens to a square of natural transformations of functors to Cat_∞, which we can identify with a natural transformation in the square

$$\begin{tikzcd} \mathcal{E} \arrow[r, "j_{\mathcal{E}}"] \arrow[d, "p"'] & \mathsf{Cat}_{\infty/\mathcal{E}} \arrow[d, "p_*"] \arrow[dl, Rightarrow] \\ \mathcal{C} \arrow[r, "j_{\mathcal{C}}"'] & \mathsf{Cat}_{\infty/\mathcal{C}} \end{tikzcd}$$

whose component at $x \in \mathcal{E}$ is given by the square

$$\begin{tikzcd} \mathcal{E}_{/x} \arrow[r] \arrow[d] & \mathcal{C}_{/px} \arrow[d] \\ \mathcal{E} \arrow[r, "p"'] & \mathcal{C}. \end{tikzcd}$$

If p is a right fibration, then the functor $\mathcal{E}_{/x} \to \mathcal{C}_{/px}$ is an equivalence, so that this natural transformation is invertible, and we get a commutative square

$$\begin{tikzcd} \mathcal{E} \arrow[r, "j_{\mathcal{E}}"] \arrow[d, "p"'] & \mathsf{RFib}(\mathcal{E}) \arrow[d, "p_*"] \\ \mathcal{C} \arrow[r, "j_{\mathcal{C}}"'] & \mathsf{RFib}(\mathcal{C}). \end{tikzcd} \tag{4.2}$$

Proposition 4.4.4. *For $p\colon \mathcal{E} \to \mathcal{C}$ a right fibration, the commutative square (4.2) is a pullback.*

Proof. We know from Proposition 4.4.1 that the horizontal functors are both fully faithful. It then follows from Exercise 2.16 that the square is a pullback if and only if it is a pullback on π_0 of cores. Thus we have a pullback if and only if the objects in the image of $j_{\mathcal{E}}$ are precisely those whose value under p_* lies in the image of $j_{\mathcal{C}}$. In other words, we need to show that a right fibration over $\mathcal{E}$ is representable if and only if its composition with p is a representable right fibration over $\mathcal{C}$. This is clear since both conditions amount to having a terminal object. □

Combining Proposition 4.4.4 with Observation 4.4.2, we see:

Corollary 4.4.5. *For a right fibration $p\colon \mathcal{E} \to \mathcal{C}$, there is a pullback square*

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{RFib}(\mathcal{C})_{/p} \arrow[d] \\ \mathcal{C} \arrow[r, "j"'] & \mathsf{RFib}(\mathcal{C}). \end{tikzcd}$$

If F is the presheaf corresponding to p under straightening, then this gives an equivalence

$$F \simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}(-), F)$$

in $\mathsf{PSh}(\mathcal{C})$.

Question 4.4.6. Can we make this Yoneda equivalence functorial in F? (This amounts to making the straightening construction in Observation 4.4.3 functorial in right fibrations...)

Chapter 5

Limits and colimits

5.1 Joins and (co)cones

In this section we introduce the ∞-categories of *(co)cones* on a diagram, which we will use to define (co)limits in the next section. We also explain how these can be described in terms of *joins* of ∞-categories.

Definition 5.1.1. For a functor $p \colon \mathcal{I} \to \mathcal{C}$, the ∞-category of *cones* of p is the pullback

$$\begin{array}{ccc} \mathcal{C}_{/p} & \longrightarrow & \mathsf{Fun}(\mathcal{I}, \mathcal{C})_{/p} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathsf{Fun}(\mathcal{I}, \mathcal{C}), \end{array}$$

where the bottom horizontal functor is the constant diagram functor (given by composition with $\mathcal{I} \to [0]$). Dually, the ∞-category $\mathcal{C}_{p/}$ of *cocones* of p is the pullback

$$\begin{array}{ccc} \mathcal{C}_{p/} & \longrightarrow & \mathsf{Fun}(\mathcal{I}, \mathcal{C})_{p/} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathsf{Fun}(\mathcal{I}, \mathcal{C}). \end{array}$$

Definition 5.1.2. The *join* of two ∞-categories $\mathcal{K}, \mathcal{L}$ is the pushout

$$\mathcal{K} \star \mathcal{L} := \mathcal{K} \amalg_{\mathcal{K} \times \mathcal{L} \times \{0\}} \mathcal{K} \times \mathcal{L} \times [1] \amalg_{\mathcal{K} \times \mathcal{L} \times \{1\}} \mathcal{L}.$$

(Note that this is *not* symmetric in $\mathcal{K}$ and $\mathcal{L}$, though we have $(\mathcal{K} \star \mathcal{L})^{\mathrm{op}} \simeq \mathcal{L}^{\mathrm{op}} \star \mathcal{K}^{\mathrm{op}}$.)

Example 5.1.3. The left and right cones on $\mathcal{K}$ can be described as the joins

$$\mathcal{K}^{\triangleleft} \simeq [0] \star \mathcal{K}, \qquad \mathcal{K}^{\triangleright} \simeq \mathcal{K} \star [0].$$

Remark 5.1.4. Just as for the cones, it can be shown (e.g. using quasicategories) that we have

$$(\mathcal{K} \star \mathcal{L})^{\simeq} \simeq \mathcal{K}^{\simeq} \amalg \mathcal{L}^{\simeq},$$

$$(\mathcal{K} \star \mathcal{L})(x, y) \simeq \begin{cases} *, & x \in \mathcal{K}, y \in \mathcal{L}, \\ \emptyset, & x \in \mathcal{L}, y \in \mathcal{K}, \\ \mathcal{K}(x, y), & x, y \in \mathcal{K}, \\ \mathcal{L}(x, y), & x, y \in \mathcal{L}. \end{cases}$$

We can generalize Lemma 3.3.5 to:

Proposition 5.1.5. *For a functor $p\colon \mathcal{I} \to \mathcal{C}$ and an ∞-category $\mathcal{K}$, we have canonical pullback squares*

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{K}, \mathcal{C}_{/p}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K} \star \mathcal{I}, \mathcal{C}) \arrow[d] \\ \{p\} \arrow[r] & \mathsf{Fun}(\mathcal{I}, \mathcal{C}), \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Fun}(\mathcal{K}, \mathcal{C}_{p/}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{I} \star \mathcal{K}, \mathcal{C}) \arrow[d] \\ \{p\} \arrow[r] & \mathsf{Fun}(\mathcal{I}, \mathcal{C}). \end{tikzcd}$$

Proof. We prove the first case. From the definition of $\mathcal{C}_{/p}$, we get a pullback

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{K}, \mathcal{C}_{/p}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K} \times [1], \mathsf{Fun}(\mathcal{I}, \mathcal{C})) \arrow[d] \\ \mathsf{Fun}(\mathcal{K}, \mathcal{C}) \times \{p\} \arrow[r] & \mathsf{Fun}(\mathcal{K} \times \{0, 1\}, \mathsf{Fun}(\mathcal{I}, \mathcal{C})). \end{tikzcd}$$

On the other hand, from the definition of the join we have a pullback

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{K} \star \mathcal{I}, \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K} \times [1], \mathsf{Fun}(\mathcal{I}, \mathcal{C})) \arrow[d] \\ \mathsf{Fun}(\mathcal{K}, \mathcal{C}) \times \mathsf{Fun}(\mathcal{I}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\mathcal{K} \times \{0, 1\}, \mathsf{Fun}(\mathcal{I}, \mathcal{C})). \end{tikzcd}$$

We can then combine these into a commutative diagram

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{K}, \mathcal{C}_{/p}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K} \star \mathcal{I}, \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K} \times [1], \mathsf{Fun}(\mathcal{I}, \mathcal{C})) \arrow[d] \\ \mathsf{Fun}(\mathcal{K}, \mathcal{C}) \times \{p\} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K}, \mathcal{C}) \times \mathsf{Fun}(\mathcal{I}, \mathcal{C}) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K} \times \{0, 1\}, \mathsf{Fun}(\mathcal{I}, \mathcal{C})) \\ \{p\} \arrow[r] & \mathsf{Fun}(\mathcal{I}, \mathcal{C}), & \end{tikzcd}$$

where the top left square is a pullback by 3-for-2. Hence the composite square in the first row is a pullback, which completes the proof. □

Observation 5.1.6. In particular, taking $\mathcal{K} = [0]$ above, we have pullback squares

$$\begin{tikzcd} \mathcal{C}_{/p} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{C}) \arrow[d] \\ \{p\} \arrow[r] & \mathsf{Fun}(\mathcal{I}, \mathcal{C}), \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_{p/} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \arrow[d] \\ \{p\} \arrow[r] & \mathsf{Fun}(\mathcal{I}, \mathcal{C}). \end{tikzcd}$$

5.2 Limits and colimits

We can now define (co)limits in an ∞-category. Using straightening we can then provide concrete descriptions of these in the ∞-category Gpd_∞, which allows us to deduce further characterizations of (co)limits.

Definition 5.2.1. A *limit* of p is an terminal object in the ∞-category $\mathcal{C}_{/p}$ of cones. Dually, a *colimit* is an initial object in $\mathcal{C}_{p/}$.

Notation 5.2.2. We write $\lim_{\mathcal{I}} p$ and $\operatorname{colim}_{\mathcal{I}} p$ for the objects of $\mathcal{C}$ that are the value at the cone point of the limit cone and colimit cocone of p, when these exist.

Proposition 5.2.3. *Suppose $F\colon \mathcal{C} \to \mathsf{Gpd}_\infty$ is a functor that corresponds to the left fibration $\mathcal{F} \to \mathcal{C}$. Then:*

- *The limit of F is given by the ∞-groupoid $\mathsf{Map}_{/\mathcal{C}}(\mathcal{C}, \mathcal{F})$ of sections of $\mathcal{F}$.*
- *The colimit of F is given by the localization $\|\mathcal{F}\|$.*

Proof. By definition, the limit is a terminal object in $\mathsf{Gpd}_{\infty/F}$. Using the equivalence $\mathsf{Fun}(\mathcal{C}, \mathsf{Gpd}_\infty) \simeq \mathsf{LFib}(\mathcal{C})$ this ∞-category is the pullback

$$\begin{tikzcd} \mathsf{Gpd}_{\infty/F} \arrow[r] \arrow[d] & \mathsf{LFib}(\mathcal{C})_{/\mathcal{F}} \arrow[d] \\ \mathsf{Gpd}_\infty \arrow[r, "-\times\mathcal{C}"'] & \mathsf{LFib}(\mathcal{C}). \end{tikzcd}$$

Hence $\mathsf{Gpd}_{\infty/F} \to \mathsf{Gpd}_\infty$ is the right fibration for the functor

$$X \mapsto \mathsf{Map}_{/\mathcal{C}}(\mathcal{C} \times X, \mathcal{F}) \simeq \mathsf{Map}(X, \mathsf{Map}_{/\mathcal{C}}(\mathcal{C}, \mathcal{F})),$$

where we have used that $\mathsf{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{F})$ is an ∞-groupoid, since it's a fibre of the left fibration $\mathsf{Fun}(\mathcal{C}, \mathcal{F}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$. Thus $\mathsf{Gpd}_{\infty/F}$ is equivalent to $\mathsf{Gpd}_{\infty/\mathsf{Map}_{/\mathcal{C}}(\mathcal{C},\mathcal{F})}$, which indeed has a terminal object given by $\mathsf{Map}_{/\mathcal{C}}(\mathcal{C}, \mathcal{F})$.

To identify the colimit, we similarly observe that $\mathsf{Gpd}_{\infty,F/} \to \mathsf{Gpd}_\infty$ is the left fibration for the functor

$$X \mapsto \mathsf{Map}_{/\mathcal{C}}(\mathcal{F}, \mathcal{C} \times X) \simeq \mathsf{Map}(\mathcal{F}, X) \simeq \mathsf{Map}(\|\mathcal{F}\|, X).$$

Thus $\mathsf{Gpd}_{\infty,F/} \simeq \mathsf{Gpd}_{\infty,\|\mathcal{F}\|/}$ which indeed has the required initial object. $\square$

Exercise 5.1. Let $q\colon \mathcal{E} \to \mathcal{C}^{\mathrm{op}}$ be the right fibration for F. Show that the limit of F can also be described as the ∞-groupoid of sections of q, and the colimit of F is also $\|\mathcal{E}\|$.

Definition 5.2.4. Given morphisms $x \xrightarrow{f} y \xleftarrow{g} z$ in an ∞-category $\mathcal{C}$, their *pullback*, if it exists, is the limit of the corresponding diagram $\Lambda^2_2 \to \mathcal{C}$, where $\Lambda^2_2 := \{0 < 2\} \amalg_2 \{1 < 2\}$ is the cospan category

$$0 \to 2 \leftarrow 1.$$

Corollary 5.2.5. *Pullbacks in the ∞-category Gpd_∞ are pullbacks in the "external" sense.*

Proof. Given a diagram $\Lambda^2_2 \to \mathsf{Gpd}_\infty$ with corresponding left fibration $\mathcal{E} \to \Lambda^2_2$, we know that the limit is $\mathsf{Map}_{/\Lambda^2_2}(\Lambda^2_2, \mathcal{E})$. Now using that Λ^2_2 is a pushout, we get

$$\begin{aligned}\mathsf{Map}_{/\Lambda^2_2}(\Lambda^2_2, \mathcal{E}) &\simeq \mathsf{Map}_{/\Lambda^2_2}(\{0 < 2\}, \mathcal{E}) \times_{\mathsf{Map}_{/\Lambda^2_2}(\{2\},\mathcal{E})} \mathsf{Map}_{/\Lambda^2_2}(\{1 < 2\}, \mathcal{E}) \\ &\simeq \mathsf{Map}_{/[1]}([1], \mathcal{E}_{0<2}) \times_{\mathcal{E}_2} \mathsf{Map}_{/[1]}([1], \mathcal{E}_{1<2}) \\ &\simeq \mathcal{E}_0 \times_{\mathcal{E}_2} \mathcal{E}_1,\end{aligned}$$

where we have used that

$$\mathsf{Map}_{/[1]}([1], \mathcal{E}_{i<2}) \simeq \mathsf{Ar}(\mathcal{E}_{i<2})_{i<2} \xrightarrow{\mathrm{ev}_0} \mathcal{E}_i$$

is an equivalence since $\mathcal{E}_{i<2} \to [1]$ is a left fibration. □

5.3 Localizations and (co)limits of ∞-categories

We now wish to describe (co)limits in Cat_∞ in terms of (co)cartesian fibrations. To state this, we first need to briefly introduce localizations of ∞-categories:

Definition 5.3.1. Suppose $\mathcal{C}$ is an ∞-category and $S \subseteq \mathcal{C}[1]$ is an ∞-groupoid of morphisms that generate a subcategory $\mathcal{C}_S$ via Fact 2.11.1. Then the *localization* $\mathcal{C}[S^{-1}]$ of $\mathcal{C}$ at S is the ∞-category given by the pushout

$$\begin{tikzcd} \mathcal{C}_S \arrow[r] \arrow[d] & \|\mathcal{C}_S\| \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{C}[S^{-1}]. \end{tikzcd}$$

Observation 5.3.2. For an ∞-category $\mathcal{D}$, we get a pullback square

$$\begin{tikzcd} \mathsf{Map}(\mathcal{C}[S^{-1}], \mathcal{D}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \mathsf{Map}(\mathcal{C}_S, \mathcal{D}^{\simeq}) \arrow[r] & \mathsf{Map}(\mathcal{C}_S, \mathcal{D}). \end{tikzcd}$$

In other words, a functor $\mathcal{C} \to \mathcal{D}$ factors (uniquely) through $\mathcal{C}[S^{-1}]$ if and only if it takes the morphisms in S to equivalences in $\mathcal{D}$.

Exercise 5.2. Let $L\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the localization functor at $S \subseteq \mathcal{C}[1]$, and take $\bar{S} \subseteq \mathcal{C}[1]$ to be the ∞-groupoid of morphisms that are taken to equivalences by L. Show that then L also exhibits $\mathcal{C}[S^{-1}]$ as the localization of $\mathcal{C}$ at $\bar{S}$.

Remark 5.3.3. It follows that a functor $F\colon \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ at some collection of morphisms if and only if it is the localization at those morphisms that are inverted by F. Thus "being a localization" is a *property* of a functor. (Unfortunately it is a property that we often want to establish but can be very hard to verify!)

Proposition 5.3.4. *Suppose $F\colon \mathcal{C} \to \mathsf{Cat}_\infty$ is a functor that corresponds to the cocartesian fibration $\mathcal{F} \xrightarrow{p} \mathcal{C}$. Then:*

- *The limit of F is given by the ∞-category $\mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{F})$ of cocartesian sections of $\mathcal{F}$, i.e. the full subcategory of $\mathsf{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{F})$ spanned by functors that take all morphisms of $\mathcal{C}$ to p-cocartesian morphisms in F.*
- *The colimit of F is given by the localization $\mathcal{F}[S^{-1}]$, where S is the collection of p-cocartesian morphisms.*

Proof. By definition, the limit is a terminal object in $\mathsf{Cat}_{\infty/F}$. Using the equivalence $\mathsf{Fun}(\mathcal{C}, \mathsf{Cat}_\infty) \simeq \mathsf{Cocart}(\mathcal{C})$, this ∞-category is the pullback

$$\begin{tikzcd}
\mathsf{Cat}_{\infty/F} \arrow[r] \arrow[d] & \mathsf{Cocart}(\mathcal{C})_{/\mathcal{F}} \arrow[d] \\
\mathsf{Cat}_\infty \arrow[r, "-\times\mathcal{C}"'] & \mathsf{Cocart}(\mathcal{C}).
\end{tikzcd}$$

Hence $\mathsf{Cat}_{\infty/F} \to \mathsf{Cat}_\infty$ is the right fibration for the functor

$$\mathcal{K} \mapsto \mathsf{Map}^{\mathrm{coct}}_{/\mathcal{C}}(\mathcal{C} \times \mathcal{K}, \mathcal{F}) \simeq \mathsf{Map}(\mathcal{K}, \mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{F})).$$

Thus $\mathsf{Cat}_{\infty/F}$ is equivalent to $\mathsf{Cat}_{\infty/\mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{C}}(\mathcal{C},\mathcal{F})}$, which indeed has a terminal object given by $\mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{F})$.

To identify the colimit, we similarly observe that $\mathsf{Cat}_{\infty,F/} \to \mathsf{Cat}_\infty$ is the left fibration for the functor

$$\mathcal{K} \mapsto \mathsf{Map}^{\mathrm{coct}}_{/\mathcal{C}}(\mathcal{F}, \mathcal{C} \times \mathcal{K}).$$

Now under the equivalence

$$\mathsf{Map}_{/\mathcal{C}}(\mathcal{F}, \mathcal{C} \times \mathcal{K}) \simeq \mathsf{Map}(\mathcal{F}, \mathcal{C}),$$

the sub-∞-groupoid $\mathsf{Map}^{\mathrm{coct}}_{/\mathcal{C}}(\mathcal{F}, \mathcal{C} \times \mathcal{K})$ corresponds to the ∞-groupoid of morphisms $\mathcal{F} \to \mathcal{C}$ that take the p-cocartesian morphisms to equivalences, since the cocartesian morphisms in $\mathcal{C} \times \mathcal{K}$ are precisely those whose component in $\mathcal{K}$ is an equivalence. Thus we have a natural equivalence

$$\mathsf{Map}^{\mathrm{coct}}_{/\mathcal{C}}(\mathcal{F}, \mathcal{C} \times \mathcal{K}) \simeq \mathsf{Map}(\mathcal{F}[S^{-1}], \mathcal{K}),$$

so that $\mathsf{Cat}_{\infty,F/} \simeq \mathsf{Cat}_{\infty,\mathcal{F}[S^{-1}]/}$, which indeed has the required initial object. $\square$

Exercise 5.3. Let $q \colon \mathcal{E} \to \mathcal{B}^{\mathrm{op}}$ be the cartesian fibration for F. Show that the limit of F can also be described as the ∞-category of cartesian sections of q, and the colimit of F is also the localization of $\mathcal{E}$ at the cartesian morphisms.

Exercise 5.4. Generalize the argument from Corollary 5.2.5 to show that pullbacks in Cat_∞ are pullbacks in the "external" sense.

5.4 (Co)limits in subcategories

In this section we will prove the following criterion for identifying (co)limits in a subcategory:

Proposition 5.4.1. *Let $i \colon \mathcal{C}' \to \mathcal{C}$ be a subcategory, and $p \colon \mathcal{K} \to \mathcal{C}'$ a diagram such that ip has a limit in $\mathcal{C}$ for which the limit cone $q \colon \mathcal{K}^{\triangleleft} \to \mathcal{C}$ factors through $\mathcal{C}'$. Suppose a morphism $x \to \lim_{\mathcal{K}} ip$ with $x \in \mathcal{C}'$ such that the composites $x \to \lim_{\mathcal{K}} ip \xrightarrow{q(-\infty \to k)} p(k)$ are in $\mathcal{C}'$ for all $k \in \mathcal{K}$ lies in $\mathcal{C}'$. Then q is also a limit cone in $\mathcal{C}'$.*

Remark 5.4.2. Less formally, this says that the limit of p in $\mathcal{C}$ is also the limit of p in $\mathcal{C}'$ provided that $\lim_{\mathcal{K}} ip$ is contained in $\mathcal{C}'$, and a morphism $x \to \lim_{\mathcal{K}} ip$ with $x \in \mathcal{C}'$ lies in $\mathcal{C}'$ if and only if all the composites $x \to \lim_{\mathcal{K}} ip \to p(k)$ are in $\mathcal{C}'$.

Observation 5.4.3. Suppose $i \colon \mathcal{C}' \to \mathcal{C}$ is a subcategory, and $\mathcal{C}$ has a terminal object x. If x lies in $\mathcal{C}'$ and the unique morphism $c \to x$ lies in $\mathcal{C}'$ for all $c \in \mathcal{C}'$, then x is also a terminal object in $\mathcal{C}'$, since we have

$$\emptyset \neq \mathcal{C}'(c, x) \hookrightarrow \mathcal{C}(c, x) \simeq *.$$

Proposition 5.4.4. *Suppose $i \colon \mathcal{C}' \to \mathcal{C}$ is a subcategory. Given a diagram $p \colon \mathcal{K} \to \mathcal{C}'$, the induced functor*

$$i_{/p} \colon \mathcal{C}'_{/p} \to \mathcal{C}_{/ip}$$

is also a monomorphism, exhibiting $\mathcal{C}'_{/p}$ as the subcategory of $\mathcal{C}_{/ip}$ whose objects are the cones $q \colon \mathcal{K}^{\triangleleft} \to \mathcal{C}$ such that $q(-\infty)$ is an object of $\mathcal{C}'$ and $q(-\infty) \to q(k)$ is a morphism in $\mathcal{C}'$ for every $k \in \mathcal{K}$, and whose morphisms among these are those whose component at the cone point also lies in $\mathcal{C}'$.

Proof. For any ∞-category $\mathcal{L}$ we have the commutative cube

$$\begin{tikzcd}
\mathrm{Map}(\mathcal{L}, \mathcal{C}'_{/p}) \arrow[rr] \arrow[dr, "(i_{/p})_*"] \arrow[dd] & & \mathrm{Map}(\mathcal{L} \star \mathcal{K}, \mathcal{C}') \arrow[dr, hook] \arrow[dd] & \\
& \mathrm{Map}(\mathcal{L}, \mathcal{C}_{/ip}) \arrow[rr] \arrow[dd] & & \mathrm{Map}(\mathcal{L} \star \mathcal{K}, \mathcal{C}) \arrow[dd] \\
\{p\} \arrow[rr] \arrow[dr, "="] & & \mathrm{Map}(\mathcal{K}, \mathcal{C}') \arrow[dr, hook] & \\
& \{ip\} \arrow[rr] & & \mathrm{Map}(\mathcal{K}, \mathcal{C}).
\end{tikzcd}$$

Here we know the front and back faces are pullbacks, and the horizontal morphisms in the right square are both monomorphisms. Moreover, the bottom face is a pullback (as a fibre of a monomorphism is a point if it is not empty), so the top face is also a pullback by the 3-for-2 property. This means that $(i_{/p})_*$ is a monomorphism of ∞-groupoids for every $\mathcal{L}$, so that $i_{/p}$ is a monomorphism of ∞-categories, as required. Taking $\mathcal{L} = [0]$ we see that the objects of $\mathcal{C}'_{/p}$ are those cones $q\colon \mathcal{K}^{\triangleleft} \to \mathcal{C}$ that factor through $\mathcal{C}'$, i.e. those that take the cone point to an object of $\mathcal{C}'$ and the morphism $-\infty \to k$ to a morphism in $\mathcal{C}'$ for every $k \in \mathcal{K}$; taking $\mathcal{L} = [1]$ we get that the morphisms of $\mathcal{C}'_{/p}$ are those morphisms among such cones whose component at the cone point lie in $\mathcal{C}'$. □

As a special case, we note:

Corollary 5.4.5. *Suppose $i\colon \mathcal{C}' \to \mathcal{C}$ is a full subcategory. Given a diagram $p\colon \mathcal{K} \to \mathcal{C}'$, the induced functor*

$$i_{/p}\colon \mathcal{C}'_{/p} \to \mathcal{C}_{/ip}$$

is also fully faithful, with image the cones whose cone point lies in $\mathcal{C}'$. In other words, we have a pullback square

$$\begin{tikzcd}
\mathcal{C}'_{/p} \arrow[r] \arrow[d] & \mathcal{C}_{/ip} \arrow[d] \\
\mathcal{C}' \arrow[r] & \mathcal{C}.
\end{tikzcd}$$

Proof of Proposition 5.4.1. By Proposition 5.4.4 we have a subcategory inclusion

$$i_{/p}\colon \mathcal{C}'_{/p} \to \mathcal{C}_{/ip}$$

where $\mathcal{C}_{/ip}$ has a terminal object q that is in the image of $i_{/p}$. From the description of this subcategory in Proposition 5.4.4, we see that the given assumptions imply that for any cone $q' \in \mathcal{C}'_{/p}$, the unique morphism $i_{/p}(q') \to q$ is contained in $\mathcal{C}'_{/p}$. Then q is also a terminal object in $\mathcal{C}'_{/p}$ by Observation 5.4.3. □

Corollary 5.4.6. *Suppose $i\colon \mathcal{C}' \to \mathcal{C}$ is a full subcategory, and $p\colon \mathcal{K} \to \mathcal{C}'$ is a diagram such that ip has a limit in $\mathcal{C}$. If $\lim_{\mathcal{C}} ip$ is an object of $\mathcal{C}'$ then it is also the limit of p in $\mathcal{C}'$.* □

5.5 (Co)limits of functors and iterated (co)limits

In this section we prove two useful results about (co)limits: a (co)limit over a product can be computed as an iterated (co)limit, and (co)limits in functor ∞-categories are computed objectwise. The following result on terminal sections is the key input for both:

Proposition 5.5.1. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration such that each fibre $\mathcal{E}_b$ has a terminal object. Then the ∞-category $\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})$ of sections of p has a terminal object, which is the unique section s such that $s(b)$ is terminal in $\mathcal{E}_b$ for all b.*

We first show that there is indeed such a unique section by making some observations on fibrewise initial objects:

Observation 5.5.2. Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cartesian fibration, and $\emptyset_x$ is an initial object in the fibre $\mathcal{E}_x$ over some $x \in \mathcal{B}$. Then for any object $e \in \mathcal{E}$ over $y \in \mathcal{B}$, the induced map

$$\mathcal{E}(\emptyset_x, e) \to \mathcal{B}(x, y)$$

is an equivalence, since its fibre at $f\colon x \to y$ is equivalent to

$$\mathcal{E}_x(\emptyset_x, f^*e) \simeq *,$$

where $f^*e \to e$ is a cartesian lift of f at e. In particular, if x is an initial object of $\mathcal{B}$, then an initial object in $\mathcal{E}_x$ is also initial in $\mathcal{E}$.

Lemma 5.5.3. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cartesian fibration such that for every $b \in \mathcal{B}$, the fibre $\mathcal{E}_b$ has an initial object, and let $\mathcal{E}_{\text{init}} \subseteq \mathcal{E}$ be the full subcategory spanned by the fibrewise initial objects. Then the restriction of p to a functor*

$$p'\colon \mathcal{E}_{\text{init}} \to \mathcal{B}$$

is an equivalence. In particular, there exists a unique section s of p such that $s(b)$ is initial in $\mathcal{E}_b$ for every $b \in \mathcal{B}$.

Proof. Since every fibre of p has an initial object, p' is essentially surjective. It therefore suffices to show that p' is fully faithful, but this follows from Observation 5.5.2. A section s of p whose values are fibrewise initial objects is then precisely an inverse of the equivalence p', and so is indeed unique. □

To complete the proof of Proposition 5.5.1 we first need an alternative description of the overcategories $\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})_{/s}$.

Notation 5.5.4. For $p\colon \mathcal{E} \to \mathcal{B}$, we define $\mathsf{Ar}_{\mathcal{B}}(\mathcal{E})$ to be the full subcategory of $\mathsf{Ar}(\mathcal{E})$ on the arrows that lie over equivalences in $\mathcal{B}$, i.e. the pullback

$$\begin{tikzcd}
\mathsf{Ar}_{\mathcal{B}}(\mathcal{E}) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{E}) \arrow[d, "\mathsf{Ar}(p)"] \\
\mathcal{B} \arrow[r, hook] & \mathsf{Ar}(\mathcal{B});
\end{tikzcd}$$

note that ev_i for $i = 0, 1$ then restricts to a functor $\mathsf{Ar}_{\mathcal{B}}(\mathcal{E}) \to \mathcal{E}$ over $\mathcal{B}$. For a section $s\colon \mathcal{E} \to \mathcal{B}$, we then define $\mathcal{E}_{//s}$ by the pullback

$$\begin{tikzcd} \mathcal{E}_{//s} \arrow[r] \arrow[d, "p_{/s}"] & \mathsf{Ar}_{\mathcal{B}}(\mathcal{E}) \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{B} \arrow[r, "s"'] & \mathcal{E}. \end{tikzcd}$$

Then ev_0 induces a functor $\mathcal{E}_{//s} \to \mathcal{E}$.

Exercise 5.5. Show that there is a natural equivalence

$$\mathsf{Ar}(\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})) \simeq \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathsf{Ar}_{\mathcal{B}}(\mathcal{E}))$$

and a pullback square

$$\begin{tikzcd} \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E}_{//s}) \arrow[r] \arrow[d] & \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathsf{Ar}_{\mathcal{B}}(\mathcal{E})) \arrow[d] \\ \{s\} \arrow[r] & \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E}), \end{tikzcd}$$

and conclude that there is a canonical equivalence

$$\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})_{/s} \simeq \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E}_{//s})$$

over $\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})$

Proposition 5.5.5. *If $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration, then $\mathrm{ev}_1\colon \mathsf{Ar}_{\mathcal{B}}(\mathcal{E}) \to \mathcal{E}$ is a cocartesian fibration, with the cocartesian morphisms being those that map to p-cocartesian morphisms under ev_0.*

Proof. We first prove a morphism α of $\mathsf{Ar}_{\mathcal{B}}(\mathcal{E})$, given by a commutative square

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "s"'] & x' \arrow[d, "t"] \\ y \arrow[r, "g"'] & y' \end{tikzcd}$$

such that f is p-cocartesian, is a cocartesian morphism. For this we consider, for an object $q\colon z \to w$ in $\mathsf{Ar}_{\mathcal{B}}(\mathcal{E})$, the commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small] \mathsf{Ar}_{\mathcal{B}}(\mathcal{E})(t, q) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Ar}_{\mathcal{B}}(\mathcal{E})(s, q) \arrow[dr] \arrow[dd] & \\ & \mathcal{E}(x', z) \arrow[rr] \arrow[dd] & & \mathcal{E}(x, z) \arrow[dd] \\ \mathcal{E}(y', w) \arrow[rr] \arrow[dr] & & \mathcal{E}(y, w) \arrow[dr] & \\ & \mathcal{E}(x', w) \arrow[rr] & & \mathcal{E}(x, w), \end{tikzcd}$$

from Exercise 3.9, where the left and right faces are pullbacks by Proposition 3.6.1. To show that α is a cocartesian morphisms, we need to prove that the back face of the cube is a pullback for any q. We claim that the front face of the cube is a pullback, so that this follows from the 3-for-2 property. To see this, observe that from p we get a commutative cube

$$\begin{tikzcd}
\mathcal{E}(x',z) \arrow[rr, "f^*"] \arrow[dr, "q_*"] \arrow[dd] & & \mathcal{E}(x,z) \arrow[dr, "q_*"] \arrow[dd] & \\
& \mathcal{E}(x',w) \arrow[rr, "f^*"] \arrow[dd] & & \mathcal{E}(x,w) \arrow[dd] \\
\mathcal{B}(px',pz) \arrow[rr, "p(f)^*"] \arrow[dr, "p(q)_*"'] & & \mathcal{B}(px,pz) \arrow[dr, "p(q)_*"] & \\
& \mathcal{B}(px',pw) \arrow[rr, "p(f)^*"] & & \mathcal{B}(px,pw).
\end{tikzcd}$$

Here the front and back faces are pullbacks since f is p-cocartesian, as is the bottom face since $p(q)$ is an equivalence. Hence the top face is indeed also a pullback by the 3-for-2 property.

It remains to check that for any object $x \xrightarrow{s} y$ of $\mathsf{Ar}_{\mathcal{B}}(\mathcal{E})$ and morphism $g\colon y \to y'$, there exists a cocartesian lift of g at s. For this we choose a p-cocartesian lift $f\colon x \to x'$ of $p(gs)$ at x; then gs factors uniquely as $x \xrightarrow{f} x' \xrightarrow{t} y$ where t lies over an equivalence in $\mathcal{B}$. Then the commutative square

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "s"'] & x' \arrow[d, "t"] \\
y \arrow[r, "g"'] & z
\end{tikzcd}$$

is a morphism in $\mathsf{Ar}_{\mathcal{B}}(\mathcal{E})$ where f is p-cocartesian, so that this gives the required cocartesian lift. $\square$

Combining this with Proposition 3.5.15, we get:

Corollary 5.5.6. *If $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration and s is a section of p, then $p_{/s}\colon \mathcal{E}_{//s} \to \mathcal{B}$ is a cocartesian fibration, with the cocartesian morphisms being those that map to p-cocartesian morphisms under* ev_0. $\square$

Proof of Proposition 5.5.1. We want to show that the (unique) fibewise terminal section s is a terminal object in $\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B},\mathcal{E})$, i.e. that the forgetful functor

$$\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B},\mathcal{E})_{/s} \to \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B},\mathcal{E})$$

is an equivalence. By Exercise 5.5, we can identify this with the functor

$$\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B},\mathcal{E}_{//s}) \to \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B},\mathcal{E})$$

given by composition with $\mathrm{ev}_{1/s}\colon \mathcal{E}_{//s} \to \mathcal{E}$. It therefore suffices to show that $\mathrm{ev}_{1/s}$ is an equivalence. But from Corollary 5.5.6 we know that this is a morphism of cocartesian fibrations over $\mathcal{B}$. By Proposition 3.5.14 it is therefore enough to show it gives an equivalence on all fibres over $\mathcal{B}$. But over $b \in \mathcal{B}$ the definition unpacks to give the forgetful functor $\mathcal{E}_{b/s(b)} \to \mathcal{E}_b$, which is an equivalence by our assumption that $s(b)$ is a terminal object in $\mathcal{E}_b$. □

We can extend these constructions to a parametrized version of more general slices:

Notation 5.5.7. For $p\colon \mathcal{E} \to \mathcal{B}$, let $\mathcal{E}^{\mathcal{K}}_{(\mathcal{B})}$ denote the pullback

$$\begin{tikzcd}
\mathcal{E}^{\mathcal{K}}_{(\mathcal{B})} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K}, \mathcal{E}) \arrow[d] \\
\mathcal{B} \arrow[r] & \mathsf{Fun}(\mathcal{K}, \mathcal{B}).
\end{tikzcd}$$

A section of $\mathcal{E}^{\mathcal{K}}_{(\mathcal{B})}$ over $\mathcal{B}$ then corresponds to a functor $\phi\colon \mathcal{B} \times \mathcal{K} \to \mathcal{E}$ over $\mathcal{B}$. We define $\mathcal{E}_{//\phi}$ by the pullback

$$\begin{tikzcd}
\mathcal{E}_{//\phi} \arrow[r] \arrow[d] & (\mathcal{E}^{\mathcal{K}}_{(\mathcal{B})})_{//\phi} \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{E}^{\mathcal{K}}_{(\mathcal{B})}
\end{tikzcd}$$

over $\mathcal{B}$, where the bottom horizontal functor corresponds to the projection $\mathcal{E} \times \mathcal{K} \to \mathcal{E}$.

Observation 5.5.8. If $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration, then for any functor $\phi\colon \mathcal{B}\times\mathcal{K} \to \mathcal{E}$ over $\mathcal{B}$, the ∞-category $\mathcal{E}_{//\phi}$ is a pullback of cocartesian fibrations over $\mathcal{B}$ along morphisms of cocartesian fibrations, and so is itself a cocartesian fibration over $\mathcal{B}$.

Exercise 5.6. Show that there are natural equivalences

$$\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B} \times \mathcal{K}, \mathcal{E}) \simeq \mathsf{Fun}(\mathcal{K}, \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})) \simeq \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E}^{\mathcal{K}}_{(\mathcal{B})}).$$

Use this to show that for $\phi\colon \mathcal{B} \times \mathcal{K} \to \mathcal{E}$ over $\mathcal{B}$, we have

$$\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})_{/\phi} \simeq \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E}_{//\phi}).$$

Corollary 5.5.9. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a cocartesian fibration and consider a functor $\phi\colon \mathcal{K} \to \mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})$, which corresponds to a commutative triangle*

$$\begin{tikzcd}
\mathcal{K} \times \mathcal{B} \arrow[rr] \arrow[dr, "\mathrm{pr}_{\mathcal{B}}"'] & & \mathcal{E} \arrow[dl, "p"] \\
& \mathcal{B}. &
\end{tikzcd}$$

Suppose that for every $b \in \mathcal{B}$, the induced functor on fibres $\phi_b \colon \mathcal{K} \to \mathcal{E}_b$ has a limit in $\mathcal{E}_b$. Then the functor ϕ has a limit in $\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})$, given by the unique cone on ϕ that selects the limit cone in each fibre.

Proof. We want to show that the ∞-category $\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E})_{/\phi}$ has a terminal object. Using Exercise 5.6, we can identify this as the ∞-category $\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E}_{//\phi})$ of sections of the cocartesian fibration $p_{/\phi} \colon \mathcal{E}_{//\phi} \to \mathcal{B}$. Unpacking the definition, the fibre of this over $b \in \mathcal{B}$ is $\mathcal{E}_{b/\phi_b}$, which by assumption has a terminal object. By Proposition 5.5.1, the fibration $p_{/\phi}$ therefore has a unique section given by these terminal objects fibrewise, and this is the terminal object of $\mathsf{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{E}_{//\phi})$. □

Corollary 5.5.10. *Suppose $\phi \colon \mathcal{K} \to \mathsf{Fun}(\mathcal{B}, \mathcal{C})$ is a functor such that $\phi(-, b)$ has a limit in $\mathcal{C}$ for all $b \in \mathcal{B}$. Then ϕ has a limit in $\mathsf{Fun}(\mathcal{B}, \mathcal{C})$, given by the unique cone on ϕ that selects the limit cone of $\phi(-, b)$ for each $b \in \mathcal{B}$.*

Proof. Apply Corollary 5.5.9 to the cocartesian fibration $\mathrm{pr}_{\mathcal{B}} \colon \mathcal{C} \times \mathcal{B} \to \mathcal{B}$. □

Corollary 5.5.11. *Suppose $\phi \colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor such that $\phi(a, -) \colon \mathcal{B} \to \mathcal{C}$ has a limit for all $a \in \mathcal{A}$. Let $\psi \colon \mathcal{B} \to \mathcal{C}$ be the limit of ϕ viewed as a functor $\mathcal{B} \to \mathsf{Fun}(\mathcal{A}, \mathcal{C})$, which exists by Corollary 5.5.10. Then ϕ has a limit in $\mathcal{C}$ if and only if ψ does so, and these limits are equivalent if either exists. In other words, we have an equivalence*

$$\lim_{\mathcal{A} \times \mathcal{B}} \phi \simeq \lim_{a \in \mathcal{A}} \left(\lim_{\mathcal{B}} \phi(a, -) \right).$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{/\phi} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{B}, \mathcal{C})_{/\phi} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C})_{/\phi} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathsf{Fun}(\mathcal{B}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C})
\end{tikzcd}$$

where the right and composite squares are pullbacks (and we abuse notation by writing ϕ also when we view this as a functor $\mathcal{A} \to \mathsf{Fun}(\mathcal{B}, \mathcal{C})$). Then the left square is also a pullback by the 3-for-2 property. But here ψ is the limit of ϕ in $\mathsf{Fun}(\mathcal{B}, \mathcal{C})$, so we have an equivalence $\mathsf{Fun}(\mathcal{B}, \mathcal{C})_{/\phi} \simeq \mathsf{Fun}(\mathcal{B}, \mathcal{C})_{/\psi}$ over $\mathsf{Fun}(\mathcal{B}, \mathcal{C})$. This pulls back to an equivalence $\mathcal{C}_{/\phi} \simeq \mathcal{C}_{/\psi}$ of right fibrations over $\mathcal{C}$. The limit of ϕ is a terminal object of $\mathcal{C}_{/\phi}$; this exists if and only if the equivalent ∞-category $\mathcal{C}_{/\psi}$ has a terminal object, which gives the required equivalence of limits. □

Remark 5.5.12. The formula for iterated limits in Corollary 5.5.11 can be generalized to describe limits of the form $\lim_{a \in \mathcal{A}} \lim_{\Phi(a)} \phi_a$, where Φ is a functor $\mathcal{A}^{\mathrm{op}} \to \mathsf{Cat}_\infty$, as limits indexed over the cartesian fibration for Φ. It is possible to generalize the proof strategy we have used to also cover this case, but it requires more involved constructions (cf. [Lur09, §4.2.2]), using in particular the right adjoint to pullback along a cartesian fibration, which we have not introduced. We will instead prove this in a different way in §7.4

Observation 5.5.13. Given a functor $\phi\colon \mathcal{A}\times\mathcal{B} \to \mathcal{C}$ such that $\phi(a,-)$ and $\phi(-,b)$ have limits in $\mathcal{C}$, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we get an equivalence

$$\lim_{a\in\mathcal{A}} \lim_{\mathcal{B}} \phi(a,-) \simeq \lim_{\mathcal{A}\times\mathcal{B}} \phi \simeq \lim_{b\in\mathcal{B}} \lim_{\mathcal{A}} \phi(-,b)$$

if any of these three limits exist (since then they all exist and are equivalent).

As another application of Corollary 5.5.9, we can describe limits in the limit of a diagram of ∞-categories:

Corollary 5.5.14. *Suppose $\mathcal{C}$ is the limit of a functor $F\colon \mathcal{K} \to \mathsf{Cat}_\infty$ and consider a functor $\phi\colon \mathcal{L} \to \mathcal{C}$. If the composite $\phi_k\colon \mathcal{L} \to \mathcal{C} \to F(k)$ has a limit for every k, and these limits are preserved by the functor $F(f)$ for every morphism f in $\mathcal{K}$, then ϕ has a limit in $\mathcal{C}$, and the limit cone is the unique cone that maps to the limit cone of ϕ_k for each k.*

Proof. Let $p\colon \mathcal{E} \to \mathcal{K}$ be the cocartesian fibration for F. Then the limit $\mathcal{C}$ is the ∞-category $\mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{K}}(\mathcal{K},\mathcal{E})$ by Proposition 5.3.4, which is a full subcategory of $\mathsf{Fun}_{/\mathcal{K}}(\mathcal{K},\mathcal{E})$. By Corollary 5.5.9, ϕ has a limit in $\mathsf{Fun}_{/\mathcal{K}}(\mathcal{K},\mathcal{E})$, given by the unique cone that picks out the limit cone in each fibre $\mathcal{E}_k \simeq F(k)$. By assumption, this limit is furthermore a cocartesian section of p, so that it is also the limit in the full subcategory $\mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{K}}(\mathcal{K},\mathcal{E})$ by Corollary 5.4.6. □

Corollary 5.5.15. *Suppose $F\colon \mathcal{K} \to \mathsf{Cat}_\infty$ is a functor such that the ∞-category $F(k)$ has $\mathcal{L}$-shaped limits for all $k \in \mathcal{K}$, and the functor $F(f)$ preserves these for all morphisms f in $\mathcal{K}$. Then the ∞-category $\lim_{\mathcal{K}} F$ also has $\mathcal{L}$-shaped limits, and the functors $\lim_{\mathcal{K}} F \to F(k)$ in the limit cone preserve these.* □

5.6 (Co)limits in slices

In this section we will describe limits in over- and undercategories. We start with overcategories, for which we use the following:

Proposition 5.6.1. *Consider functors $p\colon \mathcal{K} \to \mathcal{C}$ and $q\colon \mathcal{L} \to \mathcal{C}_{/p}$, and let q' be the diagram $\mathcal{L} \star \mathcal{K} \to \mathcal{C}$ corresponding to q. Then there is an equivalence of ∞-categories*

$$(\mathcal{C}_{/p})_{/q} \simeq \mathcal{C}_{/q'}$$

over $\mathcal{C}$.

Proof. For any ∞-category $\mathcal{A}$, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Map}(\mathcal{A}, (\mathcal{C}_{/p})_{/q}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{A} \star \mathcal{L}, \mathcal{C}_{/p}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{A} \star \mathcal{L} \star \mathcal{K}, \mathcal{C}) \arrow[d] \\
\{q\} \arrow[r] & \mathsf{Map}(\mathcal{L}, \mathcal{C}_{/p}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{L} \star \mathcal{K}, \mathcal{C}) \arrow[d] \\
 & \{p\} \arrow[r] & \mathsf{Map}(\mathcal{K}, \mathcal{C})
\end{tikzcd}$$

where all the squares are pullbacks. But the composite square in the top row is also the pullback square for $\mathsf{Map}(\mathcal{A}, \mathcal{C}_{/q'})$, so (since the Yoneda embedding is fully faithful) we have the required equivalence. □

Corollary 5.6.2. *Let p, q, q' be as above. Then a cone $\bar{q}\colon \mathcal{L}^{\triangleleft} \to \mathcal{C}_{/p}$ for q is a limit cone if and only if the associated cone $\bar{q}'\colon \mathcal{L}^{\triangleleft} \star \mathcal{K} \to \mathcal{C}$ is a limit cone for q'. In particular, q has a limit in $\mathcal{C}_{/p}$ if and only if q' has a limit in $\mathcal{C}$.* □

As an important special case, we note:

Corollary 5.6.3. *For $c \in \mathcal{C}$, consider $q\colon \mathcal{L} \to \mathcal{C}_{/c}$ corresponding to $q'\colon \mathcal{L}^{\triangleright} \to \mathcal{C}$. A cone $\bar{q}$ for q is a limit cone if and only if the associated cone $\bar{q}'$ is a limit cone for q'. In particular, q has a limit in $\mathcal{C}_{/c}$ if and only if q' has a limit in $\mathcal{C}$.* □

Corollary 5.6.4. *The ∞-category $\mathcal{C}_{/c}$ has all limits of shape $\mathcal{L}$ if and only if $\mathcal{C}$ has limits for all diagrams of shape $\mathcal{L}^{\triangleright}$ that take the cone point to c.* □

Example 5.6.5. For $c \in \mathcal{C}$, the product of $x \to c$ and $y \to c$ in $\mathcal{C}_{/c}$ (a limit over $\{0,1\}$) is the pullback $x \times_c y$ in $\mathcal{C}$ (a limit over $\{0,1\}^{\triangleright}$).

Now we turn to undercategories, for which we need (the dual of) the following observation about initial objects:

Proposition 5.6.6. *The following are equivalent for an object x of an ∞-category $\mathcal{C}$:*

(1) x is initial.

(2) There exists a section of the forgetful functor $\mathcal{C}_{x/} \to \mathcal{C}$ that takes x to id_x.

(3) There exists a cone $\gamma\colon \mathcal{C}^{\triangleleft} \to \mathcal{C}$ on $\mathrm{id}_{\mathcal{C}}$ that takes $-\infty \to x$ to id_x.

Proof. The object x is initial if and only if the forgetful functor $p\colon \mathcal{C}_{x/} \to \mathcal{C}$ is an equivalence. A section s of p is an inverse to p if and only if the composite $\mathcal{C}_{x/} \xrightarrow{p} \mathcal{C} \xrightarrow{s} \mathcal{C}_{x/}$ is homotopic to the identity. But this functor is uniquely determined by where it sends id_x by Corollary 4.2.2, which means that s is inverse to p if and only if $s(x) \simeq \mathrm{id}_x$. Thus the first two conditions are equivalent. But a section of p corresponds to a cone $\mathcal{C}^{\triangleleft} \to \mathcal{C}$ that restricts to the identity on $\mathcal{C}$ and takes the cone point to x. Such a cone then corresponds to an inverse of p precisely when it takes the map $-\infty \to x$ to id_x. □

Corollary 5.6.7. *For any ∞-category $\mathcal{K}$, the cone point $-\infty$ in $\mathcal{K}^{\triangleleft}$ is an initial object.*

Proof. We must construct $\gamma\colon (\mathcal{K}^{\triangleleft})^{\triangleleft} \to \mathcal{K}^{\triangleleft}$ that restricts to the identity on $\mathcal{K}^{\triangleleft}$ and takes the map between the two cone points in the source to $\mathrm{id}_{-\infty}$. But here

$$(\mathcal{K}^{\triangleleft})^{\triangleleft} \simeq [1] \star \mathcal{K},$$

and we can take the map $s_0 \star \mathrm{id}_{\mathcal{K}}\colon [1] \star \mathcal{K} \to [0] \star \mathcal{K}$. □

Proposition 5.6.8. *Suppose $\mathcal{C}$ has a terminal object x. Then for any functor $p\colon \mathcal{K} \to \mathcal{C}$, the ∞-category $\mathcal{C}_{p/}$ has a terminal object, which is the unique object that lies over x.*

Proof. Since $\mathcal{C}_{/x} \to \mathcal{C}$ is an equivalence, any diagram $\mathcal{L} \to \mathcal{C}$ has a unique extension $\mathcal{L}^{\triangleright} \to \mathcal{C}$ taking the cone point to x. In particular, there is a unique cocone p' on p taking the cone point to x, and we need to prove that this is a terminal object of $\mathcal{C}_{p/}$. For this we consider the extension of the canonical functor $\mathcal{K} \star \mathcal{C}_{p/} \to \mathcal{C}$ over $\mathcal{K} \star \mathcal{C}_{p/} \star [0] \to \mathcal{C}$, which we can regard as a cocone

$$\gamma\colon (\mathcal{C}_{p/})^{\triangleright} \to \mathcal{C}_{p/},$$

which takes the cone point to p'. Applying (the dual of) Proposition 5.6.6 it suffices to show that γ takes the map $p' \to \infty$ to the identity of p'. But this map corresponds to a functor

$$(\mathcal{K}^{\triangleright})^{\triangleright} \to \mathcal{C}$$

that restricts to p' and takes the cone point to x. By uniqueness this must correspond to $\mathrm{id}_{p'}$. □

Observation 5.6.9. Consider $f\colon \mathcal{L} \star \mathcal{K} \to \mathcal{C}$ and let $q := f|_{\mathcal{L}}$, $p := f|_{\mathcal{K}}$; then we get view f as both a diagram

$$\tilde{p}\colon \mathcal{K} \to \mathcal{C}_{q/}$$

and as

$$\tilde{q}\colon \mathcal{L} \to \mathcal{C}_{/p}.$$

In this case there is an equivalence

$$(\mathcal{C}_{q/})_{/\tilde{p}} \simeq (\mathcal{C}_{/p})_{\tilde{q}/}$$

over $\mathcal{C}$, since maps from an ∞-category $\mathcal{A}$ to either correspond to maps

$$\mathcal{L} \star \mathcal{A} \star \mathcal{K} \to \mathcal{C}$$

that restrict to f on $\mathcal{L} \star \mathcal{K}$.

Corollary 5.6.10. *For $q\colon \mathcal{L} \to \mathcal{C}$ and $p\colon \mathcal{K} \to \mathcal{C}_{q/}$, let $p'\colon \mathcal{K} \to \mathcal{C}$ be the underlying diagram in $\mathcal{C}$. If p' has a limit in $\mathcal{C}$, then p has a limit in $\mathcal{C}_{q/}$, and the limit cone for p is the unique lift of the limit cone for p'.*

Proof. By Observation 5.6.9 we have an equivalence

$$(\mathcal{C}_{q/})_{/p} \simeq (\mathcal{C}_{/p'})_{q'/},$$

where q' is p viewed as a functor $\mathcal{L} \to \mathcal{C}_{/p'}$. The result then follows by applying Proposition 5.6.8 to $\mathcal{C}_{/p'}$. ◻

Corollary 5.6.11. *Suppose the ∞-category $\mathcal{C}$ has limits for all diagrams of shape $\mathcal{K}$. Then so does $\mathcal{C}_{q/}$ for any diagram $q\colon \mathcal{L} \to \mathcal{C}$, and these are preserved by the forgetful functor to $\mathcal{C}$.* ◻

5.7 (⋆) More on localizations

In this section we collect some further results on localizations.

Lemma 5.7.1. *For any ∞-category $\mathcal{K}$, we have*

$$\|\mathcal{K}^{\mathrm{op}}\| \simeq \|\mathcal{K}\|$$

Proof. For an ∞-groupoid X, we get natural equivalences

$$\begin{aligned} \mathsf{Map}(\|\mathcal{K}^{\mathrm{op}}\|, X) &\simeq \mathsf{Map}(\mathcal{K}^{\mathrm{op}}, X) \\ &\simeq \mathsf{Map}(\mathcal{K}, X^{\mathrm{op}}) \\ &\simeq \mathsf{Map}(\mathcal{K}, X) \\ &\simeq \mathsf{Map}(\|\mathcal{K}\|, X), \end{aligned}$$

since $X^{\mathrm{op}} \simeq X$. ◻

Proposition 5.7.2. *If $\mathcal{C}$ is an ∞-category and X is an ∞-groupoid, then $\mathsf{Fun}(\mathcal{C}, X)$ is an ∞-groupoid. Hence in the commutative square*

$$\begin{tikzcd} \mathsf{Map}(\|\mathcal{C}\|, X) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathsf{Map}(\mathcal{C}, X) \arrow[d, "\sim"] \\ \mathsf{Fun}(\|\mathcal{C}\|, X) \arrow[r, "\sim"'] & \mathsf{Fun}(\mathcal{C}, X), \end{tikzcd}$$

all the morphisms are equivalences.

Proof. We have a commutative triangle

$$\begin{tikzcd} & \mathsf{Map}(\mathcal{C}, X) \arrow[dl] \arrow[dr, "\sim"] & \\ \mathsf{Map}([1], \mathsf{Fun}(\mathcal{C}, X)) \arrow[rr, "\sim"] & & \mathsf{Map}(\mathcal{C}, \mathsf{Ar}(X)), \end{tikzcd}$$

where the horizontal and right diagonal maps are equivalences. Hence the left diagonal is also an equivalence, as required. ◻

Corollary 5.7.3. *For ∞-categories $\mathcal{C}, \mathcal{D}$, the canonical map $\|\mathcal{C} \times \mathcal{D}\| \to \|\mathcal{C}\| \times \|\mathcal{D}\|$ is an equivalence.*

Proof. For an ∞-groupoid X we have natural equivalences

$$\begin{aligned} \mathsf{Map}(\|\mathcal{C} \times \mathcal{D}\|, X) &\simeq \mathsf{Map}(\mathcal{C} \times \mathcal{D}, X) \\ &\simeq \mathsf{Map}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, X)) \\ &\simeq \mathsf{Map}(\mathcal{C}, \mathsf{Map}(\|\mathcal{D}\|, X)) \\ &\simeq \mathsf{Map}(\|\mathcal{C}\|, \mathsf{Map}(\|\mathcal{D}\|, X)) \\ &\simeq \mathsf{Map}(\|\mathcal{C}\| \times \|\mathcal{D}\|, X). \end{aligned}$$

Tracing this through, we see that this given by composition with the the canonical map $\|\mathcal{C} \times \mathcal{D}\| \to \|\mathcal{C}\| \times \|\mathcal{D}\|$, so that this is an equivalence. □

Proposition 5.7.4. *Consider a functor $L\colon \mathcal{C} \to \mathcal{C}'$ and a collection S of morphisms in $\mathcal{C}$ that are taken to equivalences by L. Then the following are equivalent:*

(1) L exhibits $\mathcal{C}'$ as the localization $\mathcal{C}[S^{-1}]$.

(2) For any ∞-category $\mathcal{D}$, composition with L gives a fully faithful functor

$$L^*\colon \mathsf{Fun}(\mathcal{C}', \mathcal{D}) \hookrightarrow \mathsf{Fun}(\mathcal{C}, \mathcal{D}),$$

whose image is precisely the functors that take the morphisms in S to equivalences.

(3) Composition with L gives a fully faithful functor

$$L^*\colon \mathsf{Fun}(\mathcal{C}', \mathsf{Gpd}_\infty) \hookrightarrow \mathsf{Fun}(\mathcal{C}, \mathsf{Gpd}_\infty),$$

whose image is precisely the functors that take the morphisms in S to equivalences.

Lemma 5.7.5. *Let $\ell\colon \mathcal{C} \to \|\mathcal{C}\|$ be the localization of an ∞-category $\mathcal{C}$ to an ∞-groupoid. Then the functor*

$$\ell^*\colon \mathsf{Fun}(\|\mathcal{C}\|, \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

given by composition with ℓ is fully faithful for any ∞-category $\mathcal{D}$, with image those functors that take all morphisms in $\mathcal{C}$ to equivalences in $\mathcal{D}$.

Proof. Unpacking the definitions, to see that ℓ^* is fully faithful we must show that the square

$$\begin{tikzcd} \mathsf{Map}(\mathcal{C}, \mathsf{Ar}(\mathcal{D})^{\simeq}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{C}, \mathsf{Ar}(\mathcal{D})) \arrow[d] \\ \mathsf{Map}(\mathcal{C}, \mathcal{D}^{\simeq} \times \mathcal{D}^{\simeq}) \arrow[r] & \mathsf{Map}(\mathcal{C}, \mathcal{D} \times \mathcal{D}) \end{tikzcd}$$

is a pullback. This holds because $\mathsf{Ar}(\mathcal{D}) \to \mathcal{D} \times \mathcal{D}$ is conservative by Proposition 2.5.5. □

Proof of Proposition 5.7.4. Since L takes the morphisms in S to equivalences, we have for any ∞-category $\mathcal{D}$ a commutative square

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{C}', \mathcal{D}) \arrow[r, "L^*"] \arrow[d] & \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \mathsf{Fun}(\|\mathcal{C}_S\|, \mathcal{D}) \arrow[r, hook] & \mathsf{Fun}(\mathcal{C}_S, \mathcal{D}). \end{tikzcd}$$

Here we know the bottom horizontal map is fully faithful by Lemma 5.7.5. If (1) holds then this square is a pullback, so that L^* is also fully faithful (by the dual of Lemma 2.4.7), and its image is as required by the pullback square on cores. Conversely, if (2) holds, then Exercise 2.16 implies that the square is a pullback if and only if it is one on sets after applying $\pi_0(-)^{\simeq}$, and this is true by the assumed description of the image of L^*; then we have in particular a pullback on cores, which gives (1).

(3) is a special case of (2), so it remains to prove that it implies the general case. Using the Yoneda embedding $\mathcal{D} \hookrightarrow \mathsf{PSh}(\mathcal{D})$, we get a fully faithful functor

$$\mathsf{Fun}(\mathcal{A}, \mathcal{D}) \hookrightarrow \mathsf{Fun}(\mathcal{A}, \mathsf{PSh}(\mathcal{D})) \simeq \mathsf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathsf{Fun}(\mathcal{A}, \mathsf{Gpd}_\infty)),$$

so that L^* fits in a commutative diagram

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{C}', \mathcal{D}) \arrow[r, "L^*"] \arrow[d] & \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \mathsf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathsf{Fun}(\mathcal{C}', \mathsf{Gpd}_\infty)) \arrow[r, "\mathsf{Fun}(\mathcal{D}^{\mathrm{op}}, L^*)"'] & \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Gpd}_\infty)), \end{tikzcd}$$

where the bottom horizontal functor is fully faithful by Corollary 2.6.4. It follows that L^* is fully faithful, and its image is those functor $F\colon \mathcal{C} \to \mathcal{D}$ such that for every $d \in \mathcal{D}$ the functor $\mathcal{D}(F(-), d)$ takes the morphisms in S to equivalences; but by Lemma 2.9.6 this is equivalent to F taking these morphisms to equivalences in $\mathcal{D}$. ◻

Lemma 5.7.6. *Localizations are left orthogonal to conservative functors.*

Proof. From Lemma 2.4.7 we know left orthogonal maps are closed under cobase change, so it suffices to show that a conservative functor $F\colon \mathcal{C} \to \mathcal{D}$ is right orthogonal to $\mathcal{K} \to \|\mathcal{K}\|$ for any ∞-category $\mathcal{K}$. This amounts to the square

$$\begin{tikzcd} \mathsf{Map}(\mathcal{K}, \mathcal{C}^{\simeq}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{K}, \mathcal{D}^{\simeq}) \arrow[d] \\ \mathsf{Map}(\mathcal{K}, \mathcal{C}) \arrow[r] & \mathsf{Map}(\mathcal{K}, \mathcal{D}) \end{tikzcd}$$

being a pullback, which follows from Exercise 2.13. ◻

Lemma 5.7.7. *For ∞-categories $\mathcal{K}, \mathcal{L}$ we have a pushout*

$$\begin{tikzcd} \mathcal{L} \times \mathcal{K}^{\simeq} \arrow[r] \arrow[d] & \|\mathcal{L}\| \times \mathcal{K}^{\simeq} \arrow[d] \\ \mathcal{L} \times \mathcal{K} \arrow[r] & \|\mathcal{L}\| \times \mathcal{K}. \end{tikzcd}$$

Proof. We must show that we get a pullback on maps to any ∞-category $\mathcal{C}$. The resulting square we can rewrite as

$$\begin{tikzcd} \mathsf{Map}(\|\mathcal{L}\|, \mathsf{Fun}(\mathcal{K}, \mathcal{C})) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{L}, \mathsf{Fun}(\mathcal{K}, \mathcal{C})) \arrow[d] \\ \mathsf{Map}(\|\mathcal{L}\|, \mathsf{Fun}(\mathcal{K}^{\simeq}, \mathcal{C})) \arrow[r] & \mathsf{Map}(\mathcal{L}, \mathsf{Fun}(\mathcal{K}^{\simeq}, \mathcal{C})). \end{tikzcd}$$

This is a pullback since composition with the essentially surjective functor $\mathcal{K}^{\simeq} \to \mathcal{K}$ gives a conservative functor $\mathsf{Fun}(\mathcal{K}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{K}^{\simeq}, \mathcal{C})$by Exercise 2.18, and localizations are left orthogonal to conservative functors by Lemma 5.7.6. □

Lemma 5.7.8. *Consider a localization $L\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$. For any ∞-category $\mathcal{K}$, the product $\mathcal{C} \times \mathcal{K} \to \mathcal{C}[S^{-1}] \times \mathcal{K}$ is the localization of $\mathcal{C} \times \mathcal{K}$ at the morphisms whose components in $\mathcal{C}$ lie in S.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{C}_S \times \mathcal{K}^{\simeq} \arrow[r] \arrow[d] & \|\mathcal{C}_S\| \times \mathcal{K}^{\simeq} \arrow[d] \\ \mathcal{C}_S \times \mathcal{K} \arrow[r] \arrow[d] & \|\mathcal{C}_S\| \times \mathcal{K} \arrow[d] \\ \mathcal{C} \times \mathcal{K} \arrow[r] & \mathcal{C}[S^{-1}] \times \mathcal{K}. \end{tikzcd}$$

Here the bottom square is a pushout since $(-) \times \mathcal{K}$ preserves these, and the top square is a pushout by Lemma 5.7.7. This exhibits $\mathcal{C}[S^{-1}] \times \mathcal{K}$ as a localization since $\|\mathcal{C}_S\| \times \mathcal{K}^{\simeq} \simeq \|\mathcal{C}_S \times \mathcal{K}^{\simeq}\|$ by Corollary 5.7.3. □

Proposition 5.7.9. *Suppose $L\colon \mathcal{D} \to \mathcal{C}$ is a functor such that there exist*

- *a fully faithful functor $i\colon \mathcal{C} \to \mathcal{D}$,*
- *a natural equivalence $Li \simeq \mathrm{id}_{\mathcal{C}}$,*
- *and a natural transformation $\eta\colon \mathrm{id}_{\mathcal{D}} \to iL$ such that $L\eta$ is an equivalence.*[1]

Then L is a localization.

[1] We may also take $\eta\colon iL \to \mathrm{id}_{\mathcal{D}}$ here.

Proof. Let $\Lambda\colon \mathcal{D} \to \mathcal{D}[S^{-1}]$ be the localization of $\mathcal{D}$ at the collection S of morphisms inverted by L; then L factors as

$$\mathcal{D} \xrightarrow{\Lambda} \mathcal{D}[S^{-1}] \xrightarrow{L'} \mathcal{C};$$

we want to show that L' is an equivalence with inverse $i' := \Lambda i$. By assumption we have $L'i' \simeq Li \simeq \mathrm{id}_{\mathcal{C}}$. For the other direction we consider the natural transformation $\Lambda\eta\colon \Lambda \to \Lambda i L \simeq i'L'\Lambda$. By Lemma 5.7.8 this factors as

$$\mathcal{D} \times [1] \xrightarrow{\Lambda \times \mathrm{id}} \mathcal{D}[S^{-1}] \times [1] \xrightarrow{\eta'} \mathcal{D}[S^{-1}],$$

where η' is a transformation $\mathrm{id}_{\mathcal{D}[S^{-1}]} \to i'L'$. But since all components of η also lie in S, the transformation η' is in fact a natural equivalence, as required. □

Chapter 6

Free fibrations, adjunctions and cofinality

6.1 Free fibrations

In this section we study free (co)cartesian fibrations, which turn out to be useful surprisingly often.

Notation 6.1.1. Given functors $F\colon \mathcal{A} \to \mathcal{C}$, $G\colon \mathcal{B} \to \mathcal{C}$, we write $\mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}$ for the pullback

$$\begin{tikzcd} \mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B} \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{C}) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \mathcal{A}\times\mathcal{B} \arrow[r, "F\times G"'] & \mathcal{C}\times\mathcal{C}. \end{tikzcd}$$

An object of $\mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}$ then consists of $a \in \mathcal{A}$, $b \in \mathcal{B}$ and a morphism $F(a) \to G(b)$ in $\mathcal{C}$. This construction has a number of names in the literature: it is often called the *oriented*, *directed* or *lax* pullback of F and G; for mysterious historical reasons it is also referred to as the "comma construction".

Definition 6.1.2. Given a functor $F\colon \mathcal{A} \to \mathcal{B}$ we write

$$\mathfrak{F}_{\mathrm{coct}}(F) := \mathcal{A}\overrightarrow{\times}_{\mathcal{B}}\mathcal{B} \simeq \mathcal{A} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \to \mathcal{B},$$

where the pullback is over F and ev_0 and the map to $\mathcal{B}$ is induced by ev_1. Dually, we write

$$\mathfrak{F}_{\mathrm{cart}}(F) := \mathcal{B}\overrightarrow{\times}_{\mathcal{B}}\mathcal{A} \to \mathcal{B},$$

with the functor to $\mathcal{B}$ induced by ev_0.

Observation 6.1.3. By Lemma 4.3.10, the functor $\mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B} \to \mathcal{A}\times\mathcal{B}$ is a bifibration for any F, G. In particular, $\mathfrak{F}_{\mathrm{coct}}(F)$ is a cocartesian fibration over $\mathcal{B}$, with the cocartesian morphisms being those that map to equivalences in $\mathcal{A}$. Dually,

$\mathfrak{F}_{\mathrm{cart}}(F)$ is a cartesian fibration, with the cartesian morphisms being those that map to equivalences in $\mathcal{A}$.

The functor $s_0^*\colon \mathcal{B} \to \mathsf{Ar}(\mathcal{B})$ given by composition with the degeneracy $s_0\colon [1] \to [0]$ pulls back along any functor F to a section $\eta_F\colon \mathcal{A} \to \mathfrak{F}_{\mathrm{(co)cart}}(F)$. We will show that this exhibits $\mathfrak{F}_{\mathrm{(co)cart}}(F)$ as the free (co)cartesian fibration on F, in the following sense:

Theorem 6.1.4. *For any functor $F\colon \mathcal{A} \to \mathcal{B}$ and any (co)cartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$, composition with η_F restricts to an equivalence*

$$\eta_F^*\colon \mathsf{Fun}_{/\mathcal{B}}^{\mathrm{(co)cart}}(\mathfrak{F}_{\mathrm{(co)cart}}(F), \mathcal{E}) \xrightarrow{\sim} \mathsf{Fun}_{/\mathcal{B}}(\mathcal{A}, \mathcal{E}).$$

We will prove this by giving an explicit inverse. To describe this we first make some observations about $\mathfrak{F}_{\mathrm{(co)cart}}(p)$ when p is already a (co)cartesian fibration:

Observation 6.1.5. When p is a cocartesian fibration, we can interpret Proposition 3.6.4 as providing an equivalence

$$\mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \xrightarrow{\sim} \mathfrak{F}_{\mathrm{coct}}(p)$$

over $\mathcal{B}$, via the pullback square

$$\begin{tikzcd} \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathsf{Ar}(p)"'] & \mathcal{E} \arrow[d, "p"] \\ \mathsf{Ar}(\mathcal{B}) \arrow[r, "\mathrm{ev}_0"'] & \mathcal{B}. \end{tikzcd}$$

Under this equivalence a cocartesian morphism in $\mathfrak{F}_{\mathrm{coct}}(p)$ corresponds to a commutative square

$$\begin{tikzcd} \bullet \arrow[r, "\sim"] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

where the vertical morphisms are p-cocartesian and the top horizontal morphism is an equivalence. By the 3-for-2 property from Lemma 3.5.3 it follows that the bottom horizontal morphism is also cocartesian, so that

$$\mathrm{ev}_1\colon \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \to \mathcal{E}$$

defines a morphism of cocartesian fibrations over $\mathcal{B}$; we think of this as a morphism

$$\epsilon_p\colon \mathfrak{F}_{\mathrm{coct}}(p) \to p$$

in $\mathsf{Cocart}(\mathcal{B})$. We also note that under the equivalence, the map η_p corresponds to the inclusion $\mathcal{E} \hookrightarrow \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E})$ of the equivalences, so that the composite

$$\mathcal{E} \xrightarrow{\eta_p} \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}) \xrightarrow{\epsilon_p} \mathcal{E}$$

is homotopic to the identity. Moreover, the morphisms ϵ_p are natural in morphisms of cocartesian fibrations — for any morphism of cocartesian fibrations

$$\begin{tikzcd} \mathcal{E}_1 \arrow[rr, "\phi"] \arrow[dr, "p_1'"'] & & \mathcal{E}_2 \arrow[dl, "p_2"] \\ & \mathcal{B}, & \end{tikzcd}$$

there is a natural commutative square

$$\begin{tikzcd} \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}_1) \arrow[r, "\mathsf{Ar}_{\mathrm{coct}}(\phi)"] \arrow[d, "\mathrm{ev}_1"'] & \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E}_2) \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{E}_1 \arrow[r, "\phi"'] & \mathcal{E}_2, \end{tikzcd}$$

which we can interpret as a square

$$\begin{tikzcd} \mathfrak{F}_{\mathrm{coct}}(p_1) \arrow[r, "\mathfrak{F}_{\mathrm{coct}}(\phi)"] \arrow[d, "\epsilon_{p_1}"'] & \mathfrak{F}_{\mathrm{coct}}(p_2) \arrow[d, "\epsilon_{p_2}"] \\ p_1 \arrow[r, "\phi"'] & p_2 \end{tikzcd}$$

of cocartesian fibrations over $\mathcal{B}$.

Proof of Theorem 6.1.4. We will prove the cocartesian case, by showing that η_F^* has an inverse τ, which takes

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\phi"] \arrow[dr, "F"'] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{B} & \end{tikzcd}$$

to the composite

$$\begin{tikzcd} \mathcal{A}\overrightarrow{\times}_{\mathcal{B}}\mathcal{B} \arrow[r, "\phi\overrightarrow{\times}_{\mathcal{B}}\mathcal{B}"] \arrow[dr, "\mathfrak{F}_{\mathrm{coct}}(F)"'] & \mathcal{E}\overrightarrow{\times}_{\mathcal{B}}\mathcal{B} \arrow[r, "\epsilon_p"] \arrow[d, "\mathfrak{F}_{\mathrm{coct}}(p)"] & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{B}. & \end{tikzcd}$$

We will show that the composites of this with η_F^* in both directions give the identity. On the one hand, for a morphism $\phi\colon F \to p$ we have a natural diagram over $\mathcal{B}$

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, "\eta_F"'] & p \arrow[d, "\eta_p"] \arrow[dr, "="] & \\ \mathfrak{F}_{\mathrm{coct}}(F) \arrow[r, "\mathfrak{F}_{\mathrm{coct}}(\phi)"'] & \mathfrak{F}_{\mathrm{coct}}(p) \arrow[r, "\epsilon_p"'] & p, \end{tikzcd}$$

which shows that $\eta_F^*\tau(\phi) \simeq \phi$, naturally in ϕ. For the other direction, we apply the naturality of ϵ in morphisms of cocartesian fibrations to a morphism $\mathfrak{F}_{\mathrm{coct}}(F) \to p$. Note that here $\mathfrak{F}^2_{\mathrm{coct}}(F) \simeq \mathsf{Ar}_{\mathrm{coct}}(\mathfrak{F}_{\mathrm{coct}}(F))$ can be identified with $\mathcal{A} \times_{\mathcal{B}} \mathsf{Fun}([2], \mathcal{B})$, with $\epsilon_{\mathfrak{F}_{\mathrm{coct}}(F)}\colon \mathfrak{F}^2_{\mathrm{coct}}(F) \to \mathfrak{F}_{\mathrm{coct}}(F)$ given by restriction along $d_1\colon \{0 < 2\} \hookrightarrow [2]$ and $\mathfrak{F}_{\mathrm{coct}}(\eta_F)\colon \mathfrak{F}_{\mathrm{coct}}(F) \to \mathfrak{F}^2_{\mathrm{coct}}(F)$ given by composition with $s_1\colon [2] \to [1]$. Since $d_1 \circ s_1 = \mathrm{id}_{[1]}$, we see that for $\phi\colon \mathfrak{F}_{\mathrm{coct}}(F) \to p$ there is a natural commutative diagram

$$\begin{tikzcd} \mathfrak{F}_{\mathrm{coct}}(F) \arrow[r, "\mathfrak{F}_{\mathrm{coct}}(\eta_F)"] \arrow[dr, "="'] & \mathfrak{F}^2_{\mathrm{coct}}(F) \arrow[r, "\mathfrak{F}_{\mathrm{coct}}(\phi)"] \arrow[d, "\epsilon_{\mathfrak{F}_{\mathrm{coct}}(F)}"] & \mathfrak{F}_{\mathrm{coct}}(p) \arrow[d, "\epsilon_p"] \\ & \mathfrak{F}_{\mathrm{coct}}(F) \arrow[r, "\phi"'] & p, \end{tikzcd}$$

which shows that $\tau\eta_F^* \simeq \mathrm{id}$. □

6.2 Representable bifibrations

As a consequence of Theorem 6.1.4, the free (co)cartesian fibrations also have universal properties as bifibrations:

Proposition 6.2.1. *Let $p\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ be a bifibration.*

(i) Given a functor $F\colon \mathcal{A} \to \mathcal{B}$, restriction along

$$\eta_F\colon \mathcal{A} \to \mathcal{A} \overrightarrow{\times}_{\mathcal{B}} \mathcal{B} \simeq (F, \mathrm{id})^* \mathsf{Ar}(\mathcal{B})$$

induces an equivalence

$$\mathsf{Map}_{/\mathcal{A}\times\mathcal{B}}((F, \mathrm{id})^*\mathsf{Ar}(\mathcal{B}), \mathcal{E}) \simeq \mathsf{Map}_{/\mathcal{A}\times\mathcal{B}}(\mathcal{A}, \mathcal{E}).$$

(ii) Given a functor $G\colon \mathcal{B} \to \mathcal{A}$, restriction along

$$\eta_G\colon \mathcal{A} \to \mathcal{A} \overrightarrow{\times}_{\mathcal{A}} \mathcal{B} \simeq (\mathrm{id}, G)^* \mathsf{Ar}(\mathcal{A})$$

induces an equivalence

$$\mathsf{Map}_{/\mathcal{A}\times\mathcal{B}}((\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{A}), \mathcal{E}) \simeq \mathsf{Map}_{/\mathcal{A}\times\mathcal{B}}(\mathcal{B}, \mathcal{E}).$$

Proof. We prove the first case. Here we know that $\mathcal{A}\overrightarrow{\times}_{\mathcal{B}}\mathcal{B} \to \mathcal{B}$ is the free cocartesian fibration on F, so that

$$\begin{tikzcd}
\mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{B}}(\mathcal{A}\overrightarrow{\times}_{\mathcal{B}}\mathcal{B}, \mathcal{E}) \arrow[r, "\sim"] \arrow[d] & \mathsf{Fun}_{/\mathcal{B}}(\mathcal{A}, \mathcal{E}) \arrow[d] \\
\mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{B}}(\mathcal{A}\overrightarrow{\times}_{\mathcal{B}}\mathcal{B}, \mathcal{A}\times\mathcal{B}) \arrow[r, "\sim"'] & \mathsf{Fun}_{/\mathcal{B}}(\mathcal{A}, \mathcal{A}\times\mathcal{B})
\end{tikzcd}$$

with the horizontal maps being equivalences. We then get an equivalence on fibres

$$\mathsf{Map}_{/\mathcal{A}\times\mathcal{B}}(\mathcal{A}\overrightarrow{\times}_{\mathcal{B}}\mathcal{B}, \mathcal{E}) \simeq \mathsf{Map}_{/\mathcal{A}\times\mathcal{B}}(\mathcal{A}, \mathcal{E}),$$

since any functor $\mathcal{A}\overrightarrow{\times}_{\mathcal{B}}\mathcal{B} \to \mathcal{E}$ over $\mathcal{A}\times\mathcal{B}$ automatically preserves cartesian morphisms over $\mathcal{A}$ by Observation 4.3.4. □

As a special case, we can identify the *free* bifibration on an ∞-category $\mathcal{C}$:

Corollary 6.2.2. *For any ∞-category $\mathcal{C}$, $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C}\times\mathcal{C}$ is the free bifibration on $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}\times\mathcal{C}$, in the sense that composition with s_0^* gives an equivalence*

$$\mathsf{Fun}_{/\mathcal{C}\times\mathcal{C}}(\mathsf{Ar}(\mathcal{C}), \mathcal{E}) \xrightarrow{\sim} \mathsf{Fun}_{/\mathcal{C}\times\mathcal{C}}(\mathcal{C}, \mathcal{E})$$

for any bifibration $\mathcal{E} \to \mathcal{C}\times\mathcal{C}$. □

Definition 6.2.3. We say a bifibration $p\colon \mathcal{E} \to \mathcal{A}\times\mathcal{B}$ is

- *corepresentable* if for every $a \in \mathcal{A}$, the left fibration $\mathcal{E}_a \to \mathcal{B}$ is corepresentable, i.e. $\mathcal{E}_a$ has an initial object,
- *representable* if for every $b \in \mathcal{B}$, the right fibration $\mathcal{E}_b \to \mathcal{A}$ is representable, i.e. $\mathcal{E}_b$ has a terminal object,
- *birepresentable* if it is both representable and corepresentable.

Proposition 6.2.4. *For a bifibration $p\colon \mathcal{E} \to \mathcal{A}\times\mathcal{B}$, we have:*

- *p is corepresentable if and only if there exists a functor $F\colon \mathcal{A} \to \mathcal{B}$ and an equivalence $\mathcal{E} \simeq (F, \mathrm{id})^*\mathsf{Ar}(\mathcal{B})$ over $\mathcal{A}\times\mathcal{B}$.*
- *p is representable if and only if there exists a functor $G\colon \mathcal{B} \to \mathcal{A}$ and an equivalence $\mathcal{E} \simeq (\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{A})$ over $\mathcal{A}\times\mathcal{B}$.*

Proof. We prove the first case. To start with, we note that $(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{B})$ is always corepresentable, since its fibre at $a \in \mathcal{A}$ is $\mathcal{B}_{F(a)/} \to \mathcal{B}$. Now we suppose that p is a corepresentable bifibration. Let $\mathcal{E}_0$ denote the full subcategory of $\mathcal{E}$ on the fibrewise initial objects over $\mathcal{A}$; then $p_{\mathcal{A}}$ is a cartesian fibration with fibrewise initial objects, so that the restriction of $p_{\mathcal{A}}$ to $q\colon \mathcal{E}_0 \to \mathcal{A}$ is an equivalence by Lemma 5.5.3. If we set $F := p_{\mathcal{B}} q^{-1}$ we then have a commutative triangle

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "q^{-1}"] \arrow[dr, "(\mathrm{id},F)"'] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{A} \times \mathcal{B}. & \end{tikzcd}$$

By Proposition 6.2.1 this extends uniquely to a commutative triangle

$$\begin{tikzcd} (F,\mathrm{id})^*\mathsf{Ar}(\mathcal{B}) \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{E} \arrow[dl, "p"] \\ & \mathcal{A} \times \mathcal{B}, & \end{tikzcd}$$

and we claim that ϕ is an equivalence. It suffices to check this on fibres over $a \in \mathcal{A}$, where we get a functor $\mathcal{B}_{F(a)/} \to \mathcal{E}_a$ over $\mathcal{B}$; this is an equivalence since by construction it takes $\mathrm{id}_{F(a)}$ to the initial object of $\mathcal{E}_a$. □

In the next section we will see that birepresentable bifibrations correspond to *adjunctions*. We can also identify the bifibrations that correspond to equivalences:

Proposition 6.2.5. *The following are equivalent for a bifibration $p\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$:*

(1) $\mathcal{E}$ is both corepresentable and representable, and an object of $\mathcal{E}$ is fibrewise terminal over $\mathcal{A}$ if and only if it is fibrewise initial over $\mathcal{B}$.

(2) There exists an ∞-category $\mathcal{C}$ and an equivalence of bifibrations

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}) \arrow[r, "\sim"] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\alpha \times \beta"'] & \mathcal{A} \times \mathcal{B} \end{tikzcd}$$

where α and β are equivalences.

Proof. Given (1), we take $\mathcal{C}$ to be the full subcategory of $\mathcal{E}$ on the fibrewise initial/terminal objects. Then $\alpha := p_{\mathcal{A}}|_{\mathcal{C}}$ and $\beta := p_{\mathcal{B}}|_{\mathcal{C}}$ are both equivalences by Lemma 5.5.3, so we have a commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\alpha \times \beta"'] & \mathcal{A} \times \mathcal{B}. \end{tikzcd}$$

Now Corollary 6.2.2 implies that we can extend the top horizontal morphism over $\mathsf{Ar}(\mathcal{C})$ to

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\alpha\times\beta"'] & \mathcal{A} \times \mathcal{B}, \end{tikzcd}$$

and we want to prove that the functor $\mathsf{Ar}(\mathcal{C}) \to \mathcal{E}$ is an equivalence. For this it suffices to check on fibres over each $x \in \mathcal{C}$ in the second variable, where we get a morphism of right fibrations

$$\begin{tikzcd} \mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathcal{E}_{\beta(x)} \arrow[d] \\ \mathcal{C} \arrow[r, "\alpha"'] & \mathcal{A}. \end{tikzcd}$$

But this is an equivalence since x is terminal in $\mathcal{E}_{\beta(x)}$. This proves (2). For the converse, it suffices to observe that the fibrewise terminal and initial objects in $\mathsf{Ar}(\mathcal{C})$ are both given by the objects that are equivalences in $\mathcal{C}$. $\square$

Observation 6.2.6. Let $p\colon \mathcal{E} \to \mathcal{C} \times \mathcal{C}$ be the bifibration obtained from unstraightening $\mathcal{C}(-,-)$ in the "wrong" order (first to a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{LFib}(\mathcal{C})$ and then to a cartesian fibration $\mathcal{E} \to \mathcal{C}$ over $\mathcal{C} \times \mathcal{C}$). Then from naturality of straightening we can conclude that Proposition 6.2.5 applies to p. It follows that p straightens to $\mathcal{C}(-,-)$, but potentially composed with an autoequivalence of $\mathcal{C}$. Since I can't prove that this is homotopic to $\mathrm{id}_{\mathcal{C}}$, we have to assume the following:

Fact 6.2.7. *Straightening the bifibration $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ in both orders produces the same functor*

$$\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Gpd}_\infty.$$

For our discussion of adjunctions below the following special case of Proposition 6.2.1 (which I learned from [RV22, §3.5]) will be particularly useful:

Corollary 6.2.8.

(1) For functors $G\colon \mathcal{C} \to \mathcal{B}$, $F\colon \mathcal{B} \to \mathcal{A}$, $P\colon \mathcal{C} \to \mathcal{A}$, the ∞-groupoid

$$\mathsf{Map}_{/\mathcal{B}\times\mathcal{C}}((\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{B}), (F,P)^*\mathsf{Ar}(\mathcal{A}))$$

is equivalent to the ∞-groupoid $\mathsf{Nat}_{\mathcal{C},\mathcal{A}}(FG, P)$ of natural transformations. The equivalence takes a natural transformation $\alpha\colon FG \to P$ to a map over $\mathcal{B} \times \mathcal{C}$ given at (b,c) by the composite

$$\mathcal{B}(b, Gc) \xrightarrow{(F)} \mathcal{A}(Fb, FGc) \xrightarrow{\alpha_{c,*}} \mathcal{A}(Fb, Pc).$$

(2) For functors $F\colon \mathcal{B} \to \mathcal{C}$, $Q\colon \mathcal{B} \to \mathcal{A}$, $G\colon \mathcal{C} \to \mathcal{A}$, the ∞-groupoid

$$\mathsf{Map}_{/\mathcal{B}\times\mathcal{C}}((F,\mathrm{id})^*\mathsf{Ar}(\mathcal{C}), (Q,G)^*\mathsf{Ar}(\mathcal{A}))$$

is equivalent to the ∞-groupoid $\mathsf{Nat}_{\mathcal{C},\mathcal{A}}(Q, GF)$ of natural transformations. The equivalence takes a natural transformation $\beta\colon Q \to GF$ to a map over $\mathcal{B}\times\mathcal{C}$ given at (b,c) by the composite

$$\mathcal{C}(Fb, c) \xrightarrow{(G)} \mathcal{A}(GFb, Gc) \xrightarrow{\beta_b^*} \mathcal{A}(Qb, Gc).$$

Proof. We prove the first part. By Proposition 6.2.1 there is an equivalence

$$\mathsf{Map}_{/\mathcal{B}\times\mathcal{C}}((\mathrm{id},G)^*\mathsf{Ar}(\mathcal{B}), (F,P)^*\mathsf{Ar}(\mathcal{A})) \simeq \mathsf{Map}_{/\mathcal{B}\times\mathcal{C}}(\mathcal{C}, (F,P)^*\mathsf{Ar}(\mathcal{A}))$$

where we can further identify the right-hand side as

$$\mathsf{Map}_{/\mathcal{A}\times\mathcal{A}}(\mathcal{C}, \mathsf{Ar}(\mathcal{A})) \simeq \left\{ \begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr, "{(FG,P)}"'] & & \mathsf{Ar}(\mathcal{A}) \arrow[dl] \\ & \mathcal{A}\times\mathcal{A}. & \end{tikzcd} \right\}$$

Using that $\mathsf{Ar}(\mathsf{Fun}(\mathcal{C},\mathcal{A})) \simeq \mathsf{Fun}(\mathcal{C}, \mathsf{Ar}(\mathcal{A}))$, we can identify this as the mapping ∞-groupoid from FG to P in $\mathsf{Fun}(\mathcal{C},\mathcal{A})$, i.e. the ∞-groupoid of natural transformations $FG \to P$. It remains to show that the inverse of this equivalence is as described. From the proof of Theorem 6.1.4 and the description of cocartesian transport for $\mathsf{Ar}(\mathcal{A})$ in Example 3.6.8, we can identify the functor corresponding to $\alpha\colon \mathcal{C} \to \mathsf{Ar}(\mathcal{A})$ over FG, P as the composite

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{C} \arrow[r, "\mathsf{Ar}(F)\times\alpha"] \arrow[d] & \mathsf{Ar}(\mathcal{A}) \times_{\mathcal{A}} \mathsf{Ar}(\mathcal{A}) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{A}) \arrow[d] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{A} \arrow[r, "="] & \mathcal{A} \end{tikzcd}$$

where the second map is given by composition in $\mathcal{A}$. The horizontal composite thus takes $b \xrightarrow{\phi} Gc$ to $Fb \xrightarrow{F\phi} FGc \xrightarrow{\alpha_c} Pc$ as required. $\square$

Exercise 6.1. Prove that the equivalence in Corollary 6.2.8 is compatible with composition, in the following way: Given functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$, $P\colon \mathcal{D} \to \mathcal{B}$, $Q\colon \mathcal{C} \to \mathcal{B}$ and morphisms of bifibrations

$$(F,\mathrm{id})^*\mathsf{Ar}(\mathcal{D}) \to (\mathrm{id},G)^*\mathsf{Ar}(\mathcal{C}) \to (P,Q)^*\mathsf{Ar}(\mathcal{B})$$

corresponding to natural transformations $\alpha\colon \mathrm{id} \to GF$, $\beta\colon QG \to P$, show that the composite corresponds to the natural transformation

$$Q \xrightarrow{Q\alpha} QGF \xrightarrow{\beta F} PF.$$

Proposition 6.2.9. *For $p\colon \mathcal{E} \to [1]$, let $i_s\colon \mathcal{E}_s \to \mathcal{E}$ be the inclusion of the fibre at $s = 0, 1$; we then have a bifibration*[1]

$$(i_0, i_1)^*\mathsf{Ar}(\mathcal{E}) \xrightarrow{q} \mathcal{E}_0 \times \mathcal{E}_1.$$

(1) p is a cocartesian fibration if and only if q is corepresentable, in which case q is equivalent to $(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{E}_1)$ where $F\colon \mathcal{E}_0 \to \mathcal{E}_1$ is the straightening of p.

(2) p is a cartesian fibration if and only if q is representable, in which case q is equivalent to $(\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{E}_0)$ where $G\colon \mathcal{E}_1 \to \mathcal{E}_0$ is the straightening of p.

Proof. We prove the cocartesian case. Here q is corepresentable if and only if for all $x \in \mathcal{E}_0$, the left fibration

$$\mathcal{E}_{1,x/} := \mathcal{E}_{x/} \times_{\mathcal{E}} \mathcal{E}_1 \to \mathcal{E}_1$$

is corepresentable. This left fibration corresponds to the copresheaf $\mathcal{E}(x, -)$ on $\mathcal{E}_1$, while a cocartesian morphism $f\colon x \to y$ in $\mathcal{E}$ over $0 \to 1$ is one such that composing with it gives an equivalence

$$\mathcal{E}_1(y, e) \to \mathcal{E}(x, e)$$

for $e \in \mathcal{E}_1$, i.e. f exhibits $\mathcal{E}(x, -)$ as represented by y. Thus the full subcategory of $(i_0, i_1)^*\mathsf{Ar}(\mathcal{E})$ of fibrewise initial objects is precisely the full subcategory of $\mathsf{Ar}(\mathcal{E})$ on the cocartesian morphisms over $0 \to 1$, so that q is corepresented by the composite

$$\mathcal{E}_0 \xleftarrow{\sim} \mathsf{Ar}_{\mathrm{coct}}(\mathcal{E})_{0\to 1} \to \mathcal{E}_1,$$

which is also the straightening of p. □

6.3 Adjunctions

Definition 6.3.1. We say functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are *adjoint* (with F *left adjoint* to G and G *right adjoint* to F) if there exist natural transformations

$$\eta\colon \mathrm{id}_{\mathcal{C}} \to GF, \quad \epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$$

(the *unit* and *counit* of the adjunction) such that the composite transformations

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

are equivalent to identities (the "triangle identities").

Notation 6.3.2. We will sometimes write $F \dashv G$ to mean "F is left adjoint to G".

[1]This construction is in fact part of an equivalence between bifibrations and ∞-categories over $[1]$, with the inverse taking a bifibration $q\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ to $\mathcal{E} \times [1] \amalg_{\mathcal{E}\times\{0,1\}} (\mathcal{A} \amalg \mathcal{B})$.

Warning 6.3.3. We can interpret this definition as saying that an adjunction of ∞-categories is an adjunction in the homotopy 2-category of ∞-categories. If one wants to define the ∞-groupoid of coherent adjunction data correctly one must be a bit more careful. One option is to specify only one of the two transformations, say the unit, and require that this satisfies a *property* as follows:

Observation 6.3.4. For a functor $F\colon \mathcal{C} \to \mathcal{D}$ we have a commutative square

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C}) \arrow[r, "\mathsf{Ar}(F)"] \arrow[d] & \mathsf{Ar}(\mathcal{D}) \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "F\times F"'] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

which straightens to a natural transformation

$$\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Gpd}_\infty$.

Definition 6.3.5. Given functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$, a natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ is a *unit transformation* if the composite

$$\mathcal{D}(Fx, y) \to \mathcal{C}(GFx, Gy) \to \mathcal{C}(x, Gy) \tag{6.1}$$

is an equivalence for all $x \in \mathcal{C}, y \in \mathcal{D}$, where the first map comes from G as in Observation 6.3.4 and the second is given by composition with η.

Exercise 6.2. Consider a natural transformation $\alpha\colon F \to G$ of functors $F, G\colon \mathcal{A} \to \mathcal{B}$. By viewing α as a functor $\mathcal{A} \times [1] \to \mathcal{B}$, we get a morphism of bifibrations

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{A}) \times \mathsf{Ar}([1]) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{B}) \arrow[d] \\ \mathcal{A} \times [1] \times \mathcal{A} \times [1] \arrow[r] & \mathcal{B} \times \mathcal{B}. \end{tikzcd}$$

Show that from this we can extract a natural commutative square

$$\begin{tikzcd} \mathcal{C}(x, y) \arrow[r, "(F)"] \arrow[d, "(G)"'] & \mathcal{D}(Fx, Fy) \arrow[d, "\alpha_{y,*}"] \\ \mathcal{D}(Gx, Gy) \arrow[r, "\alpha_x^*"'] & \mathcal{D}(Fx, Gy). \end{tikzcd}$$

Proposition 6.3.6. *The following are equivalent for functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$:*

(1) F is left adjoint to G.

(2) There exists a unit transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$.

(3) *There exists a natural equivalence* $\mathcal{D}(F-,-) \simeq \mathcal{C}(-,G-)$ *of functors* $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Gpd}_\infty$.

Proof. Suppose F is left adjoint to G, so there exists a unit η and counit ϵ. We claim that the composite

$$\mathcal{C}(x, Gy) \to \mathcal{D}(Fx, FGy) \to \mathcal{D}(Fx, y)$$

where the first map is given by applying F and the second by composition with ϵ, is inverse to (6.1). Indeed, by naturality (applying Exercise 6.2 to ϵ) we have a commutative diagram

$$\begin{tikzcd}
\mathcal{D}(Fx,y) \arrow[r, "(G)"] \arrow[rrr, bend left, "\epsilon^*_{Fx}"] & \mathcal{C}(GFx,Gy) \arrow[r, "(F)"] \arrow[d, "\eta^*_x"'] & \mathcal{D}(FGFx,FGy) \arrow[r, "\epsilon_{y,*}"] \arrow[d, "F\eta^*_x"] & \mathcal{D}(FGFx,y) \arrow[d, "F\eta^*_x"] \\
 & \mathcal{C}(x,Gy) \arrow[r, "(F)"'] & \mathcal{D}(Fx,FGy) \arrow[r, "\epsilon_{y,*}"] & \mathcal{D}(Fx,y)
\end{tikzcd}$$

where the composite of our two maps is the composition along the bottom, and the composition along the top gives the identity by one of the triangle identities. Similarly, the composition in the other order is also the identity, using the other triangle identity.

Now suppose η is a unit transformation. Then η corresponds under Corollary 6.2.8 to a morphism

$$(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D}) \to (\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{C})$$

of bifibrations over $\mathcal{C} \times \mathcal{D}$. Moreover, this is given over (c,d) by the composite

$$\mathcal{D}(Fc, d) \to \mathcal{C}(GFc, Gd) \to \mathcal{C}(c, Gd),$$

which is an equivalence since η is a unit transformation. Thus we have an equivalence of bifibrations, which straightens to an equivalence as in (3).

Given the equivalence (3), we can conversely unstraighten this to an equivalence of bifibrations

$$\phi \colon (F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D}) \xrightarrow{\sim} (\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{C})$$

over $\mathcal{C} \times \mathcal{D}$. By Corollary 6.2.8, ϕ corresponds to a natural transformation $\eta \colon \mathrm{id}_{\mathcal{C}} \to GF$, while ϕ^{-1} corresponds to $\epsilon \colon FG \to \mathrm{id}_{\mathcal{D}}$. The composite

$$(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D}) \xrightarrow{\phi} (\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{C}) \xrightarrow{\phi^{-1}} (F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D})$$

is the identity, but also corresponds to the composition

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

by Exercise 6.1, so this gives one of the triangle identities. The composite in the other order similarly gives the other identity. $\square$

Corollary 6.3.7. *The following are equivalent for a functor $F\colon \mathcal{C} \to \mathcal{D}$:*

(1) F is a left adjoint (i.e. F has a right adjoint).

(2) The corepresentable bifibration $(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D})$ is also representable.

(3) The ∞-category $\mathcal{C}_{/d} := \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d}$ has a terminal object for all $d \in \mathcal{D}$.

(4) The presheaf $\mathcal{D}(F(-), d)$ on $\mathcal{C}$ is representable for all $d \in \mathcal{D}$.

(5) The cocartesian fibration $\mathcal{F} \to [1]$ corresponding to F is also a cartesian fibration.

Dually, the following are equivalent for a functor $G\colon \mathcal{D} \to \mathcal{C}$:

(1') G is a right adjoint (i.e. G has a left adjoint).

(2') The representable bifibration $(\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{C})$ is also corepresentable.

(3') The ∞-category $\mathcal{D}_{c/} := \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{c/}$ has an initial object for all $c \in \mathcal{C}$.

(4') The copresheaf $\mathcal{C}(c, G(-))$ on $\mathcal{D}$ is corepresentable for all $c \in \mathcal{C}$.

(5') The cartesian fibration $\mathcal{G} \to [1]$ corresponding to G is also a cocartesian fibration.

Proof. By Proposition 6.3.6, a functor G is right adjoint to F if and only if there is a natural equivalence of bifibrations

$$(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D}) \simeq (\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{C});$$

The existence of such a G is equivalent to $(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D})$ being a representable bifibration by Proposition 6.2.4, and by definition this means that $(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D})_d \simeq \mathcal{C}_{/d}$ has a terminal object for all $d \in \mathcal{D}$. Since $\mathcal{C}_{/d} \to \mathcal{C}$ is the right fibration for $\mathcal{D}(F(-), d)$, such a terminal object exists if and only if this presheaf is representable. Finally, it follows from Proposition 6.2.9 that the bifibration $(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D})$ is representable if and only if the cocartesian fibration for F is also a cartesian fibration. □

Exercise 6.3. Show that if we have an adjunction $F \dashv G$ then on opposite ∞-categories we have $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$.

Lemma 6.3.8. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ has a right adjoint G. Then for any ∞-category $\mathcal{A}$,*

- *the functor $F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{A}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{A})$ has as left adjoint G^*,*
- *the functor $F_*\colon \mathsf{Fun}(\mathcal{A}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{D})$ has as right adjoint G_*.*

Proof. Suppose we have a unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and counit $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$. Then composition with these induce the required unit and counit transformations for $G^* \dashv F^*$, $F_* \dashv G_*$. □

Lemma 6.3.9. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ and $F'\colon \mathcal{D} \to \mathcal{E}$ have right adjoints G' and G, respectively. Then the composite $F'F$ is left adjoint to GG'.*

Proof. We can either see that unit and counit transformations for the adjunctions $F \dashv G$ and $F' \dashv G'$ can be combined to produce a unit and counit for the composites, or observe that we have equivalences

$$\begin{aligned}
(F'F, \mathrm{id})^*\mathsf{Ar}(\mathcal{E}) &\simeq (F, \mathrm{id})^*(F', \mathrm{id})^*\mathsf{Ar}(\mathcal{E}) \\
&\simeq (F, \mathrm{id})^*(\mathrm{id}, G')^*\mathsf{Ar}(\mathcal{D}) \\
&\simeq (F, G')^*\mathsf{Ar}(\mathcal{D}) \\
&\simeq (\mathrm{id}, G')^*(F, \mathrm{id})^*\mathsf{Ar}(\mathcal{D}) \\
&\simeq (\mathrm{id}, G')^*(\mathrm{id}, G)^*\mathsf{Ar}(\mathcal{C}) \\
&\simeq (\mathrm{id}, GG')^*\mathsf{Ar}(\mathcal{C}),
\end{aligned}$$

which implies $F'F \dashv GG'$. □

Exercise 6.4. Show that if $\mathcal{C}$ has all limits of shape $\mathcal{K}$, then the constant diagram functor $\mathcal{C} \to \mathsf{Fun}(\mathcal{K}, \mathcal{C})$ has a right adjoint, given by taking limits of such diagrams. Dually, if $\mathcal{C}$ has all $\mathcal{K}$-shaped colimits, the same functor has a left adjoint.

Proposition 6.3.10. *Suppose F is left adjoint to G with unit η and counit ϵ. Then:*

(1) F is fully faithful if and only if η is a natural equivalence.

(2) G is fully faithful if and only if ϵ is a natural equivalence.

Proof. We prove the first case. Here we have from Exercise 6.2 a natural commutative diagram

$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r, "(F)"] \arrow[ddr, "\eta_{y,*}"'] & \mathcal{D}(Fx,Fy) \arrow[d, "(G)"] \\
 & \mathcal{C}(GFx,GFy) \arrow[d, "\eta_x^*"] \\
 & \mathcal{C}(x,GFy)
\end{tikzcd}$$

for all $x, y \in \mathcal{C}$. Here the vertical composite is an equivalence, so that η_x^* is an equivalence if and only if the top horizontal map is an equivalence. This holds for all y if and only if η_x is an equivalence in $\mathcal{C}$, and this in turn holds for all x if and only if η is a natural equivalence. Thus η is a natural equivalence if and only if F is fully faithful, as required. □

Corollary 6.3.11. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$.*

(1) If G is fully faithful, then F is a localization.

(2) If F is fully faithful, then G is a localization.

Proof. We prove the first case; the second is proved similarly, or by observing that localizations are closed under taking op. By Proposition 6.3.10 the counit $\epsilon\colon FG \to \mathrm{id}$ is a natural equivalence. For the unit η we then have that $F\eta$ is also a natural equivalence, since the composite

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

is the identity and the second map is an equivalence. The conditions of Proposition 5.7.9 therefore hold, so that F is a localization. ◻

6.4 (⋆) Free left and right fibrations

In this section we will describe the free left and right fibrations on a functor, i.e. the left adjoints to the fully faithful inclusions

$$\mathsf{LFib}(\mathcal{B}),\ \mathsf{RFib}(\mathcal{B}) \hookrightarrow \mathsf{Cat}_{\infty/\mathcal{B}},$$

and look at the morphisms that are inverted by these.

Observation 6.4.1. The universal property of the localization $\|{-}\|\colon \mathsf{Cat}_\infty \to \mathsf{Gpd}_\infty$ says precisely that this is a left adjoint of the inclusion $\mathsf{Gpd}_\infty \hookrightarrow \mathsf{Cat}_\infty$. By Lemma 6.3.8, this induces for any ∞-category $\mathcal{K}$ an adjunction

$$\mathsf{Fun}(\mathcal{K}, \mathsf{Cat}_\infty) \rightleftarrows \mathsf{Fun}(\mathcal{K}, \mathsf{Gpd}_\infty).$$

In terms of fibrations, this means that the fully faithful inclusions

$$\mathsf{LFib}(\mathcal{K}) \hookrightarrow \mathsf{Cocart}(\mathcal{K}), \quad \mathsf{RFib}(\mathcal{K}) \hookrightarrow \mathsf{Cart}(\mathcal{K})$$

have left adjoints $(-)^\ell$ and $(-)^r$, respectively. For a cocartesian fibration $\mathcal{E} \to \mathcal{K}$, the left fibration $\mathcal{E}^\ell \to \mathcal{K}$ has fibre $\|\mathcal{E}_k\|$ at $k \in \mathcal{K}$.[2]

Lemma 6.4.2. *The fully faithful inclusions* $\mathsf{LFib}(\mathcal{K}), \mathsf{RFib}(\mathcal{K}) \hookrightarrow \mathsf{Cat}_{\infty/\mathcal{K}}$ *have left adjoints, given by* $L^\ell_{\mathcal{K}} := \mathfrak{F}_{\mathrm{coct}}(-)^\ell$ *and* $L^r_{\mathcal{K}} := \mathfrak{F}_{\mathrm{cart}}(-)^r$*, respectively.*

Proof. The inclusion $\mathsf{LFib}(\mathcal{K}) \hookrightarrow \mathsf{Cat}_{\infty/\mathcal{K}}$ factors as

$$\mathsf{LFib}(\mathcal{K}) \hookrightarrow \mathsf{Cocart}(\mathcal{K}) \to \mathsf{Cat}_{\infty/\mathcal{K}};$$

its left adjoint is therefore the composite of the left adjoints of these two functors by Lemma 6.3.9. ◻

Definition 6.4.3. A morphism f in $\mathsf{Cat}_{\infty/\mathcal{K}}$ is called a *contravariant equivalence* if $L^r_{\mathcal{K}}(f)$ is an equivalence in $\mathsf{RFib}(\mathcal{K})$. (Dually, we say f is a *covariant equivalence* if $L^\ell_{\mathcal{K}}(f)$ is an equivalence in $\mathsf{LFib}(\mathcal{K})$, where $L^\ell_{\mathcal{K}}$ is the localization to left fibrations.)

[2] One can show that $\mathcal{E}^\ell$ is in fact the localization of $\mathcal{E}$ at the morphisms that lie over equivalences in $\mathcal{K}$, but we will hopefully not need this…

Lemma 6.4.4. *A morphism*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[dl, "q"] \\ & \mathcal{K} & \end{tikzcd}$$

in $\mathsf{Cat}_{\infty/\mathcal{K}}$ *is a contravariant equivalence if and only if for every* $k \in \mathcal{K}$*, the induced morphism of* ∞*-groupoids*

$$\|\mathcal{C} \times_{\mathcal{K}} \mathcal{K}_{k/}\| \to \|\mathcal{D} \times_{\mathcal{K}} \mathcal{K}_{k/}\|$$

is an equivalence.

Proof. Since $L^r(f)$ is a morphism of right fibrations, we can check that it is an equivalence on fibres. At $k \in \mathcal{K}$ the formula for $\mathfrak{F}_{\mathrm{cart}}(-)^r$ tells us that we get precisely the give morphism of ∞-groupoids. □

Proposition 6.4.5. *Contravariant equivalences are closed under base change along left fibrations. In other words, if the morphism* $f\colon \mathcal{C} \to \mathcal{D}$ *over* $\mathcal{B}$ *is a contravariant equivalence in* $\mathsf{Cat}_{\infty/\mathcal{B}}$ *and* $p\colon \mathcal{E} \to \mathcal{B}$ *is a left fibration, then* $p^*f\colon \mathcal{C} \times_{\mathcal{B}} \mathcal{E} \to \mathcal{D} \times_{\mathcal{B}} \mathcal{E}$ *is a contravariant equivalence in* $\mathsf{Cat}_{\infty/\mathcal{E}}$*.*

Proof. We use the criterion of Lemma 6.4.4. For $e \in \mathcal{E}$ we note

$$(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}) \times_{\mathcal{E}} \mathcal{E}_{e/} \simeq \mathcal{C} \times_{\mathcal{B}} \mathcal{E}_{e/}$$

and that we have a commutative square

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{B}} \mathcal{E}_{e/} \arrow[r] \arrow[d, "\sim"'] & \mathcal{D} \times_{\mathcal{B}} \mathcal{E}_{e/} \arrow[d, "\sim"] \\ \mathcal{C} \times_{\mathcal{B}} \mathcal{B}_{pe/} \arrow[r] & \mathcal{D} \times_{\mathcal{B}} \mathcal{B}_{pe/}. \end{tikzcd}$$

Here the vertical maps are equivalences since $\mathcal{E}_{e/} \simeq \mathcal{B}_{pe/}$ by Lemma 3.3.10 as p is by assumption a left fibration. It follows that the top horizontal map gives an equivalence on localizations if the bottom horizontal map does so. □

Corollary 6.4.6. *Suppose we have a pullback square*

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "f"] \arrow[d, "p'"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B}' \arrow[r, "g"] & \mathcal{B} \end{tikzcd}$$

where p is a left fibration. Then the induced square

$$\begin{tikzcd} L^r_{\mathcal{E}}(f) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\ L^r_{\mathcal{B}}(g) \arrow[r] & \mathcal{B} \end{tikzcd}$$

is also a pullback.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "p^*\eta"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & p^*L^r_{\mathcal{B}}(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B}' \arrow[r, "\eta"] & L^r_{\mathcal{B}}(g) \arrow[r] & \mathcal{B}, \end{tikzcd}$$

where the right-hand square is a pullback by definition, and the left square is a pullback by 3-for-2. Here $\eta\colon \mathcal{B}' \to L^r_{\mathcal{B}}(g)$ is a contravariant equivalence in $\mathsf{Cat}_{\infty/\mathcal{B}}$, so that $p^*\eta$ is a contravariant equivalence in $\mathsf{Cat}_{\infty/\mathcal{E}}$ by Proposition 6.4.5. But the target of $p^*\eta$ is also a right fibration over $\mathcal{E}$, so this means that it exhibits $p^*L^r_{\mathcal{B}}(g)$ as $L^r_{\mathcal{E}}(f)$, as required. □

Proposition 6.4.7. *Suppose a functor $f\colon \mathcal{C} \to \mathcal{D}$ over $\mathcal{B}$ is a contravariant equivalence in $\mathsf{Cat}_{\infty/\mathcal{B}}$. Then $\|f\|\colon \|\mathcal{C}\| \to \|\mathcal{D}\|$ is an equivalence.*

Proof. The functor

$$\mathsf{Cat}_{\infty/\mathcal{B}} \xrightarrow{\text{forget}} \mathsf{Cat}_\infty \xrightarrow{\|-\|} \mathsf{Gpd}_\infty$$

is left adjoint to

$$\mathsf{Gpd}_\infty \hookrightarrow \mathsf{Cat}_\infty \xrightarrow{(-)\times\mathcal{B}} \mathsf{Cat}_{\infty/\mathcal{B}}.$$

Here the right adjoint factors through the full subcategory $\mathsf{RFib}(\mathcal{B})$, which means that the left adjoint factors through $L^r_{\mathcal{B}}$, as required. □

Lemma 6.4.8. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a right fibration. Then a morphism*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f"] \arrow[dr, "g"'] & & \mathcal{D} \arrow[dl, "h"] \\ & \mathcal{E} & \end{tikzcd}$$

is a contravariant equivalence in $\mathsf{Cat}_{\infty/\mathcal{E}}$ if and only if the morphism

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f"] \arrow[dr, "pg"'] & & \mathcal{D} \arrow[dl, "ph"] \\ & \mathcal{B} & \end{tikzcd}$$

is a contravariant equivalence in $\mathsf{Cat}_{\infty/\mathcal{B}}$.

Proof. By Observation 8.1.1 we have an adjunction

$$p_! : \mathsf{Cat}_{\infty/\mathcal{E}} \rightleftarrows \mathsf{Cat}_{\infty/\mathcal{B}} : p^*$$

where $p_!$ is given by composition with p and p^* by pullback along p. Since p is a right fibration, this restricts the full subcategories of right fibrations as an adjunction

$$p_! : \mathsf{RFib}(\mathcal{E}) \rightleftarrows \mathsf{RFib}(\mathcal{B}) : p^*$$

We then have a commutative square

$$\begin{tikzcd}
\mathsf{RFib}(\mathcal{B}) \arrow[r, hook] \arrow[d, "p^*"'] & \mathsf{Cat}_{\infty/\mathcal{B}} \arrow[d, "p^*"] \\
\mathsf{RFib}(\mathcal{E}) \arrow[r, hook] & \mathsf{Cat}_{\infty/\mathcal{E}}
\end{tikzcd}$$

of right adjoints. Since left adjoints compose and are unique, it follows that we also have a commutative square of left adjoints

$$\begin{tikzcd}
\mathsf{Cat}_{\infty/\mathcal{E}} \arrow[r, "L^r_{\mathcal{E}}"] \arrow[d, "p_!"'] & \mathsf{RFib}(\mathcal{E}) \arrow[d, "p_!"] \\
\mathsf{Cat}_{\infty/\mathcal{B}} \arrow[r, "L^r_{\mathcal{B}}"] & \mathsf{RFib}(\mathcal{B}).
\end{tikzcd}$$

Since equivalences in $\mathsf{RFib}(\mathcal{B})$ are detected in Cat_∞, it follows that for our morphism $f\colon g \to h$ in $\mathsf{Cat}_{\infty/\mathcal{E}}$, $L^r_{\mathcal{E}}(f)$ is an equivalence if and only if $p \circ L^r_{\mathcal{E}}(f) \simeq L^r_{\mathcal{B}}(p \circ f)$ is an equivalence. □

6.5 Cofinal functors

In this section we introduce cofinal and coinitial functors among ∞-categories, and show that these have a variety of useful characterizations.

Definition 6.5.1. An ∞-category $\mathcal{C}$ is *weakly contractible* if $\|\mathcal{C}\| \simeq *$.

Definition 6.5.2. A functor $p\colon \mathcal{E} \to \mathcal{B}$ is *cofinal* if for all $b \in \mathcal{B}$, the ∞-category $\mathcal{E}_{b/} := \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{b/}$ is weakly contractible. Dually, p is *coinitial* if for all $b \in \mathcal{B}$, the ∞-category $\mathcal{E}_{/b} := \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{/b}$ is weakly contractible.

Warning 6.5.3. The term *cofinal* comes from "cofinal subsequences" in analysis — it does *not* mean these functors are dual to some class of "final" functors. Unfortunately many authors ignore this, and in general there is a wide variety of different naming conventions for cofinal and coinitial functors in the literature (see Table 6.1 for an (incomplete) list thereof).

	$\lVert\mathcal{E}_{b/}\rVert \simeq *$	$\lVert\mathcal{E}_{/b}\rVert \simeq *$
This text, [Lan21]	cofinal	coinitial
[Lur09]	cofinal	(none)
[Lur17]	left cofinal	right cofinal
[Ker]	right cofinal	left cofinal
[RV22], [ML98]	final	initial
[Cis19]	final	cofinal

Table 6.1: Some terminology for cofinal/coinitial functors

For the rest of this section we will state and prove results for cofinal functors, but of course the dual results for coinitial functors also hold, with the same proofs.

Proposition 6.5.4. *A functor $p\colon \mathcal{E} \to \mathcal{B}$ is cofinal if and only if $L^r_{\mathcal{B}}(p)$ is an equivalence (where p is viewed as a morphism $p \to \mathrm{id}_{\mathcal{B}}$ in $\mathsf{Cat}_{\infty/\mathcal{B}}$).*

Proof. By Lemma 6.4.2 we can identify $L^r_{\mathcal{B}}(p)$ as the morphism of right fibrations

$$\begin{tikzcd} (\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}))^r \arrow[rr] \arrow[dr] & & \mathcal{B} \arrow[dl, "="] \\ & \mathcal{B}, & \end{tikzcd}$$

where on the left the map to $\mathcal{B}$ comes from ev_0. This is an equivalence if and only if it is so on all fibres over $b \in \mathcal{B}$, where we get

$$\lVert\mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{b/}\rVert \to \{b\},$$

which is by definition an equivalence for all b if and only if p is cofinal. □

Observation 6.5.5. Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a right fibration. Combining Proposition 6.5.4 with Lemma 6.4.8 we see that a functor $f\colon \mathcal{C} \to \mathcal{E}$ is cofinal if and only if f is a contravariant equivalence when viewed as a morphism $pf \to p$ in $\mathsf{Cat}_{\infty/\mathcal{B}}$. For any functor $g\colon \mathcal{C} \to \mathcal{B}$, we can consider the unit morphism

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\eta"] \arrow[dr, "g"'] & & \mathcal{E} \arrow[dl, "L^r_{\mathcal{B}}(g)"] \\ & \mathcal{B}. & \end{tikzcd}$$

Here the adjunction identities imply that η is a contravariant equivalence over $\mathcal{B}$; since $L^r_{\mathcal{B}}(g)$ is a right fibration, it follows that η is cofinal. Thus any functor has a factorization as a cofinal functor followed by a right fibration.

Corollary 6.5.6. *A functor $p\colon \mathcal{E} \to \mathcal{B}$ is cofinal if and only if it is left orthogonal to all right fibrations.*

Proof. The functor p is left orthogonal to a right fibration $q\colon \mathcal{P} \to \mathcal{C}$ when the commutative square

$$\begin{tikzcd} \mathsf{Map}(\mathcal{B},\mathcal{P}) \arrow[r] \arrow[d] & \mathsf{Map}(\mathcal{B},\mathcal{C}) \arrow[d] \\ \mathsf{Map}(\mathcal{E},\mathcal{P}) \arrow[r] & \mathsf{Map}(\mathcal{E},\mathcal{C}) \end{tikzcd}$$

is a pullback, which we can check on fibres over each map $f\colon \mathcal{B} \to \mathcal{C}$. The map on fibres at f we can identify as the morphism

$$\mathsf{Map}_{/\mathcal{B}}(\mathcal{B}, f^*\mathcal{P}) \to \mathsf{Map}_{/\mathcal{B}}(\mathcal{E}, f^*\mathcal{P})$$

given by composition with p. It follows that p is left orthogonal to all right fibrations if and only if for any right fibration $\mathcal{Q} \to \mathcal{B}$, composition with p gives an equivalence

$$\mathsf{Map}_{/\mathcal{B}}(\mathcal{B}, \mathcal{Q}) \to \mathsf{Map}_{/\mathcal{B}}(\mathcal{E}, \mathcal{Q}).$$

By adjunction, this is equivalent to $L^r_{\mathcal{B}}(p)$ being an equivalence, which we saw characterized p as cofinal in Proposition 6.5.4. □

Corollary 6.5.7. *If $p\colon \mathcal{E} \to \mathcal{B}$ is cofinal, then for any functor $\phi\colon \mathcal{B} \to \mathcal{C}$, the induced morphism of overcategories*

$$\mathcal{C}_{\phi/} \to \mathcal{C}_{\phi p/}$$

is an equivalence.

Proof. This is a morphism of left fibrations over $\mathcal{C}$, so it suffices to show it gives an equivalence on fibres over each $x \in \mathcal{C}$. The fibre of $\mathcal{C}_{\phi/}$ at x we can identify with the ∞-groupoid of maps $\mathcal{B}^{\triangleright} \to \mathcal{C}$ that restrict to ϕ on $\mathcal{B}$ and take the cone point to x; this is also the fibre at ϕ of

$$\mathsf{Fun}(\mathcal{B}, \mathcal{C}_{/x}) \to \mathsf{Fun}(\mathcal{B}, \mathcal{C}),$$

i.e. $\mathsf{Map}_{/\mathcal{B}}(\mathcal{B}, \phi^*\mathcal{C}_{/x})$, and we can identify the morphism on fibres at x as the map

$$\mathsf{Map}_{/\mathcal{B}}(\mathcal{B}, \phi^*\mathcal{C}_{/x}) \to \mathsf{Map}_{/\mathcal{B}}(\mathcal{E}, \phi^*\mathcal{C}_{/x})$$

given by composition with p. If p is cofinal, then this is an equivalence by Proposition 6.5.4, as required. □

Corollary 6.5.8. *If $p\colon \mathcal{E} \to \mathcal{B}$ is cofinal, then a functor $\phi\colon \mathcal{B} \to \mathcal{C}$ has a colimit in $\mathcal{C}$ if and only if ϕp has a colimit, in which case the canonical map*

$$\operatorname{colim}_{\mathcal{E}} \phi p \to \operatorname{colim}_{\mathcal{B}} \phi$$

is an equivalence.

Proof. By Corollary 6.5.7 composition with p induces an equivalence : $\mathcal{C}_{\phi/} \xrightarrow{\sim} \mathcal{C}_{\phi p/}$ over $\mathcal{C}$. Thus $\mathcal{C}_{\phi/}$ has an initial object if and only if $\mathcal{C}_{\phi p/}$ does so. The canonical map on colimits is the image in $\mathcal{C}$ of the unique map from the initial object of $\mathcal{C}_{\phi p/}$ to the image of that of $\mathcal{C}_{\phi/}$, so this is moreover an equivalence if these initial objects exist. □

We now consider some first examples of coinitial and cofinal functors:

Proposition 6.5.9. *Any localization is both cofinal and coinitial.*

Proof. Right and left fibrations are in particular conservative (Exercise 3.1), so a localization is left orthogonal to these by Lemma 5.7.6. □

Proposition 6.5.10. *For an object $x \in \mathcal{C}$, the functor $\{x\} \to \mathcal{C}$ is cofinal if and only if x is a terminal object, and coinitial if and only if x is an initial object.*

Proof. The functor $\{x\} \to \mathcal{C}$ is cofinal if for all $c \in \mathcal{C}$, the ∞-category

$$\{x\} \times_{\mathcal{C}} \mathcal{C}_{c/} \simeq \mathcal{C}(c, x)$$

is weakly contractible, i.e. contractible as this is an ∞-groupoid; this says precisely that x is a terminal object. □

Corollary 6.5.11. *If $\mathcal{K}$ has a terminal object x, then any functor $F\colon \mathcal{K} \to \mathcal{C}$ has a colimit, given by the object $F(x) \in \mathcal{C}$.* □

Observation 6.5.12. Suppose the ∞-category $\mathcal{K}$ has a terminal object x. Since $\{x\} \to \mathcal{K}$ is cofinal, we have an equivalence

$$* \simeq \operatorname{colim}_{\{x\}} * \xrightarrow{\sim} \operatorname{colim}_{\mathcal{K}} * \simeq \|\mathcal{K}\|.$$

Using Lemma 5.7.1, this implies that $\|\mathcal{K}\|$ is also contractible when $\mathcal{K}$ has an initial object.

Putting this together, we have almost proved the following:

Theorem 6.5.13. *The following are equivalent for a functor $p\colon \mathcal{E} \to \mathcal{B}$:*

(1) p is cofinal, i.e. $\mathcal{E}_{b/}$ is weakly contractible for all $b \in \mathcal{B}$.

(2) $L^r_{\mathcal{B}}(p)$ is an equivalence.

(3) p is left orthogonal to all right fibrations.

(4) For any functor $\phi\colon \mathcal{B} \to \mathcal{C}$, the induced functor

$$\mathcal{C}_{\phi/} \to \mathcal{C}_{\phi p/}$$

is an equivalence.

(5) *A functor $\phi\colon \mathcal{B} \to \mathcal{C}$ has a colimit if and only if ϕp has a colimit, in which case the induced map of colimits is an equivalence.*

(6) *For every functor $\phi\colon \mathcal{B} \to \mathsf{Gpd}_\infty$, the canonical map*

$$\operatorname{colim}_{\mathcal{E}} \phi p \to \operatorname{colim}_{\mathcal{B}} \phi$$

is an equivalence.

Proof. We have already seen that the first three items are equivalent and imply the next two. The sixth item is a special case of the fifth, so we only need to show that it implies that p is cofinal. This is true since we can identify $\|\mathcal{E}_{b/}\| \simeq \operatorname{colim}_{\mathcal{E}} \mathcal{B}(b, p(-))$, and $\operatorname{colim}_{\mathcal{B}} \mathcal{B}(b,-) \simeq \|\mathcal{B}_{b/}\| \simeq *$ by Observation 6.5.12, since $\mathcal{B}_{b/}$ has an initial object. □

Proposition 6.5.14. *Any left adjoint is coinitial, and any right adjoint is cofinal.*

Proof. We prove the first case. By Corollary 6.3.7, if $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint then $\mathcal{C}_{/d}$ has a terminal object for all $d \in \mathcal{D}$ and so is weakly contractible by Observation 6.5.12. □

Observation 6.5.15. We have characterized the cofinal functors as those that are left orthogonal to right fibrations. This implies that cofinal functors are closed under several constructions:

- If F is cofinal then a composite GF is cofinal if and only if G is cofinal (Lemma 2.4.5).
- Pushouts of cofinal functors are cofinal (Lemma 2.4.6)
- Cofinal functors are closed under cobase change (Lemma 2.4.7).
- Cofinal functors are closed under retracts (Lemma 2.4.9).
- Cofinal functors are closed under products. (It suffices to show that if F is cofinal then so is $F \times \mathcal{K}$ for a fixed ∞-category $\mathcal{K}$, which follows from right fibrations being closed under exponentiation, Corollary 3.2.5.)

Proposition 6.5.16 ("Quillen's Theorem A"). *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is cofinal. Then $\|F\|\colon \|\mathcal{C}\| \to \|\mathcal{D}\|$ is an equivalence of ∞-groupoids.*

Proof. Here $\|F\|$ is the map on colimits induced by F for the constant functor to Gpd_∞ with value $*$. □

Exercise 6.5. Assuming the functor $\mathcal{K} \to \mathcal{K}^{\triangleleft}$ is fully faithful (cf. Corollary 8.8.3), show that it is cofinal if and only if $\mathcal{K}$ is weakly contractible. Conclude that weakly contractible colimits in $\mathcal{C}_{x/}$ are computed in $\mathcal{C}$.

Exercise 6.6. Show by hand that $\Lambda^2_0 \to (\Lambda^2_0)^{\triangleleft}$ is cofinal, where $\Lambda^2_0 \simeq \{0 < 1\} \amalg_{\{0\}} \{0 < 2\} \simeq \{1,2\}^{\triangleleft}$. Conclude that pushouts in $\mathcal{C}_{x/}$ are given by pushouts in $\mathcal{C}$.

Proposition 6.5.4 says that a functor $p\colon \mathcal{E} \to \mathcal{B}$ is cofinal if and only if it is a contravariant equivalence when viewed as a morphism $p \to \mathrm{id}_{\mathcal{B}}$ in $\mathsf{Cat}_{\infty/\mathcal{B}}$. We thus have the following as a special case of Proposition 6.4.5:

Corollary 6.5.17. *Cofinal functors are closed under base change along left fibrations. Dually, coinitial functors are closed under base change along right fibrations.* $\square$

Remark 6.5.18. More generally, cofinal functors are closed under base change along cocartesian fibrations, but this is a more involved result to prove.

Chapter 7

Weighted colimits

7.1 Representable criterion for (co)limits

Above we defined the limit of a functor $F \colon \mathcal{I} \to \mathcal{C}$ as a terminal cone on F, i.e. as a terminal object in $\mathcal{C}_{/F}$. We now want to prove that we can also recognize limits on mapping ∞-groupoids, namely by a natural equivalence

$$\mathcal{C}(x, \lim_{\mathcal{I}} F) \simeq \lim_{\mathcal{I}} \mathcal{C}(x, F).$$

This amounts to showing that the right fibration $\mathcal{C}_{/F} \to \mathcal{C}$ is also the fibration for the functor on the right. The left fibration for the functor $\mathcal{C}(x, F(-))$ is $F^*\mathcal{C}_{x/} \to \mathcal{I}$, so using our description of limits in Gpd_∞ we can identify the right-hand side as

$$\mathsf{Map}_{/\mathcal{I}}(\mathcal{I}, F^*\mathcal{C}_{x/}) \simeq \mathsf{Map}_{/\mathcal{C}}(\mathcal{I}, \mathcal{C}_{x/}).$$

Our goal is therefore to identify the fibration for a functor of this form. We will build this up in several stages, where we relate constructions on functors to constructions on fibrations.

Observation 7.1.1. For an ∞-category $\mathcal{K}$ we have a functor $- \times \mathcal{K} \colon \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ (for example given by restricting the limit functor from Exercise 6.4). For any ∞-category $\mathcal{C}$, the presheaf $\mathsf{Map}(- \times \mathcal{K}, \mathcal{C})$ is represented by $\mathsf{Fun}(\mathcal{K}, \mathcal{C})$, so the functor $- \times \mathcal{K}$ has a right adjoint, given by $\mathsf{Fun}(\mathcal{K}, -)$. Using Lemma 6.3.8 we then get for any ∞-category $\mathcal{C}$ an adjunction

$$(- \times \mathcal{K})_* : \mathsf{Fun}(\mathcal{C}, \mathsf{Cat}_\infty) \rightleftarrows \mathsf{Fun}(\mathcal{C}, \mathsf{Cat}_\infty) : \mathsf{Fun}(\mathcal{K}, -).$$

Lemma 7.1.2. *For $F \colon \mathcal{C} \to \mathsf{Cat}_\infty$, the functor $F \times \mathcal{K} := (- \times \mathcal{K}) \circ F$ is the product of F and $\mathrm{const}_{\mathcal{K}}$ in $\mathsf{Fun}(\mathcal{C}, \mathsf{Cat}_\infty)$.*

Proof. We have natural maps

$$\mathrm{const}_* \times \mathcal{K} \leftarrow F \times \mathcal{K} \to F \times *.$$

Here

$$F \times * = (- \times *)_* \circ F \simeq F$$

as $- \times * \simeq \mathrm{id}_{\mathsf{Cat}_\infty}$, while

$$\mathrm{const}_* \times \mathcal{K} \simeq (- \times \mathcal{K}) \circ \mathrm{const}_* \simeq \mathrm{const}_{\mathcal{K}}.$$

This diagram exhibits $F \times \mathcal{K}$ as a product by Corollary 5.5.10. □

Proposition 7.1.3. *Suppose we have morphisms of cocartesian fibrations*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr, "p"'] & \mathcal{G} \arrow[d, "q"] & \mathcal{F}' \arrow[l] \arrow[dl, "p'"] \\ & \mathcal{B} & \end{tikzcd}$$

that straighten to natural transformations

$$F \to G \leftarrow F'$$

of functors $\mathcal{B} \to \mathsf{Cat}_\infty$. Then the pullback $F \times_G F'$ in functors corresponds to the fibration

$$\pi := p \times_q p' \colon \mathcal{F} \times_{\mathcal{G}} \mathcal{F}' \to \mathcal{B}.$$

Proof. Since straightening is an equivalence of ∞-categories, it suffices to prove that π is a pullback in the ∞-category $\mathsf{Cocart}(\mathcal{B})$ of cocartesian fibrations. This is a (non-full) subcategory of $\mathsf{Cat}_{\infty/\mathcal{B}}$, so we first use Corollary 5.6.3 and Exercise 6.6 to conclude that π is a pullback in $\mathsf{Cat}_{\infty/\mathcal{B}}$. Then we want to apply Proposition 5.4.1 to conclude that π is also a pullback in $\mathsf{Cocart}(\mathcal{B})$. For this we need to check that for a triangle

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\phi"] \arrow[dr, "r"'] & & \mathcal{F} \times_{\mathcal{G}} \mathcal{F}' \arrow[dl, "\pi"] \\ & \mathcal{B}, & \end{tikzcd}$$

the functor ϕ preserves cocartesian morphisms if and only if its composites with the projections to $\mathcal{F}, \mathcal{G}$ and $\mathcal{F}'$ all do so. But this follows from the description of π-cocartesian morphisms in Exercise 3.7 as those that map to p- and p'-cocartesian morphisms in $\mathcal{F}$ and $\mathcal{F}'$, respectively. □

Proposition 7.1.4. *If $p \colon \mathcal{F} \to \mathcal{B}$ is the cocartesian fibration for a functor F, then the cocartesian fibration for $\mathsf{Fun}(\mathcal{K}, F)$ is $\mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{F}) \to \mathcal{B}$.*

Proof. In $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)$, the object $\mathsf{Fun}(\mathcal{K}, F)$ represents the presheaf $\mathsf{Map}(- \times \mathcal{K}, F)$ by Observation 7.1.1, so we can identify the corresponding fibration by understanding this presheaf fibrationally. Here the functor $G \times \mathcal{K}$ is naturally equivalent to the product $G \times \mathrm{const}_{\mathcal{K}}$. We know that $\mathcal{B} \times \mathcal{K} \to \mathcal{B}$ is the fibration for $\mathrm{const}_{\mathcal{K}}$ (since pullback corresponds to composition), so by Proposition 7.1.3 we know that if $\mathcal{E} \to \mathcal{B}$ is the fibration for G, then that for $G \times \mathcal{K}$ is $\mathcal{E} \times \mathcal{K} \simeq \mathcal{E} \times_{\mathcal{B}} (\mathcal{B} \times \mathcal{K}) \to \mathcal{B}$. Now we observe that we have a natural equivalence

$$\mathsf{Fun}_{/\mathcal{B}}(\mathcal{E} \times \mathcal{K}, \mathcal{F}) \simeq \mathsf{Fun}(\mathcal{K}, \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F})) \simeq \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{F}))$$

by a variant of Exercise 5.6; we need to show that this restricts to the full subcategories of functors that preserve cocartesian morphisms. On the one hand, a cocartesian morphism in $\mathcal{E} \times \mathcal{K}$ is one that projects to a cocartesian morphism in $\mathcal{E}$ and an equivalence in $\mathcal{K}$, so that a functor $\mathcal{E} \times \mathcal{K} \to \mathcal{F}$ over $\mathcal{B}$ preserves cocartesian morphisms if and only if its restriction to $\mathcal{E} \times \{k\} \to \mathcal{F}$ does so for every $k \in \mathcal{K}$. On the other hand, a cocartesian morphism in $\mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{F})$ is also one whose component at each $k \in \mathcal{K}$ is cocartesian in $\mathcal{F}$ by Corollary 3.6.5 and Proposition 3.5.15; the equivalence above therefore restricts to

$$\mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{B}}(\mathcal{E} \times \mathcal{K}, \mathcal{F}) \simeq \mathsf{Fun}(\mathcal{K}, \mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F})) \simeq \mathsf{Fun}^{\mathrm{coct}}_{/\mathcal{B}}(\mathcal{E}, \mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{F})).$$

Thus $\mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{F})$ represents the presheaf $\mathsf{Map}^{\mathrm{coct}}_{/\mathcal{B}}(- \times \mathcal{K}, \mathcal{F})$ on $\mathsf{Cocart}(\mathcal{B})$. If $\mathcal{F}$ straightens to F, it follows that $\mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{F})$ is the cocartesian fibration for $\mathsf{Fun}(\mathcal{K}, F)$. □

Corollary 7.1.5. *Suppose $p \colon \mathcal{F} \to \mathcal{B}$ is the cocartesian fibration for a functor F. Given a natural transformation $\alpha \colon F \to \mathrm{const}_{\mathcal{C}}$ and a functor $\phi \colon \mathcal{K} \to \mathcal{C}$, the cocartesian fibration for the functor $\mathsf{Fun}_{/\mathcal{C}}(\mathcal{K}, F(-))$ is the pullback of*

$$\mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{E}) \to \mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{B} \times \mathcal{C}) \leftarrow \mathcal{B}$$

where the first map comes from composition with the functor $\mathcal{E} \to \mathcal{B} \times \mathcal{C}$ over $\mathcal{B}$ that corresponds to the natural transformation α, and the second is adjoint to $\mathrm{id}_{\mathcal{B}} \times \phi$.

Proof. As in the proof of Lemma 7.1.2 we know that since limits in $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)$ are computed pointwise (Corollary 5.5.10), the functor $\mathsf{Fun}_{/\mathcal{C}}(\mathcal{K}, F(-))$ is the pullback

$$\begin{tikzcd}
\mathsf{Fun}_{/\mathcal{C}}(\mathcal{K}, F(-)) \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K}, F(-)) \arrow[d] \\
\mathrm{const}_* \arrow[r] & \mathsf{Fun}(\mathcal{K}, \mathrm{const}_{\mathcal{C}})
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)$. This corresponds to a pullback in $\mathsf{Cocart}(\mathcal{B})$, which by Proposition 7.1.3 is computed by a pullback in Cat_∞. Now Proposition 7.1.4 shows that the fibrations for the functors in this square are as claimed. □

Observation 7.1.6. Suppose $\mathcal{Q} \to \mathcal{B}$ is the cocartesian fibration for the functor $\mathsf{Fun}_{/\mathcal{C}}(\mathcal{K}, F(-))$ as above. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(\mathcal{K}, \mathcal{E}) \arrow[d] \\
\mathcal{B} \arrow[r] & \mathsf{Fun}_{(\mathcal{B})}(\mathcal{K}, \mathcal{B} \times \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Fun}(\mathcal{K}, \mathcal{B} \times \mathcal{C}) \arrow[d] \\
& \mathcal{B} \arrow[r] & \mathsf{Fun}(\mathcal{K}, \mathcal{B})
\end{tikzcd}$$

where all three squares are pullbacks. It follows that $\mathcal{Q}$ fits in a pullback square

$$\begin{tikzcd}
\mathcal{Q} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K}, \mathcal{E}) \arrow[d] \\
\mathcal{B} \arrow[r, "\mathrm{const} \times \{\phi\}"] & \mathsf{Fun}(\mathcal{K}, \mathcal{B}) \times \mathsf{Fun}(\mathcal{K}, \mathcal{C})
\end{tikzcd}$$

Corollary 7.1.7. *Given a functor $F \colon \mathcal{I} \to \mathcal{C}$, we have:*

(1) The left fibration for the functor $\mathsf{Map}_{/\mathcal{C}}(\mathcal{I}, \mathcal{C}_{/-})$ is $\mathcal{C}_{F/} \to \mathcal{C}$.

(2) The right fibration for the functor $\mathsf{Map}_{/\mathcal{C}}(\mathcal{I}, \mathcal{C}_{-/})$ is $\mathcal{C}_{/F} \to \mathcal{C}$.

Proof. We prove the first statement; the second follows similarly from the duals of the results in this section. The cocartesian fibration for $x \mapsto \mathcal{C}_{/x}$ is $\mathrm{ev}_1 \colon \mathsf{Ar}(\mathcal{C}) \to \mathcal{I}$, so we get from Observation 7.1.6 that the cocartesian fibration for[1] $\mathsf{Fun}_{/\mathcal{C}}(\mathcal{I}, \mathcal{C}_{/x})$ is the pullback

$$\begin{tikzcd}
\mathcal{Q} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{I}, \mathsf{Ar}(\mathcal{C})) \arrow[d] \\
\mathcal{C} \arrow[r, "\{F\} \times \mathrm{const}"] & \mathsf{Fun}(\mathcal{I}, \mathcal{C}) \times \mathsf{Fun}(\mathcal{I}, \mathcal{C}).
\end{tikzcd}$$

which is also the definition of $\mathcal{C}_{F/}$. □

Interpreting this in terms of functors, we get:

Corollary 7.1.8. *Given a functor $F \colon \mathcal{I} \to \mathcal{C}$, we have natural equivalences*

- $\mathsf{Nat}_{\mathcal{I},\mathcal{C}}(F, \mathrm{const}_{-}) \simeq \mathsf{Nat}_{\mathcal{I}^{\mathrm{op}},\mathsf{Gpd}_\infty}(\mathrm{const}_*, \mathcal{C}(F, -)) \simeq \lim_{\mathcal{I}^{\mathrm{op}}} \mathcal{C}(F, -)$,
- $\mathsf{Nat}_{\mathcal{I},\mathcal{C}}(\mathrm{const}_{-}, F) \simeq \mathsf{Nat}_{\mathcal{I},\mathsf{Gpd}_\infty}(\mathrm{const}_*, \mathcal{C}(-, F)) \simeq \lim_{\mathcal{I}} \mathcal{C}(-, F)$. □

[1]Here $\mathsf{Fun}_{/\mathcal{C}}(\mathcal{I}, \mathcal{C}_{/-})$ is an ∞-groupoid since $\mathcal{C}_{/x} \to \mathcal{C}$ is a right fibration and so conservative — the components of a natural transformation over $\mathcal{C}$ by definition lie over equivalences in $\mathcal{C}$.

Corollary 7.1.9. *Given a diagram $F: \mathcal{I} \to \mathcal{C}$, an object of $\mathcal{C}$ is the limit of F if and only if it represents the presheaf*

$$\mathsf{Nat}_{\mathcal{I},\mathsf{Gpd}_\infty}(\mathrm{const}_*, \mathcal{C}(-, F)) \simeq \lim_{\mathcal{I}} \mathcal{C}(-, F),$$

while it is the colimit of F if and only if it corepresents the copresheaf

$$\mathsf{Nat}_{\mathcal{I}^{\mathrm{op}},\mathsf{Gpd}_\infty}(\mathrm{const}_*, \mathcal{C}(F, -)) \simeq \lim_{\mathcal{I}^{\mathrm{op}}} \mathcal{C}(F, -).$$

Exercise 7.1. Use Corollary 7.1.9 to prove that a left adjoint preserves all colimits and a right adjoint preserves all limits.

7.2 Weighted limits and colimits

We now introduce the notion of *weighted* (co)limits. As we will see, these can actually be subsumed within the ordinary notions of (co)limits, but will still be useful to formulate and prove a number of results later on.

Definition 7.2.1. The *limit* of $F: \mathcal{I} \to \mathcal{C}$ *weighted by* $W: \mathcal{I} \to \mathsf{Gpd}_\infty$ is an object of $\mathcal{C}$ that represents the presheaf

$$x \mapsto \mathsf{Nat}_{\mathcal{I},\mathsf{Gpd}_\infty}(W, \mathcal{C}(x, F)).$$

Dually, the *colimit* of F *weighted by* $U: \mathcal{I}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ is an object of $\mathcal{C}$ that corepresents the copresheaf

$$x \mapsto \mathsf{Nat}_{\mathcal{I}^{\mathrm{op}},\mathsf{Gpd}_\infty}(U, \mathcal{C}(F, x)).$$

Observation 7.2.2. From Corollary 7.1.9 we see that the ordinary (co)limit of a functor F is also the W-weighted (co)limit for $W = \mathrm{const}_*$.

Proposition 7.2.3. *If $p: \mathcal{W} \to \mathcal{I}$ is the left fibration for W, then $\lim_{\mathcal{I}}^W F$ is also $\lim_{\mathcal{W}} F \circ p$. If $q: \mathcal{U} \to \mathcal{C}$ is the right fibration for U, then $\operatorname{colim}_{\mathcal{I}}^U F$ is also $\operatorname{colim}_{\mathcal{U}} F \circ q$.*

Proof. We prove the limit version. Since the left fibration for $\mathcal{C}(x, F)$ is $F^*\mathcal{C}_{x/}$, we can rewrite the presheaf represented by $\lim_{\mathcal{I}}^W$ as

$$x \mapsto \mathsf{Map}_{/\mathcal{I}}(\mathcal{W}, F^*\mathcal{C}_{x/}) \simeq \mathsf{Map}_{/\mathcal{C}}(\mathcal{W}, \mathcal{C}_{x/}).$$

By Corollary 7.1.7 this corresponds to the right fibration $\mathcal{C}_{/Fp} \to \mathcal{C}$, which is also represented by $\lim_{\mathcal{W}} Fp$. □

Combining this with Proposition 5.2.3 and Proposition 5.3.4, we deduce the following concrete descriptions of weighted (co)limits in Gpd_∞ and Cat_∞:

Corollary 7.2.4. *Given $W: \mathcal{I} \to \mathsf{Gpd}_\infty$ with left fibration $p: \mathcal{W} \to \mathcal{I}$ and $U: \mathcal{I}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ with right fibration $q: \mathcal{U} \to \mathcal{C}$, we have:*

- *For $\phi\colon \mathcal{I} \to \mathsf{Gpd}_\infty$ with corresponding left fibration $\pi\colon \mathcal{E} \to \mathcal{I}$,*

$$\lim_{\mathcal{I}}^{W} \phi \simeq \mathsf{Map}_{/\mathcal{I}}(\mathcal{W}, \mathcal{E}) \simeq \mathsf{Nat}_{\mathcal{I},\mathsf{Gpd}_\infty}(W, \phi),$$

$$\operatorname{colim}_{\mathcal{I}}^{U} \phi \simeq \|\mathcal{U} \times_{\mathcal{I}} \mathcal{E}\|.$$

- *For $\phi\colon \mathcal{I} \to \mathsf{Cat}_\infty$ with corresponding cocartesian fibration $\pi\colon \mathcal{E} \to \mathcal{I}$,*

$$\lim_{\mathcal{I}}^{W} \phi \simeq \mathsf{Fun}_{/\mathcal{I}}^{\mathrm{coct}}(\mathcal{W}, \mathcal{E}),$$

$$\operatorname{colim}_{\mathcal{I}}^{U} \phi \simeq (\mathcal{U} \times_{\mathcal{I}} \mathcal{E})[S^{-1}],$$

where S consists of all morphisms that map to π-cocartesian morphisms in $\mathcal{E}$.

Example 7.2.5. For a representable weight $\mathcal{C}(-, c)$, we get

$$\operatorname{colim}_{\mathcal{C}}^{\mathcal{C}(-,c)} F \simeq \operatorname{colim}_{\mathcal{C}_{/c}} F \simeq F(c),$$

since $\mathcal{C}_{/c}$ has a terminal object (Proposition 6.5.10). Similarly,

$$\lim_{\mathcal{C}}^{\mathcal{C}(c,-)} F \simeq F(c).$$

7.3 The Yoneda lemma

We will now show a more natural version of the Yoneda lemma, namely that there is an equivalence

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}, \Phi) \simeq \Phi$$

that is natural in $\Phi \in \mathsf{PSh}(\mathcal{C})$. The starting point for this is some observations about weighted colimits.

Observation 7.3.1. By Corollary 7.2.4, for $\phi\colon \mathcal{C} \to \mathsf{Cat}_\infty$ with cocartesian fibration $\pi\colon \mathcal{E} \to \mathcal{C}$ and $W\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ with corresponding right fibration $\mathcal{W} \to \mathcal{C}$, there is a natural equivalence

$$\begin{aligned}\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(W, \mathsf{Map}_{\mathsf{Cat}_\infty}(\phi, \mathcal{D})) &\simeq \mathsf{Map}_{\mathsf{Cat}_\infty}(\operatorname{colim}_{\mathcal{C}}^{W} \phi, \mathcal{D}) \\ &\simeq \mathsf{Map}((\mathcal{W} \times_{\mathcal{C}} \mathcal{E})[S^{-1}], \mathcal{D}),\end{aligned}$$

where S comprises the morphisms that project to π-cocartesian ones in $\mathcal{E}$.

Observation 7.3.2. Specializing the previous observation, given a bifibration $p\colon \mathcal{E} \to \mathcal{A} \times \mathcal{B}$ corresponding to $\gamma\colon \mathcal{B} \to \mathsf{RFib}(\mathcal{A})$ and $\Gamma\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Gpd}_\infty$, we have for $W \in \mathsf{PSh}(\mathcal{B}), F \in \mathsf{PSh}(\mathcal{A})$ with corresponding right fibrations $\mathcal{W} \to \mathcal{B}, \mathcal{F} \to \mathcal{A}$, natural equivalences

$$\begin{aligned}\mathsf{Map}_{\mathsf{PSh}(\mathcal{B})}(W, \mathsf{Map}_{\mathsf{PSh}(\mathcal{A})}(\Gamma, F)) &\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{B})}(W, \mathsf{Map}_{/\mathcal{A}}(\gamma(-), \mathcal{F})) \\ &\simeq \mathsf{Map}_{/\mathcal{A}}((\mathcal{W} \times_{\mathcal{B}} \mathcal{E})[S^{-1}], \mathcal{F}), \\ &\simeq \mathsf{Map}_{/\mathcal{A}}(\mathcal{W} \times_{\mathcal{B}} \mathcal{E}, \mathcal{F}).\end{aligned}$$

Here the last equivalence uses that S consists precisely of the morphisms that project to $p_{\mathcal{B}}$-cocartesian morphisms, but these are equivalently those that project to equivalences in $\mathcal{A}$; since $\mathcal{F} \to \mathcal{A}$ is a right fibration and so in particular conservative, any functor over $\mathcal{A}$ takes these to equivalences.

Proposition 7.3.3. *For $W, F \in \mathsf{PSh}(\mathcal{C})$, there is a natural equivalence*

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(W, \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}, F)) \simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(W, F).$$

Proof. Suppose W, F correspond to the right fibrations $\mathcal{W}, \mathcal{F} \to \mathcal{C}$. Specializing Observation 7.3.2 to the bifibration $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$, we get a natural equivalence

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(W, \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}, F)) \simeq \mathsf{Map}_{/\mathcal{C}}(\mathcal{W} \times_{\mathcal{C}} \mathsf{Ar}(\mathcal{C}), \mathcal{F})$$

where the pullback $\mathcal{W} \times_{\mathcal{C}} \mathsf{Ar}(\mathcal{C})$ is over ev_1 and the map to $\mathcal{C}$ is given by ev_0. But this is precisely the free cartesian fibration on $\mathcal{W}$, and since $\mathcal{F}$ is a right fibration we get from Theorem 6.1.4 a natural equivalence

$$\mathsf{Map}_{/\mathcal{C}}(\mathcal{W} \times_{\mathcal{C}} \mathsf{Ar}(\mathcal{C}), \mathcal{F}) \simeq \mathsf{Map}_{/\mathcal{C}}(\mathcal{W}, \mathcal{F}).$$

Combined with straightening, this completes the proof. □

Since the Yoneda embedding is fully faithful also for the locally small ∞-category $\mathsf{PSh}(\mathcal{C})$, we get:

Corollary 7.3.4 (The Yoneda Lemma). *There is a natural equivalence*

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}, -) \simeq \mathrm{id}$$

of functors $\mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{C})$. □

The equivalence in Proposition 7.3.3 also has the following interpretation (sometimes known as the *co-Yoneda lemma*):

Corollary 7.3.5. *There is a natural equivalence*

$$\mathrm{colim}_{\mathcal{C}}^{W} \mathsf{y} \simeq W$$

for $W \in \mathsf{PSh}(\mathcal{C})$. □

Observation 7.3.6. If $p \colon \mathcal{E} \to \mathcal{C}$ is the right fibration for W, then Corollary 7.3.5 is equivalent via Proposition 7.2.3 to: W is the colimit of the composite

$$\mathcal{E} \xrightarrow{p} \mathcal{C} \xrightarrow{Yo} \mathsf{PSh}(\mathcal{C}).$$

7.4 More about (co)limits

In this section we will prove some further useful results about (co)limits.

Observation 7.4.1. For any ∞-category $\mathcal{K}$, Exercise 7.1 implies that the left adjoint $\mathcal{K} \times -$ preserves colimits, while the right adjoint $\mathsf{Fun}(\mathcal{K}, -)$ preserves limits.

Lemma 7.4.2. *Suppose the ∞-category $\mathcal{C}$ is the limit of a functor $F\colon \mathcal{K} \to \mathsf{Cat}_\infty$. Then for all $x, y \in \mathcal{C}$ we have*

$$\mathcal{C}(x,y) \simeq \lim_{\mathcal{K}} \mathsf{Map}_{F(k)}(x_k, y_k),$$

where x_k, y_k are the images of x, y in $F(k)$ under the functor from $\mathcal{C}$ in the limit cone.

Proof. Since $\mathsf{Ar}(\mathcal{C}) \simeq \lim_{k \in \mathcal{K}} \mathsf{Ar}(F(k))$ by Observation 7.4.1, we have a pullback square

$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r] \arrow[d] & \lim_{k\in\mathcal{K}} \mathsf{Ar}(F(k)) \arrow[d] \\
\{(x,y)\} \arrow[r] & \lim_{k\in\mathcal{K}} F(k)^{\times 2}.
\end{tikzcd}$$

Since limits commute by Corollary 5.5.11, this implies that $\mathcal{C}(x,y)$ is also a limit over $\mathcal{K}$, as required. □

Proposition 7.4.3. *For any ∞-category $\mathcal{K}$, the functor*

$$\mathsf{Fun}(-,\mathcal{K})\colon \mathsf{Cat}_\infty^{\mathrm{op}} \to \mathsf{Cat}_\infty$$

preserves limits.

Proof. For $\mathcal{L} \in \mathsf{Cat}_\infty$ and a functor $F\colon \mathcal{I} \to \mathsf{Cat}_\infty$ we have natural equivalences

$$\begin{aligned}
\mathsf{Map}(\mathcal{L}, \mathsf{Fun}(\operatorname{colim}_{\mathcal{I}} F, \mathcal{K})) &\simeq \mathsf{Map}(\mathcal{L} \times \operatorname{colim}_{\mathcal{I}} F, \mathcal{K}) \\
&\simeq \mathsf{Map}(\operatorname{colim}_{\mathcal{I}}(\mathcal{L} \times F), \mathcal{K}) \\
&\simeq \lim_{\mathcal{I}^{\mathrm{op}}} \mathsf{Map}(\mathcal{L} \times F, \mathcal{K}) \\
&\simeq \lim_{\mathcal{I}^{\mathrm{op}}} \mathsf{Map}(\mathcal{L}, \mathsf{Fun}(F, \mathcal{K})) \\
&\simeq \mathsf{Map}(\mathcal{L}, \lim_{\mathcal{I}^{\mathrm{op}}} \mathsf{Fun}(F, \mathcal{K})).
\end{aligned}$$

This implies

$$\mathsf{Fun}(\operatorname{colim}_{\mathcal{I}} F, \mathcal{K}) \simeq \lim_{\mathcal{I}^{\mathrm{op}}} \mathsf{Fun}(F, \mathcal{K}),$$

as required. □

Corollary 7.4.4. *Given a functor $\phi\colon \mathcal{K} \to \mathcal{C}$, where $\mathcal{K} := \operatorname{colim}_{\mathcal{I}} F$ for some functor $F\colon \mathcal{I} \to \mathsf{Cat}_\infty$ with $\mathcal{I}$ weakly contractible, we have*

$$\mathcal{C}_{\phi/} \simeq \lim_{\mathcal{I}^{\mathrm{op}}} \mathcal{C}_{\phi_i/}, \qquad \mathcal{C}_{/\phi} \simeq \lim_{\mathcal{I}^{\mathrm{op}}} \mathcal{C}_{/\phi_i}$$

where $\phi_i\colon F(i) \to \mathcal{C}$ is obtained by restricting ϕ along the colimit cocone.

Proof. We prove the first case. By Observation 5.1.6 we have a pullback square

$$\begin{tikzcd} \mathcal{C}_{\phi/} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \arrow[d] \\ \{\phi\} \arrow[r] & \mathsf{Fun}(\mathcal{K}, \mathcal{C}). \end{tikzcd}$$

Here $(-)^{\triangleright} := (-) \times [1] \amalg_{(-)\times\{1\}} [0]$ preserves weakly contractible colimits, since colimits commute (Corollary 5.5.11), so that $\mathsf{Fun}((-)^{\triangleright}, \mathcal{C})$ preserves weakly contractible limits. Since limits commute, this implies the result. □

Corollary 7.4.5. *Given a functor* $\phi \colon \mathcal{K} \to \mathcal{C}$*, where* $\mathcal{K} := \operatorname{colim}_{\mathcal{I}} F$ *for some functor* $F \colon \mathcal{I} \to \mathsf{Cat}_\infty$*, we have a decomposition*

$$\operatorname{colim}_{\mathcal{K}} \phi \simeq \operatorname{colim}_{\mathcal{I}} \operatorname{colim}_{F(i)} \phi_i,$$

where $\phi_i \colon F(i) \to \mathcal{C}$ *is obtained by restricting* ϕ *along the colimit cocone, provided these colimits exist.*

Proof. For $c \in \mathcal{C}$ we have natural equivalences

$$\begin{aligned} \mathcal{C}(\operatorname{colim}_{\mathcal{K}} \phi, c) &\simeq \mathsf{Map}_{\mathsf{Fun}(\mathcal{K},\mathcal{C})}(\phi, \mathrm{const}_c) \\ &\simeq \lim_{\mathcal{I}} \mathsf{Map}_{\mathsf{Fun}(F(i),\mathcal{C})}(\phi_i, \mathrm{const}_c) \\ &\simeq \lim_{\mathcal{I}} \mathcal{C}(\operatorname{colim}_{F(i)} \phi_i, c) \\ &\simeq \mathcal{C}(\operatorname{colim}_{\mathcal{I}} \operatorname{colim}_{F(i)} \phi_i, c). \end{aligned}$$

where the second uses Proposition 7.4.3 and Lemma 7.4.2. □

Exercise 7.2 (An alternative proof of Corollary 7.4.5). Show that for a functor $F \colon \mathcal{K} \to \mathcal{C}$, the ∞-groupoid $\mathsf{Map}_{/\mathcal{C}}(\mathcal{K}, \mathcal{C}_{/c})$ is naturally equivalent to $\mathsf{Map}_{\mathsf{Fun}(\mathcal{K},\mathcal{C})}(F, \mathrm{const}_c)$. Conclude that if $\mathsf{Cat}'_{\infty/\mathcal{C}}$ denotes the full subcategory of $\mathsf{Cat}_{\infty/\mathcal{C}}$ on those functors whose colimits exist in $\mathcal{C}$, then the functor

$$\mathcal{C} \xrightarrow{\mathcal{C}_{/-}} \mathsf{RFib}(\mathcal{C}) \hookrightarrow \mathsf{Cat}'_{\infty/\mathcal{C}}$$

has a left adjoint. Now deduce Corollary 7.4.5 from the fact that left adjoints preserve colimits (Exercise 7.1) together with and the description of colimits in $\mathsf{Cat}'_{\infty/\mathcal{C}}$ from Corollary 5.6.11 and Corollary 5.4.6.

7.5 (★) Ends and natural transformations

We can describe mapping ∞-groupoids in functor ∞-categories as certain weighted limits.

Definition 7.5.1. Given a functor $\Phi\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, the *coend* of Φ, if it exists, is the weighted colimit

$$\int_{\mathcal{C}} \Phi := \operatorname{colim}^{\mathcal{C}(-,-)} \Phi.$$

Dually, the *end* of a functor $\Psi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, if it exists, is the weighted limit

$$\int_{\mathcal{C}}^{*} \Psi := \lim^{\mathcal{C}(-,-)} \Psi.$$

Warning 7.5.2. We use Yoneda's original notation for coends rather than the Australian (or upside-down) convention, where the *end* is denoted as $\int_{\mathcal{C}} \Phi$ and the coend as $\int^{\mathcal{C}} \Phi$. After all, it is the coend of a functor that is (very) loosely similar to an integral, not the end.

Notation 7.5.3. The mapping ∞-groupoid functor $\mathcal{C}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Gpd}_\infty$ has a corresponding right fibration

$$\mathsf{Tw}^r(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}},$$

which can be obtained from $\mathsf{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ by first taking the cocartesian straigthening in the second variable and then taking the cartesian unstraightening of the corresponding functor. The corresponding left fibration is then

$$\mathsf{Tw}^\ell(\mathcal{C}) := \mathsf{Tw}^r(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}.$$

(Here $\mathsf{Tw}^\ell(\mathcal{C})$ and $\mathsf{Tw}^r(\mathcal{C})$ are the *twisted arrow ∞-categories* of $\mathcal{C}$, which can be described explicitly as certain simplicial ∞-groupoids; to see that these describe the mapping ∞-groupoid functor as we have constructed it requires relating the cartesian and cocartesian fibrations of a single functor via a span construction.)

Notation 7.5.4. For functors $F, G\colon \mathcal{C} \to \mathcal{D}$, we write

$$\mathsf{Nat}_{\mathcal{C},\mathcal{D}}(F,G) := \mathsf{Map}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F,G)$$

for the ∞-groupoid of natural transformations from F to G.

We can explicitly identify ends in Gpd_∞ in terms of bifibrations:

Proposition 7.5.5. *For a functor $\Phi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Gpd}_\infty$ with corresponding bifibration*[2] *$p\colon \mathcal{E} \to \mathcal{C} \times \mathcal{C}$, we have an equivalence*

$$\int_{\mathcal{C}}^{\star} \Phi \simeq \mathsf{Map}_{/\mathcal{C}\times\mathcal{C}}(\mathcal{C}, \mathcal{E}).$$

[2] Obtained, by our convention, by first unstraightening Φ to a functor $\mathcal{C} \to \mathsf{RFib}(\mathcal{C})$ and then taking the cocartesian unstraightening of this.

Proof. By Corollary 6.2.2, we have a natural equivalence

$$\mathsf{Map}_{/\mathcal{C}\times\mathcal{C}}(\mathcal{C},\mathcal{E}) \simeq \mathsf{Map}_{/\mathcal{C}\times\mathcal{C}}(\mathsf{Ar}(\mathcal{C}),\mathcal{E}).$$

This also gives an equivalence on ∞-groupoids of maps between the corresponding left fibrations, which gives the end of Φ by Corollary 7.2.4. □

Corollary 7.5.6. *For functors $F, G\colon \mathcal{C} \to \mathcal{D}$, we have*

$$\mathsf{Nat}_{\mathcal{C},\mathcal{D}}(F,G) \simeq \int_{\mathcal{C}}^{*} \mathcal{D}(F,G).$$

Proof. Since $\mathsf{Ar}(\mathsf{Fun}(\mathcal{C},\mathcal{D})) \simeq \mathsf{Fun}(\mathcal{C},\mathsf{Ar}(\mathcal{D}))$, we can can describe $\mathsf{Nat}(F,G)$ as the fibre at (F,G) of the bifibration

$$\mathsf{Fun}(\mathcal{C},\mathsf{Ar}(\mathcal{D})) \to \mathsf{Fun}(\mathcal{C},\mathcal{D}\times\mathcal{D}).$$

Thus we have

$$\mathsf{Nat}(F,G) \simeq \left\{ \begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\Delta"] & \mathsf{Ar}(\mathcal{D}) \arrow[d] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "F\times G"] & \mathcal{D}\times\mathcal{D} \end{tikzcd} \right\} \simeq \mathsf{Map}_{/\mathcal{C}\times\mathcal{C}}(\mathcal{C}, (F\times G)^{*}\mathsf{Ar}(\mathcal{D})).$$

Here $(F\times G)^{*}\mathsf{Ar}(\mathcal{D}) \to \mathcal{C}\times\mathcal{C}$ is the bifibration for $\mathcal{D}(F(-),G(-))$, so Proposition 7.5.5 identifies this with the end of this functor. □

7.6 (⋆) Relative (co)limits

In this section we prove some useful results about (co)limits in (co)cartesian fibrations. It will be convenient to express these in terms of the notion of *relative* (co)limits:

Definition 7.6.1. Given a functor $p\colon \mathcal{E} \to \mathcal{B}$, we say that a cocone $\phi\colon \mathcal{I}^{\triangleright} \to \mathcal{E}$ is a *p-colimit* if the commutative square

$$\begin{tikzcd} \mathcal{E}(\phi(\infty), e) \arrow[r] \arrow[d] & \lim_{\mathcal{I}^{\mathrm{op}}} \mathcal{E}(\phi, e) \arrow[d] \\ \mathcal{B}(p\phi(\infty), pe) \arrow[r] & \lim_{\mathcal{I}^{\mathrm{op}}} \mathcal{B}(p\phi, pe) \end{tikzcd}$$

is a pullback for all $e \in \mathcal{E}$. Dually, we say that a cone $\psi\colon \mathcal{I}^{\triangleleft} \to \mathcal{E}$ is a *p-limit* if

$$\begin{tikzcd} \mathcal{E}(e, \phi(-\infty)) \arrow[r] \arrow[d] & \lim_{\mathcal{I}} \mathcal{E}(e, \phi) \arrow[d] \\ \mathcal{B}(pe, p\phi(-\infty)) \arrow[r] & \lim_{\mathcal{I}} \mathcal{B}(pe, p\phi) \end{tikzcd}$$

Exercise 7.3. Suppose $\bar{\phi}\colon \mathcal{I}^{\triangleright} \to \mathcal{E}$ is a cocone, and set $\phi := \bar{\phi}|_{\mathcal{I}}$. Show that $\bar{\phi}$ is a p-colimit for $p\colon \mathcal{E} \to \mathcal{B}$ if and only if the commutative square

$$\begin{tikzcd} \mathcal{E}_{\bar{\phi}/} \arrow[r] \arrow[d] & \mathcal{B}_{p\bar{\phi}/} \arrow[d] \\ \mathcal{E}_{\phi/} \arrow[r] & \mathcal{B}_{p\phi/} \end{tikzcd}$$

is a pullback. (Use 5.6.7 and 6.5.10.)

Observation 7.6.2. If ϕ is a cocone such that $p\phi$ is a colimit in $\mathcal{B}$, then ϕ is a p-colimit if and only if it is a colimit in $\mathcal{E}$.

Proposition 7.6.3. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. If the fibres of p all admit $\mathcal{K}$-indexed colimits and the cocartesian transport functors preserve these, then a cocone $\phi\colon \mathcal{I}^{\triangleright} \to \mathcal{E}$ is a p-colimit if and only if its cocartesian transport to a cocone in $\mathcal{E}_{p(\phi(\infty))}$ is a colimit there.*

Proof. Let $a := \phi(\infty)$ and take $\psi\colon \mathcal{K}^{\triangleright} \to \mathcal{E}_a$ to be the cocone obtained by cocartesian transport along ϕ[3] Then for $e \in \mathcal{E}$ over $b \in \mathcal{B}$ the fibre at $f\colon a \to b$ in the square

$$\begin{tikzcd} \mathcal{E}(\phi(\infty), e) \arrow[r] \arrow[d] & \lim_{\mathcal{K}^{\mathrm{op}}} \mathcal{E}(\phi, e) \arrow[d] \\ \mathcal{B}(a, b) \arrow[r] & \lim_{\mathcal{K}^{\mathrm{op}}} \mathcal{B}(p\phi, b) \end{tikzcd}$$

can be identified as

$$\mathcal{E}_b(f_!\phi(\infty), e) \to \lim_{\mathcal{K}} \mathcal{E}_b(f_!\psi, e),$$

so that ϕ is a p-colimit if and only if this is an equivalence for all e and f. Taking $f = \mathrm{id}_a$ we see that ψ must be a colimit in $\mathcal{E}_a$, and if $f_!$ preserves $\mathcal{K}$-shaped colimits this special case implies the statement for general f, as required. □

From this we obtain the standard description of colimits in the source of a cocartesian fibration:

Corollary 7.6.4. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. If the base $\mathcal{B}$ and the fibres of p all admit $\mathcal{K}$-indexed colimits and the cocartesian transport functors preserve these, then $\mathcal{E}$ admits $\mathcal{K}$-indexed colimits and p preserves these. For a diagram $\phi\colon \mathcal{E}$, the colimit is computed by taking the cocartesian transport of ϕ to a diagram in the fibre over the colimit of $p \circ \phi$ and computing the colimit in this fibre.* □

We can also prove a rather more restrictive description of relative colimits in *cartesian* fibrations:

[3]This can be constructed as a cocartesian morphism in $\mathsf{Fun}(\mathcal{K}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{K}, \mathcal{B})$.

Proposition 7.6.5. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cartesian fibration and $\bar{\phi}\colon \mathcal{I}^{\triangleright} \to \mathcal{B}$ is a cocone such that the functor $\mathcal{B}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ associated to p takes $\bar{\phi}^{\mathrm{op}}$ to a limit cone in Cat_∞. Then every p-cartesian lift of $\phi := \bar{\phi}|_{\mathcal{I}}$ admits a p-colimit, which is a p-cartesian lift of $\bar{\phi}$.*

Proof. Let $b := \bar{\phi}(\infty)$. Then $\mathcal{E}_b \simeq \lim_{x\in\mathcal{I}} \mathcal{E}_{p\phi(x)}$, and more precisely $\mathcal{E}_b$ is identified with the ∞-category of cartesian sections of $\phi^*\mathcal{E} \to \mathcal{I}$. A p-cartesian lift ψ of ϕ therefore determines an object $e \in \mathcal{E}_b$ with p-cartesian morphisms $\psi(x) \xrightarrow{q_x} e$ extending ψ to a p-cartesian lift $\bar{\psi}$ of $\bar{\phi}$. We claim that $\bar{\psi}$ is a p-colimit. In other words, the commutative square

$$\begin{array}{ccc} \mathcal{E}(e, e') & \longrightarrow & \lim \mathcal{E}(\psi, e') \\ \big\downarrow & & \big\downarrow \\ \mathcal{B}(b, b') & \longrightarrow & \lim \mathcal{B}(\phi, b') \end{array}$$

is a pullback for all $e' \in \mathcal{E}$, with $b' := p(e')$. To see this we look on fibres over $f\colon b \to b'$, where we get

$$\mathcal{E}_b(e, f^*e') \to \lim_{x\in\mathcal{I}} \mathcal{E}_{\phi(x)}(\psi(x), q_x^*f^*e') \simeq \lim_{x\in\mathcal{I}} \mathcal{E}_{\phi(x)}(q_x^*e, q_x^*f^*e'),$$

which is an equivalence since $\mathcal{E}_b$ is the limit $\lim_{x\in\mathcal{I}} \mathcal{E}_{\phi(x)}$. □

Corollary 7.6.6. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cartesian fibration and $\bar{\phi}\colon \mathcal{I}^{\triangleright} \to \mathcal{B}$ is a colimit cocone such that the functor $\mathcal{B}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ associated to p takes $\bar{\phi}^{\mathrm{op}}$ to a limit cone in Cat_∞. Then every p-cartesian lift of $\phi := \bar{\phi}|_{\mathcal{I}}$ admits a colimit in $\mathcal{E}$, which is a p-cartesian lift of $\bar{\phi}$.*

Corollary 7.6.7. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a right fibration and $\bar{\phi}\colon \mathcal{I}^{\triangleright} \to \mathcal{B}$ is a colimit cocone such that the functor $\mathcal{B}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ associated to p takes $\bar{\phi}^{\mathrm{op}}$ to a limit cone in Gpd_∞. Then every lift of $\phi := \bar{\phi}|_{\mathcal{I}}$ admits a colimit in $\mathcal{E}$, which is a lift of $\bar{\phi}$.*

Corollary 7.6.8. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a right fibration such that $\mathcal{B}$ admits $\mathcal{I}$-indexed colimits and the associated functor $\mathcal{B}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ preserves $\mathcal{I}$-indexed limits. Then $\mathcal{E}$ admits $\mathcal{I}$-indexed colimits and p preserves these.*

7.7 (⋆) Pullbacks of localizations

We saw in Corollary 5.7.3 that localization from ∞-categories to ∞-groupoids preserves products. However, it does *not* in general preserve pullbacks. In this section we will look at some special circumstances where pullbacks are in fact preserved, which we will deduce from the observation that colimits of ∞-groupoids are *universal*, in the following sense:

Proposition 7.7.1. *For any map $f\colon X \to Y$ of ∞-groupoids, the pullback functor $f^*\colon \mathsf{Gpd}_{\infty/Y} \to \mathsf{Gpd}_{\infty/X}$ preserves colimits.*

Proof. Under straightening, this corresponds to the functor

$$f^*\colon \mathsf{Fun}(Y, \mathsf{Gpd}_\infty) \to \mathsf{Fun}(X, \mathsf{Gpd}_\infty)$$

given by composition with f. This preserves colimits since these are computed pointwise by (the dual of) Corollary 5.5.10. □

Observation 7.7.2. Given a functor $F\colon \mathcal{B} \to \mathsf{Gpd}_\infty$, a natural transformation $\alpha\colon F \to \mathrm{const}_X$ and a map $f\colon Y \to X$, Proposition 7.7.1 says that there is a pullback square of ∞-groupoids

$$\begin{tikzcd}
\mathrm{colim}_{\mathcal{B}}\, Y \times_X F \arrow[r] \arrow[d] & \mathrm{colim}_{\mathcal{B}}\, F \arrow[d] \\
Y \arrow[r, "f"'] & X.
\end{tikzcd}$$

We can interpret this via the description of colimits in Gpd_∞ as localizations of left fibrations (Proposition 5.2.3). If $p\colon \mathcal{E} \to \mathcal{B}$ is the left fibration for F, then $Y \times_X \mathcal{E}$ is the fibration for $Y \times_X F$ by Proposition 7.1.3, so that we have a pullback square

$$\begin{tikzcd}
\|Y \times_X \mathcal{E}\| \arrow[r] \arrow[d] & \|\mathcal{E}\| \arrow[d] \\
Y \arrow[r, "f"'] & X.
\end{tikzcd}$$

Here we can take the left fibration p to be $\mathrm{id}_{\mathcal{E}}$, giving:

Corollary 7.7.3. *Suppose $\mathcal{E}$ is an ∞-category, and we're given a functor $p\colon \mathcal{E} \to X$ with X an ∞-groupoid. Then a pullback square*

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "p'"] & \mathcal{E} \arrow[d, "p"] \\
X' \arrow[r] & X
\end{tikzcd}$$

induces a pullback square of ∞-groupoids

$$\begin{tikzcd}
\|\mathcal{E}'\| \arrow[r] \arrow[d, "\|p'\|"'] & \|\mathcal{E}\| \arrow[d, "\|p\|"] \\
X' \arrow[r] & X.
\end{tikzcd}$$

In particular, the fibre of $\|\mathcal{E}\| \to X$ at $x \in X$ is $\|\mathcal{E} \times_X \{x\}\|$, so that $\|p\|$ is an equivalence if and only if $\mathcal{E} \times_X \{x\}$ is weakly contractible for all $x \in X$. □

We also get a useful criterion for detecting colimits:

Corollary 7.7.4. *For a functor $F\colon \mathcal{K} \to \mathsf{Gpd}_\infty$ and a natural transformation $\alpha\colon F \to \mathrm{const}_X$, the following are equivalent:*

(1) *α is a colimit cocone, i.e. corresponds to an equivalence $\operatorname{colim} F \xrightarrow{\sim} X$.*

(2) *For any $x \in X$, if $F_x := F \times_X \{x\}$, then $\operatorname{colim} F_x \simeq *$.*

(3) *If $\mathcal{E} \to \mathcal{K}$ is the left fibration for F, then the ∞-category $\mathcal{E} \times_X \{x\}$ is weakly contractible for all $x \in X$.* □

To exploit Corollary 7.7.3 further, we now introduce the notion of a *Kan fibration* of ∞-categories. This is based on ideas of Sattler and Wärn [SW25].

Definition 7.7.5. A functor $p\colon \mathcal{E} \to \mathcal{B}$ is a *Kan fibration* if it is both a left and a right fibration.

Proposition 7.7.6. *The following are equivalent for $p\colon \mathcal{E} \to \mathcal{B}$:*

(1) p is a Kan fibration

(2) The straightening of p (as either a left or right fibration) takes every morphism in $\mathcal{B}$ to an equivalence of ∞-groupoids.

(3) There is a pullback square

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"] & Y \arrow[d] \\ \mathcal{B} \arrow[r] & X \end{tikzcd}$$

where X and Y are ∞-groupoids.

(4) The commutative square

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"'] & \|\mathcal{E}\| \arrow[d, "\|p\|"] \\ \mathcal{B} \arrow[r] & \|\mathcal{B}\| \end{tikzcd}$$

is a pullback.

Proof. If p is a Kan fibration, then it is in particular both a cocartesian and a cartesian fibration, so its covariant straightening

$$F\colon \mathcal{B} \to \mathsf{Gpd}_\infty$$

takes every morphism in $\mathcal{B}$ to a left adjoint by Corollary 6.3.7. But a left adjoint among ∞-groupoids is an equivalence (as all natural transformations among ∞-groupoids are equivalences). Thus F factors as

$$\mathcal{B} \to \|\mathcal{B}\| \xrightarrow{F'} \mathsf{Gpd}_\infty,$$

which means that we have a pullback square

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "p"'] & Y \arrow[d, "q"] \\
\mathcal{B} \arrow[r] & \|\mathcal{B}\|
\end{tikzcd}$$

where q is the straightening of F' (so that Y is an ∞-groupoid). Conversely, any morphism of ∞-groupoids is a Kan fibration, so the first three conditions are equivalent.

Suppose that the third condition holds; then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "p"'] & \|\mathcal{E}\| \arrow[r] \arrow[d, "\|p\|"] & Y \arrow[d] \\
\mathcal{B} \arrow[r] & \|\mathcal{B}\| \arrow[r] & X,
\end{tikzcd}$$

where the outer square is a pullback by assumption. This implies that the right-hand square is a pullback by Corollary 7.7.3, hence so is the left-hand square. □

Corollary 7.7.7. *Suppose we have a commutative square*

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "p'"] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}' \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where p and p' are Kan fibrations. Then this square is a pullback if and only if the induced commutative square of ∞-groupoids

$$\begin{tikzcd}
\|\mathcal{E}'\| \arrow[r] \arrow[d] & \|\mathcal{E}\| \arrow[d] \\
\|\mathcal{B}'\| \arrow[r] & \|\mathcal{B}\|
\end{tikzcd}$$

is a pullback.

Proof. Consider the commutative cube

$$\begin{tikzcd}
\mathcal{E}' \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{E} \arrow[dd] \arrow[dr] & \\
& \|\mathcal{E}'\| \arrow[rr] \arrow[dd] & & \|\mathcal{E}\| \arrow[dd] \\
\mathcal{B}' \arrow[rr] \arrow[dr] & & \mathcal{B} \arrow[dr] & \\
& \|\mathcal{B}'\| \arrow[rr] & & \|\mathcal{B}\|.
\end{tikzcd}$$

Here the left and right faces are pullbacks, so the back face is a pullback if the front face is a pullback. Conversely, if the back face is a pullback then so is the composite square

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d] & \|\mathcal{E}\| \arrow[d] \\
\mathcal{B}' \arrow[r] & \|\mathcal{B}\|,
\end{tikzcd}$$

so that the front face is a pullback by Corollary 7.7.3. □

Lemma 7.7.8. *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration, and $p^\ell\colon \mathcal{E}^\ell \to \mathcal{B}$ is the induced left fibration as in Observation 6.4.1. Then $\|\mathcal{E}\| \to \|\mathcal{E}^\ell\|$ is an equivalence.*

Proof. The localization functor $\mathsf{Cocart}(\mathcal{B}) \to \mathsf{Gpd}_\infty$ is left adjoint to the functor $(-) \times \mathcal{B}\colon \mathsf{Gpd}_\infty \to \mathsf{Cocart}(\mathcal{B})$, which factors through the inclusion $\mathsf{LFib}(\mathcal{B}) \hookrightarrow \mathsf{Cocart}(\mathcal{B})$. The left adjoint therefore factors through $(-)^\ell$, as required. □

Observation 7.7.9. Given a pullback square

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "q"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where p is a cocartesian fibration, the induced square

$$\begin{tikzcd}
\mathcal{F}^\ell \arrow[r] \arrow[d, "q^\ell"'] & \mathcal{E}^\ell \arrow[d, "p^\ell"] \\
\mathcal{C} \arrow[r] & \mathcal{B}
\end{tikzcd}$$

is also a pullback. Indeed, since p^ℓ and q^ℓ are left fibrations, it suffices to check that we get equivalences on fibres, but this is clear from the construction of $(-)^\ell$.

Corollary 7.7.10. *Suppose we have a pullback square of ∞-categories*

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "q"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C} \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where either

- *p is a cocartesian fibration, and the cocartesian transport functor over every morphism $f\colon b \to b'$ in $\mathcal{B}$ induces an equivalence $\|\mathcal{E}_b\| \to \|\mathcal{E}_{b'}\|$,*
- *or p is a cartesian fibration, and the cartesian transport functor over every morphism $f\colon b \to b'$ in $\mathcal{B}$ induces an equivalence $\|\mathcal{E}_{b'}\| \to \|\mathcal{E}_b\|$.*

Then the commutative square of ∞-groupoids

$$\begin{tikzcd} \|\mathcal{F}\| \arrow[r] \arrow[d, "\|q\|"'] & \|\mathcal{E}\| \arrow[d, "\|p\|"] \\ \|\mathcal{C}\| \arrow[r] & \|\mathcal{B}\| \end{tikzcd}$$

is a pullback.

Proof. We prove the first case. Here we have a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{F}^{\ell} \arrow[r] \arrow[d, "q^{\ell}"'] & \mathcal{E}^{\ell} \arrow[d, "p^{\ell}"] \\ \mathcal{C} \arrow[r] & \mathcal{B} \end{tikzcd}$$

where the vertical maps in the top square give equivalences on localizations by Lemma 7.7.8 and the bottom square is a pullback by Observation 7.7.9. Moreover, our assumption on p implies that p^{ℓ} is a Kan fibration via Proposition 7.7.6, since its straightening takes a morphism $f\colon b \to b'$ to $\|{-}\|$ applied to the cocartesian transport along f for p. The conclusion therefore follows from Corollary 7.7.7. □

As a special case we have the following, first proved by Steimle [Ste21]:

Corollary 7.7.11. *Suppose we have a pullback square of ∞-categories*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "q"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{C} \arrow[r] & \mathcal{B} \end{tikzcd}$$

where p is both a cartesian and a cocartesian fibration. Then the commutative square of ∞-groupoids

$$\begin{tikzcd} \|\mathcal{F}\| \arrow[r] \arrow[d, "\|q\|"'] & \|\mathcal{E}\| \arrow[d, "\|p\|"] \\ \|\mathcal{C}\| \arrow[r] & \|\mathcal{B}\| \end{tikzcd}$$

is a pullback.

Proof. If p is both a cocartesian and a cartesian fibration, then it follows from Corollary 6.3.7 that its cocartesian straigthening takes every morphism in $\mathcal{B}$ to a left adjoint. The localization of a left adjoint is always an equivalence (since the unit and counit transformations localize to natural equivalences), so the hypothesis of Corollary 7.7.10 is satisfied. □

As another application of these ideas we can prove an ∞-categorical version of Quillen's "Theorem B":

Corollary 7.7.12 (Quillen's "Theorem B")**.** *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a functor of ∞-categories such that for every map $f\colon b \to b'$ in $\mathcal{B}$, the morphism of ∞-groupoids*

$$\|\mathcal{E}_{/b}\| \to \|\mathcal{E}_{/b'}\|,$$

given by composition with f, is an equivalence. Then the fibre of $\|p\|$ at $b \in \mathcal{B}$ is equivalent to $\|\mathcal{E}_{/b}\|$.

Proof. We have a pullback square

$$\begin{tikzcd} \mathcal{E}_{/b} \arrow[r] \arrow[d] & \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \arrow[d, "\mathfrak{F}_{\mathrm{coct}}(p)"] \\ \{b\} \arrow[r] & \mathcal{B} \end{tikzcd}$$

and our assumption implies that the hypotheses of Corollary 7.7.10 hold for the free cocartesian fibration on p, so this gives a pullback square on localizations. In other words, the fibre of $\|\mathfrak{F}_{\mathrm{coct}}(p)\|$ at b is $\|\mathcal{E}_{/b}\|$. Moreover, by (the dual of) Proposition 6.4.7, the covariant equivalence $p \to L^{\ell}_{\mathcal{B}}(p) = \mathfrak{F}_{\mathrm{coct}}(p)^{\ell}$ gives an equivalence on localizations, so this also identifies the fibre of $\|p\|$, as required. □

Chapter 8

Kan extensions

8.1 Left Kan extensions in ∞-groupoids

We have seen that if all colimits over diagrams of shape $\mathcal{K}$ exist in $\mathcal{C}$, then they give a left adjoint to the constant diagram functor

$$\mathrm{const}\colon \mathcal{C} \to \mathsf{Fun}(\mathcal{K}, \mathcal{C}),$$

which we can think of as given by restriction along the unique functor $\mathcal{K} \to *$. We now want to construct *left* and *right Kan extensions*, which will give left and right adjoints to more general functors of the form

$$F^*\colon \mathsf{Fun}(\mathcal{L}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{K}, \mathcal{C})$$

for some $F\colon \mathcal{K} \to \mathcal{L}$. We start, as a warm-up, by giving a fibrational construction of left Kan extensions for functors to ∞-groupoids in this section. Then we consider right Kan extensions in ∞-groupoids and finally we use these to obtain general Kan extensions, using the Yoneda embedding.

Observation 8.1.1. If all pullbacks along a morphism $f\colon x \to y$ exist in $\mathcal{C}$, then they give a right adjoint f^* to the functor $f_!\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ given by composition with f. Indeed for $p\colon z \to x$ and $q\colon w \to y$, we have a natural commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{/x}(p, f^*q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(z, x \times_y w) \arrow[r] \arrow[d, "(f^*q)_*"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(z, w) \arrow[d, "q_*"] \\
\{p\} \arrow[r] & \mathcal{C}(z, x) \arrow[r, "f_*"] & \mathcal{C}(z, y)
\end{tikzcd}$$

where the pullback in the composite square is also $\mathcal{C}_{/y}(fp, q)$.

Proposition 8.1.2. *For a functor $f\colon \mathcal{A} \to \mathcal{B}$, the functor $f^*\colon \mathsf{LFib}(\mathcal{B}) \to \mathsf{LFib}(\mathcal{A})$ given by pullback along f has a left adjoint $f_!$, which takes a left fibration p over $\mathcal{A}$ to $\mathfrak{F}_{\mathrm{coct}}(fp)^{\ell}$.*

Proof. We saw in Lemma 6.4.2 that $\mathfrak{F}_{\mathrm{coct}}(-)^{\ell}$ is left adjoint to the inclusion $\mathsf{LFib}(\mathcal{B}) \hookrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}$. For $q \in \mathsf{LFib}(\mathcal{B})$ we thus have natural equivalences

$$\mathsf{Map}_{/\mathcal{B}}(\mathfrak{F}_{\mathrm{coct}}(fp)^{\ell}, q) \simeq \mathsf{Map}_{/\mathcal{B}}(fp, q) \simeq \mathsf{Map}_{/\mathcal{A}}(p, f^{*}q),$$

using Observation 8.1.1. This completes the proof by Corollary 6.3.7. □

Corollary 8.1.3. *For any functor $f\colon \mathcal{A} \to \mathcal{B}$, the functor $f^{*}\colon \mathsf{Fun}(\mathcal{B}, \mathsf{Gpd}_{\infty}) \to \mathsf{Fun}(\mathcal{A}, \mathsf{Gpd}_{\infty})$ given by composition with f has a left adjoint $f_!$, which takes a functor ϕ to one given by*

$$b \mapsto \operatorname*{colim}_{(a, fa \to b) \in \mathcal{A}_{/b}} \phi(a).$$

Proof. The existence of $f_!$ follows by straightening from Proposition 8.1.2. It remains to identify the functor $f_!\phi$. If $p\colon \mathcal{E} \to \mathcal{A}$ is the left fibration for ϕ, then $f_!\phi$ corresponds to

$$b \mapsto \|\mathcal{E}_{/b}\|.$$

Here $\mathcal{E}_{/b} \to \mathcal{A}_{/b}$ is the left fibration for the composite

$$\mathcal{A}_{/b} \to \mathcal{A} \xrightarrow{\phi} \mathsf{Gpd}_{\infty},$$

so by Proposition 5.2.3 we have an equivalence $\|\mathcal{E}_{/b}\| \simeq \operatorname{colim}_{\mathcal{A}_{/b}} \phi$ as required. □

Exercise 8.1. Show that if $q\colon \mathcal{F} \to \mathcal{E}$ and $p\colon \mathcal{E} \to \mathcal{B}$ are cocartesian fibrations, then so is $pq\colon \mathcal{F} \to \mathcal{B}$; a morphism in $\mathcal{F}$ is pq-cocartesian if and only if it is q-cocartesian over a p-cocartesian morphism in $\mathcal{E}$.

Observation 8.1.4. Suppose $f\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. Then we can instead compute $f_!$ on left fibrations as $p \mapsto (fp)^{\ell}$, since fp is a cocartesian fibration by Exercise 8.1 and we have

$$\mathsf{Map}_{/\mathcal{B}}((fp)^{\ell}, q) \simeq \mathsf{Map}^{\mathrm{coct}}_{/\mathcal{B}}(fp, q) \simeq \mathsf{Map}_{/\mathcal{B}}(fp, q) \simeq \mathsf{Map}_{/\mathcal{E}}(p, f^{*}q),$$

where the second equivalence uses that q is a left fibration. It follows that on functors we can compute left Kan extensions along a cocartesian fibration f by

$$f_!\phi(b) \simeq \operatorname{colim}_{e \in \mathcal{E}_b} \phi(e).$$

Exercise 8.2. Show that for a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{B}$, the inclusion $\mathcal{E}_b \to \mathcal{E}_{/b}$ is a right adjoint (with the left adjoint taking $(e, pe \xrightarrow{f} b)$ to the cocartesian transport $f_!e \in \mathcal{E}_b$); it is therefore cofinal by Proposition 6.5.14.

8.2 Right Kan extensions in ∞-groupoids

We now want to construct right Kan extensions for functors to Gpd_∞. This is a bit less straightforward than for left Kan extensions, since there is in general no right adjoint to pullbacks in Cat_∞. We will show, using our work on weighted colimits, that the presheaf

$$(\mathcal{P} \to \mathcal{B}) \;\mapsto\; \mathsf{Map}_{/\mathcal{A}}(\mathcal{A} \times_{\mathcal{B}} \mathcal{P}, \mathcal{Q})$$

on $\mathsf{RFib}(\mathcal{B})$ is representable for any functor $f\colon \mathcal{A} \to \mathcal{B}$ and $\mathcal{Q} \in \mathsf{RFib}(\mathcal{A})$. We start with some observations on the functor $(-)^r$ from Observation 6.4.1.

Observation 8.2.1. For a right fibration $p\colon \mathcal{E} \to \mathcal{B}$, the counit map

$$\epsilon_p^r\colon \mathfrak{F}_{\mathrm{cart}}(p)^r \to p$$

is an equivalence. This follows from Proposition 6.3.10 since $\mathfrak{F}_{\mathrm{cart}}(-)^r$ is left adjoint to the fully faithful inclusion $\mathsf{RFib}(\mathcal{B}) \hookrightarrow \mathsf{Cat}_{\infty/\mathcal{B}}$ by Lemma 6.4.2. We can also see this from (the dual of) Exercise 8.2 since on the fibre over b the map

$$\|\mathcal{E}_{b/}\| \to \mathcal{E}_b$$

comes from the cartesian transport map, which is right adjoint to the inclusion $\mathcal{E}_b \to \mathcal{E}_{b/}$; it is therefore cofinal, and so gives an equivalence on localizations by Proposition 6.5.16.

Observation 8.2.2. For a functor $f\colon \mathcal{A} \to \mathcal{B}$, we have a commutative square

$$\begin{tikzcd}
\mathsf{RFib}(\mathcal{B}) \arrow[r, hook] \arrow[d, "f^*"'] & \mathsf{Cart}(\mathcal{B}) \arrow[d, "f^*"] \\
\mathsf{RFib}(\mathcal{A}) \arrow[r, hook] & \mathsf{Cart}(\mathcal{A}),
\end{tikzcd}$$

which induces a canonical natural transformation (a *mate* or *Beck–Chevalley* transformation)

$$(f^*-)^r \to f^*(-)^r.$$

This is a natural equivalence, since it is clearly an equivalence on all fibres over $\mathcal{A}$.

Combining these observations, we see:

Lemma 8.2.3. *For $p\colon \mathcal{P} \to \mathcal{B}$ in $\mathsf{RFib}(\mathcal{B})$ and $f\colon \mathcal{A} \to \mathcal{B}$, the map*

$$(f^*\epsilon_p)^r\colon (f^*\mathfrak{F}_{\mathrm{cart}}(p))^r \to f^*p$$

is an equivalence. □

This means that for $\mathcal{P} \in \mathsf{RFib}(\mathcal{B})$, $\mathcal{Q} \in \mathsf{RFib}(\mathcal{A})$ and $f : \mathcal{A} \to \mathcal{B}$, we have a natural equivalence

$$\begin{aligned} \mathsf{Map}_{/\mathcal{A}}(\mathcal{A} \times_{\mathcal{B}} \mathcal{P}, \mathcal{Q}) \simeq \mathsf{Map}_{/\mathcal{A}}((\mathcal{A} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{P})^r, \mathcal{Q}) \\ \mathsf{Map}_{/\mathcal{A}}(\mathcal{A} \times_{\mathcal{B}} \mathsf{Ar}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{P}, \mathcal{Q}). \end{aligned}$$

Here we are pulling back $\mathcal{P}$ using the bifibration $(f, \mathrm{id})^*\mathsf{Ar}(\mathcal{B})$; we can therefore apply Observation 7.3.2 and conclude:

Proposition 8.2.4. *For $\mathcal{Q} \to \mathcal{A}$ in $\mathsf{RFib}(\mathcal{A})$ corresponding to $\phi \in \mathsf{PSh}(\mathcal{A})$, the presheaf*

$$(\mathcal{P} \to \mathcal{B}) \;\mapsto\; \mathsf{Map}_{/\mathcal{A}}(\mathcal{A} \times_{\mathcal{B}} \mathcal{P}, \mathcal{Q})$$

on $\mathsf{RFib}(\mathcal{B})$ is represented by the right fibration for the presheaf

$$b \mapsto \mathsf{Map}_{/\mathcal{A}}(\mathcal{A}_{/b}, \mathcal{Q}) \simeq \lim_{(\mathcal{A}_{/b})^{\mathrm{op}}} \phi$$

on $\mathcal{B}$. □

Corollary 8.2.5. *For any $f : \mathcal{A} \to \mathcal{B}$, the functor $f^* : \mathsf{Fun}(\mathcal{B}, \mathsf{Gpd}_\infty) \to \mathsf{Fun}(\mathcal{A}, \mathsf{Gpd}_\infty)$, given by composition with f, has a right adjoint*

$$f_* : \mathsf{Fun}(\mathcal{A}, \mathsf{Gpd}_\infty) \to \mathsf{Fun}(\mathcal{A}, \mathsf{Gpd}_\infty)$$

where for $\phi : \mathcal{A} \to \mathsf{Gpd}_\infty$ we have

$$(f_*\phi)(b) \simeq \lim_{\mathcal{A}_{b/}} \phi.$$

Proof. The functor f^* corresponds under straightening for right fibrations to $(f^{\mathrm{op}})^* : \mathsf{RFib}(\mathcal{B}) \to \mathsf{RFib}(\mathcal{A})$. This has a right adjoint by Proposition 8.2.4 and Corollary 6.3.7, which is given by the required formula (since $(\mathcal{A}_{b/})^{\mathrm{op}} \simeq (\mathcal{A}^{\mathrm{op}})_{/b}$). □

Observation 8.2.6. If $p : \mathcal{E} \to \mathcal{B}$ is a cartesian fibration, then the map $\mathcal{E}_b \to \mathcal{E}_{b/}$ is coinitial for all $b \in \mathcal{B}$ by (the dual of) Exercise 8.2. We can therefore write the formula for right Kan extension along p as

$$(p_*\phi)(b) \simeq \lim_{\mathcal{E}_b} \phi$$

for a fixed b.

Exercise 8.3 ((★)). Unpack the definitions to see that the unit and counit of the adjunction $f^* \dashv f_*$ from Corollary 8.2.5 are given pointwise by the map

$$(f^*f_*\phi)(a) \simeq \lim_{\mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{f(a)/}} \phi \to \lim_{\mathcal{A}_{a/}} \phi \simeq \phi(a),$$

induced by the functor $\mathcal{A}_{a/} \to \mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{f(a)/}$, and

$$\psi(b) \simeq \lim_{\mathcal{B}_{/b}} \psi \to \lim_{a \in \mathcal{A}_{/b}} \psi(fa) \simeq (f_*f^*\psi)(b),$$

induced by the projection $\mathcal{A}_{/b} \to \mathcal{B}_{/b}$. Conclude (via Corollary 6.3.11) that if f is fully faithful then so is f_*.

8.3 Kan extensions in general

Proposition 8.3.1. *For a functor $f\colon \mathcal{A} \to \mathcal{B}$ and an ∞-category $\mathcal{C}$, we have the following for the functor*

$$f^*\colon \mathsf{Fun}(\mathcal{B}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{C})$$

given by composition with f:

(1) If for every functor $\phi\colon \mathcal{A} \to \mathcal{C}$ and $b \in \mathcal{B}$ the limit $\lim_{\mathcal{A}_{b/}} \phi$ exists in $\mathcal{C}$, then f^* has a right adjoint f_*, given by the formula

$$(f_*\phi)(b) \simeq \lim_{\mathcal{A}_{/b}} \phi \simeq \lim_{\mathcal{A}}^{\mathcal{B}(b,f(-))} \phi.$$

(2) If for every functor $\phi\colon \mathcal{A} \to \mathcal{C}$ and $b \in \mathcal{B}$ the colimit $\operatorname{colim}_{\mathcal{A}_{/b}} \phi$ exists in $\mathcal{C}$, then f^* has a left adjoint $f_!$, given by the formula

$$(f_!\phi)(b) \simeq \operatorname{colim}_{\mathcal{A}_{/b}} \phi \simeq \operatorname{colim}_{\mathcal{A}}^{\mathcal{B}(f(-),b)} \phi.$$

Exercise 8.4. Suppose we have a commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[d, "F"'] & \mathcal{C}' \arrow[d, "F'"] \\ \mathcal{D} \arrow[r, "j"'] & \mathcal{D}' \end{tikzcd}$$

where i and j are fully faithful. Show that if F' has a right adjoint G' such that $G'j$ factors through $\mathcal{C}$, then this gives a right adjoint to F.

Exercise 8.5. Show that if $F \dashv G$, then $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$.

Proof of Proposition 8.3.1. We first consider right Kan extensions. From the Yoneda embedding $\mathsf{y}_{\mathcal{C}}$ we get a commutative diagram

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{B}, \mathcal{C}) \arrow[r, hook] \arrow[d, "f^*"'] & \mathsf{Fun}(\mathcal{B}, \mathsf{PSh}(\mathcal{C})) \arrow[d, "f^*"] \\ \mathsf{Fun}(\mathcal{A}, \mathcal{C}) \arrow[r, hook] & \mathsf{Fun}(\mathcal{A}, \mathsf{PSh}(\mathcal{C})) \end{tikzcd}$$

Here we can identify the right vertical functor as

$$(f^*)_*\colon \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Fun}(\mathcal{B}, \mathsf{Gpd}_\infty)) \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Fun}(\mathcal{A}, \mathsf{Gpd}_\infty)),$$

which has a right adjoint $(f_*)_*$ by Lemma 6.3.8 and Corollary 8.2.5. From Exercise 8.4 it follows that our original map f^* for functors to $\mathcal{C}$ will have a right adjoint if $(f_*)_*$ preserves the image of the Yoneda embedding for $\mathcal{C}$. In other words, given $\phi\colon \mathcal{A} \to \mathcal{C}$, we need

$$c \mapsto (f_*\mathcal{C}(c, \phi))\,(b) \simeq \lim_{\mathcal{A}_{b/}} \mathcal{C}(c, \phi)$$

to be a representable presheaf for all $b \in \mathcal{B}$. By Corollary 7.1.9, this is true if and only if the limit $\lim_{\mathcal{A}_{b/}} \phi$ exists in $\mathcal{C}$.

To deduce the statement for left Kan extensions, we use Exercise 8.5 and identify $(f^*)^{\mathrm{op}}$ as

$$(f^{\mathrm{op}})^*\colon \mathsf{Fun}(\mathcal{B}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}) \to \mathsf{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}});$$

applying our result for right Kan extensions to f^{op} and $\mathcal{C}^{\mathrm{op}}$ now gives the required statement. □

Observation 8.3.2. More generally, if $\mathcal{C}$ does not admit the required limits for all functors from $\mathcal{A}$, the argument above still shows that if for a fixed $\phi\colon \mathcal{A} \to \mathcal{C}$ and $f\colon \mathcal{A} \to \mathcal{B}$, the limit $\lim_{\mathcal{A}_{b/}} \phi$ exists in $\mathcal{C}$, then there is a functor $f_*\phi$ given by these limits, such that

$$\mathsf{Nat}_{\mathcal{B},\mathcal{C}}(-, f_*\phi) \simeq \mathsf{Nat}_{\mathcal{A},\mathcal{C}}(f^*-, \phi).$$

We call this the (*pointwise*) *right Kan extension* of ϕ along f.

Warning 8.3.3. In general, we might say that any functor $f_*\phi$ that represents the presheaf $\mathsf{Nat}_{\mathcal{A},\mathcal{C}}(\phi, f^*-)$ on $\mathsf{Fun}(\mathcal{B}, \mathcal{C})$ is a *right Kan extension* of ϕ along f. It is possible for such a right Kan extension to exist even if $\mathcal{C}$ does not have the requsite limits over $\mathcal{A}_{b/}$. Such non-pointwise Kan extensions should be considered as rather pathological, however.

Lemma 8.3.4. *Suppose $F\colon \mathcal{A} \to \mathcal{B}$ is fully faithful. Then the left and right Kan extension functors $F_!, F_*\colon \mathsf{Fun}(\mathcal{A}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{B}, \mathcal{C})$ are also fully faithful, when they exist via the construction of Proposition 8.3.1.*

Proof. It suffices to consider the case of right Kan extensions for functors to Gpd_∞, where by Proposition 6.3.10 we can equivalently show that the counit map $f^*f_*\phi \to \phi$ is an equivalence. At $a \in \mathcal{A}$ this unpacks to the map

$$\lim_{\mathcal{A}_{f(a)/}} \phi \to \lim_{\mathcal{A}_{a/}} \phi,$$

which is an equivalence since $\mathcal{A}_{a/} \simeq \mathcal{A}_{f(a)/}$ for f fully faithful, followed by identifying the right-hand side with $\phi(a)$. □

Observation 8.3.5. More generally, if $F\colon \mathcal{A} \to \mathcal{B}$ is fully faithful, we see that $F^*\colon \mathsf{Fun}(\mathcal{B}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{A}, \mathcal{C})$ restricts to an equivalence

$$\mathsf{Fun}(\mathcal{B}, \mathcal{C})^{\text{is-Kan}} \to \mathsf{Fun}(\mathcal{A}, \mathcal{C})^{\text{has-Kan}}$$

between the full subcategory of functors $\mathcal{B} \to \mathcal{C}$ that are left (or right) Kan extended from their restrictions to $\mathcal{A}$, and that of functors $\mathcal{A} \to \mathcal{C}$ that admit a left (or right) Kan extension along F.

8.4 The universal property of presheaves

We can now prove the universal property of the ∞-category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a small ∞-category $\mathcal{C}$, namely that it is the free cocomplete ∞-category on $\mathcal{C}$.

Proposition 8.4.1. *Suppose $\mathcal{C}$ is a small ∞-category and $\mathcal{D}$ is a cocomplete ∞-category.*

(i) For any functor $F\colon \mathcal{C} \to \mathcal{D}$, the left Kan extension $\mathsf{y}_!F\colon \mathsf{PSh}(\mathcal{C}) \to \mathcal{D}$ of F along the Yoneda embedding y exists and preserves colimits.

(ii) For every colimit-preserving functor $G\colon \mathsf{PSh}(\mathcal{C}) \to \mathcal{D}$, the counit map $\mathsf{y}_!\mathsf{y}^*G \to G$ is an equivalence.

Lemma 8.4.2. *For a diagram $W\colon \mathcal{I} \to \mathsf{PSh}(\mathcal{C})$, $V \in \mathsf{PSh}(\mathcal{I})$, and a functor $F\colon \mathcal{C} \to \mathcal{D}$, we have*

$$\operatorname{colim}_{\mathcal{C}}^{\operatorname{colim}_{\mathcal{I}}^{V} W} F \simeq \operatorname{colim}_{\mathcal{I}}^{V} \operatorname{colim}_{\mathcal{C}}^{W(i)} F,$$

if both sides exist in $\mathcal{D}$.

Proof. For $d \in \mathcal{D}$, we have natural equivalences

$$\begin{aligned}
\mathcal{D}(\operatorname{colim}_{\mathcal{C}}^{\operatorname{colim}_{\mathcal{I}}^{V} W} F, d) &\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\operatorname{colim}_{\mathcal{I}}^{V} W, \mathcal{D}(F,d)) \\
&\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{I})}(V, \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(W, \mathcal{D}(F,d))) \\
&\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{I})}(V, \mathcal{D}(\operatorname{colim}_{\mathcal{C}}^{W(-)} F, d)) \\
&\simeq \mathcal{D}(\operatorname{colim}_{\mathcal{I}}^{V} \operatorname{colim}_{\mathcal{C}}^{W(-)} F, d),
\end{aligned}$$

so these two objects corepresent the same functor on $\mathcal{D}$. $\square$

Proof of Proposition 8.4.1. By Proposition 8.3.1, the left Kan extension $\mathsf{y}_!F$ exists if the weighted colimit

$$\operatorname{colim}_{\mathcal{C}}^{\mathsf{Nat}(\mathsf{y}(-),\phi)} F$$

exists in $\mathcal{D}$ for every $\phi \in \mathsf{PSh}(\mathcal{C})$. By the Yoneda lemma the weight is equivalent to $\operatorname{colim}_{\mathcal{C}}^{\phi} F$; this colimit exists in $\mathcal{D}$ since it is equivalent to a colimit over the right fibration for ϕ, which is a small ∞-category. Thus $\mathsf{y}_!F$ exists; to see that it preserves colimits, we consider a diagram $\Phi\colon \mathcal{I} \to \mathsf{PSh}(\mathcal{C})$ and apply Lemma 8.4.2 to compute:

$$\mathsf{y}_!F(\operatorname{colim}_{\mathcal{I}} \Phi) \simeq \operatorname{colim}_{\mathcal{C}}^{\operatorname{colim}_{\mathcal{I}} \Phi} F \simeq \operatorname{colim}_{\mathcal{I}} \operatorname{colim}_{\mathcal{C}}^{\Phi(-)} F \simeq \operatorname{colim}_{\mathcal{I}} F(\Phi(-)).$$

Now suppose $G\colon \mathsf{PSh}(\mathcal{C}) \to \mathcal{D}$ preserves colimits. For $\phi \in \mathsf{PSh}(\mathcal{C})$, the counit map

$$\mathsf{y}_!(\mathsf{y}^*G)(\phi) \to G(\phi)$$

is the canonical map

$$\operatorname{colim}^{\phi}_{\mathcal{C}} G \circ \mathsf{y} \to G(\operatorname{colim}^{\phi}_{\mathcal{C}} \mathsf{y}),$$

which is an equivalence since G preserves small colimits. $\square$

Corollary 8.4.3. *For a small ∞-category $\mathcal{C}$ and a cocomplete ∞-category $\mathcal{D}$, the restriction functor*

$$\mathsf{y}^* \colon \mathsf{Fun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

has a fully faithful left adjoint $\mathsf{y}_!$, and the adjunction restricts to an equivalence between $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ and the full subcategory of colimit-preserving functors $\mathsf{PSh}(\mathcal{C}) \to \mathcal{D}$.

Proof. From Lemma 8.3.4 we know that $\mathsf{y}_!$ is fully faithful, since the Yoneda embedding so. It follows from Proposition 8.4.1 that $\mathsf{y}_!$ takes values in the full subcategory of colimit-preserving functors, and also that every such functor is in its image, so this must be precisely the image of $\mathsf{y}_!$. $\square$

Proposition 8.4.4. *For a functor $F \colon \mathcal{C} \to \mathcal{D}$ among small ∞-categories, we have*

$$\mathsf{y}_{\mathcal{C},!}(\mathsf{y}_{\mathcal{D}} \circ F) \simeq F^{\mathrm{op}}_!,$$

so that there is a commutative square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\mathsf{y}_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\mathsf{y}_{\mathcal{D}}"] \\
\mathsf{PSh}(\mathcal{C}) \arrow[r, "F^{\mathrm{op}}_!"'] & \mathsf{PSh}(\mathcal{D}).
\end{tikzcd}$$

Proof. The left Kan extension functor $F^{\mathrm{op}}_!$ is a left adjoint, and so preserves colimits by Exercise 7.1. By Corollary 8.4.3 it therefore suffices to identify $F^{\mathrm{op}}_! \circ \mathsf{y}_{\mathcal{C}}$ with $\mathsf{y}_{\mathcal{D}} \circ F$. To see this we compute

$$\begin{aligned}
\mathsf{Map}_{\mathsf{PSh}(\mathcal{D})}(F^{\mathrm{op}}_! \mathsf{y}_{\mathcal{C}}(-), \Phi) &\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}_{\mathcal{C}}(-), \Phi \circ F^{\mathrm{op}}) \\
&\simeq \Phi \circ F^{\mathrm{op}} \\
&\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{D})}(\mathsf{y}_{\mathcal{D}}(-), \Phi) \circ F^{\mathrm{op}} \\
&\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{D})}(\mathsf{y}_{\mathcal{D}} \circ F, \Phi)
\end{aligned}$$

for $\Phi \in \mathsf{PSh}(\mathcal{D})$. $\square$

8.5 Retracts and idempotents

Definition 8.5.1. Let Ret denote the ordinary category with two objects $0, 1$ and morphisms generated by $f \colon 0 \to 1$, $r \colon 1 \to 0$ such that $rf = \mathrm{id}_0$. (In other words, Ret is the universal category containing a retract.) The non-identity morphisms of Ret are thus r, f and $i = fr$ (which is an *idempotent* in that $i^2 = i$). Let Idem be the full subcategory of Ret on the object 1; this is the universal category containing an idempotent.

Fact 8.5.2. *The commutative square*

$$\begin{tikzcd} \{0<2\} \arrow[r] \arrow[d] & {[2]} \arrow[d] \\ * \arrow[r] & \mathsf{Ret} \end{tikzcd}$$

is a pushout of ∞-categories. (In other words, Ret *is also the universal ∞-category containing a retract.)*

Warning 8.5.3. In ordinary categories, we can split an idempotent $i\colon x \to x$ (that is, extend it to a retract diagram) by taking a *finite* colimit, e.g. the coequalizer of i and id_x. This is *not* true in ∞-categories, as this finite colimit does not take into account the higher coherence of the idempotent (e.g. if $i^2 \simeq i$ then we get a priori two different equivalences $i^3 \simeq i$, but for an Idem-shaped diagram they are the same). The following proposition says that *sequential* colimits are enough to get splittings:

Proposition 8.5.4. *Define a functor $F\colon \mathbb{N} \to \mathsf{Idem}$, where $\mathbb{N}$ is the partially ordered set of natural numbers, by $F(n) = 1$, $F(n \to n+1) = i$. Then F is cofinal.*

Proof. We must show that $\mathbb{N} \times_{\mathsf{Idem}} \mathsf{Idem}_{1/}$ is weakly contractible. This is an ordinary category with objects $(n, \epsilon) = (n, 1 \xrightarrow{\epsilon} F(n))$ where $\epsilon = \mathrm{id}$ or i, and morphisms

$$\mathsf{Hom}((n,\epsilon),(n',\epsilon')) = \begin{cases} *, & n < n' \text{ and } \epsilon' = i, \text{ or } n = n' \text{ and } \epsilon = \epsilon' \\ \emptyset, & \text{otherwise.} \end{cases}$$

In other words, $\mathbb{N} \times_{\mathsf{Idem}} \mathsf{Idem}_{1/}$ is a partially ordered set where $(n,\epsilon) \leq (n',\epsilon')$ if either $n \leq n'$ and $\epsilon = \epsilon' = i$ or $n < n'$ and $\epsilon = \mathrm{id}, \epsilon' = i$. We can depict this poset as

$$\begin{tikzcd} (n,\mathrm{id}) \arrow[dr] & (n,i) \arrow[d] \\ (n+1,\mathrm{id}) \arrow[dr] & (n+1,i) \arrow[d] \\ (n+2,\mathrm{id}) \arrow[dr] & (n+2,i) \arrow[d] \\ \cdots & \cdots. \end{tikzcd}$$

Here the copy of $\mathbb{N}$ given by $\{(n,i)\}$ is clearly cofinal, since $\mathbb{N} \times_{(\mathbb{N} \times_{\mathsf{Idem}} \mathsf{Idem}_{1/})} (\mathbb{N} \times_{\mathsf{Idem}} \mathsf{Idem}_{1/})_{(n,\epsilon)/}$ has an initial object for all (n,ϵ). Here $\mathbb{N}$ is weakly contractible as it has an initial object, so this completes the proof. □

Corollary 8.5.5. Idem *is weakly contractible.* □

Corollary 8.5.6. $\mathsf{Idem} \hookrightarrow \mathsf{Ret}$ *is cofinal.*

Proof. We must show that $\mathsf{Idem}_{0/}$ and $\mathsf{Idem}_{1/}$ are weakly contractible. The latter has an initial object, so this is weakly contractible by Observation 6.5.12. For $\mathsf{Idem}_{0/}$, the forgetful functor $\mathsf{Idem}_{0/} \to \mathsf{Idem}$ is an equivalence, and so is weakly contractible by Corollary 8.5.5. ◻

Corollary 8.5.7. *Every functor* $\mathsf{Ret} \to \mathcal{C}$ *is (pointwise) left Kan extended from* Idem*, and such a left Kan extension exists for a functor* $\mathsf{Idem} \to \mathcal{C}$ *if and only if it has a colimit.*

Proof. We must show that $\mathsf{Idem}^{\triangleright}_{/0} \to \mathsf{Ret}^{\triangleright}_{/0} \to \mathsf{Ret} \to \mathcal{C}$ is a colimit cocone. This follows because we have a commutative square

$$\begin{tikzcd} \mathsf{Idem}_{/0} \arrow[r] \arrow[d] & \mathsf{Ret}_{/0} \arrow[d] \\ \mathsf{Idem} \arrow[r] & \mathsf{Ret} \end{tikzcd}$$

where the vertical maps are equivalences and the bottom horizontal map is cofinal by Corollary 8.5.6, and the cocone $\mathsf{Ret}^{\triangleright}_{/0} \to \mathcal{C}$ is clearly a colimit as $\mathsf{Ret}_{/0}$ has a terminal object. Moreover, this argument shows that the left Kan extension of a functor from Idem exists precisely when the functor $\mathsf{Idem}_{/0} \xrightarrow{\sim} \mathsf{Idem} \to \mathcal{C}$ has a colimit. ◻

Combining this with Observation 8.3.5, we get:

Corollary 8.5.8. $\mathsf{Fun}(\mathsf{Ret}, \mathcal{C}) \to \mathsf{Fun}(\mathsf{Idem}, \mathcal{C})$ *is fully faithful, with image those functors* $\mathsf{Idem} \to \mathcal{C}$ *that have a colimit in* $\mathcal{C}$*.* ◻

Corollary 8.5.9. *If a functor* $\phi\colon \mathsf{Idem} \to \mathcal{C}$ *has a colimit in* $\mathcal{C}$*, then this is preserved by any functor* $\mathcal{C} \xrightarrow{F} \mathcal{D}$*.*

Proof. If ϕ has a colimit, then it extends uniquely to a functor $\phi'\colon \mathsf{Ret} \to \mathcal{C}$. The composition of ϕ' with F is again left Kan extended from Idem, so that the colimit of ϕ is indeed preserved. ◻

Definition 8.5.10. An idempotent in an ∞-category $\mathcal{C}$ is *split* if it is in the image of $\mathsf{Fun}(\mathsf{Ret}, \mathcal{C})$. We say that $\mathcal{C}$ is *idempotent-complete* if every idempotent is split.

Observation 8.5.11. An ∞-category $\mathcal{C}$ is idempotent-complete if and only if every functor $\mathsf{Idem} \to \mathcal{C}$ has a colimit. By Proposition 8.5.4 this is true if $\mathcal{C}$ has sequential colimits, i.e. colimits over the poset $\mathbb{N}$.

We would like to say that any ∞-category has an *idempotent-completion*, obtained by freely splitting its idempotents. We will prove this in the next section by reformulating this in terms of *absolute* colimits.

8.6 Absolute colimits and idempotent-completion

Definition 8.6.1. A presheaf $W\colon \mathcal{K}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ is an *absolute weight* if a W-weighted colimit in an ∞-category $\mathcal{C}$ is preserved by any functor $\mathcal{C} \to \mathcal{D}$.

Observation 8.6.2. $W \in \mathsf{PSh}(\mathcal{K})$ is an absolute weight if and only if for any functors $\phi\colon \mathcal{K} \to \mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{D}$, such that $\operatorname{colim}^W_{\mathcal{K}} \phi$ exists in $\mathcal{C}$, the object $F(\operatorname{colim}^W_{\mathcal{K}} \phi)$ satisfies

$$\mathcal{D}(F(\operatorname{colim}^W_{\mathcal{K}} \phi), d) \simeq \lim^W_{\mathcal{K}^{\mathrm{op}}} \mathcal{D}(\phi, d)$$

for all $d \in \mathcal{D}$. This holds if and only if the functor $\mathcal{D}(F(-), d)$ preserves the weighted colimit. To show that W is absolute it therefore suffices to show that the colimit is preserved by any functor $\mathcal{C} \to \mathsf{Gpd}_\infty^{\mathrm{op}}$.

Definition 8.6.3. Let $\mathcal{C}$ be a cocomplete ∞-category. An object c in $\mathcal{C}$ is *absolute* if $\mathcal{C}(c, -)$ preserves all colimits.

Remark 8.6.4. Absolute objects are also known as *completely compact* and *tiny* objects.

Lemma 8.6.5. *A presheaf $W \in \mathsf{PSh}(\mathcal{K})$ is an absolute weight if and only if it is an absolute object of $\mathsf{PSh}(\mathcal{K})$.*

Proof. First suppose W is an absolute weight. Since $W \simeq \operatorname{colim}^W_{\mathcal{K}} \mathsf{y}$, for a colimit $\operatorname{colim}_{\mathcal{I}} \phi$ in $\mathsf{PSh}(\mathcal{K})$ we have

$$\begin{aligned}
\mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(W, \operatorname{colim}_{\mathcal{I}} \phi) &\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(\operatorname{colim}^W_{\mathcal{C}} \mathsf{y}, \operatorname{colim}_{\mathcal{I}} \phi) \\
&\simeq \lim^W_{\mathcal{C}^{\mathrm{op}}} \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(\mathsf{y}, \operatorname{colim}_{\mathcal{I}} \phi) \\
&\simeq \lim^W_{\mathcal{C}^{\mathrm{op}}} \operatorname{colim}_{\mathcal{I}} \phi \\
&\simeq \lim^W_{\mathcal{C}^{\mathrm{op}}} \operatorname{colim}_{\mathcal{I}} \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(\mathsf{y}, \phi) \\
&\simeq \operatorname{colim}_{\mathcal{I}} \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(\operatorname{colim}^W_{\mathcal{C}} \mathsf{y}, \phi) \\
&\simeq \operatorname{colim}_{\mathcal{I}} \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(W, \phi),
\end{aligned}$$

where we have used that the functor $\operatorname{colim}_{\mathcal{I}} \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(-, \phi)$ must preserve the W-weighted colimit. Thus W is an absolute object.

Now suppose W is an absolute object in $\mathsf{PSh}(\mathcal{K})$, and that $\phi\colon \mathcal{K} \to \mathcal{C}$ is a functor such that $\operatorname{colim}^W_{\mathcal{K}} \phi$ exists in $\mathcal{C}$. By Observation 8.6.2 we want to show that for any presheaf ψ on $\mathcal{C}$, we have

$$\psi(\operatorname{colim}^W_{\mathcal{K}} \phi) \simeq \lim^W_{\mathcal{K}^{\mathrm{op}}} \psi(\phi).$$

Here the right-hand side is equivalently $\mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(W, \phi^{\mathrm{op},*}\psi)$. Since $\psi \simeq \operatorname{colim}^{\psi}_{\mathcal{C}} \mathsf{y}$, $\phi^{\mathrm{op},*}$ preserves colimits, and W is absolute, we can rewrite this as

$$\begin{aligned}
\mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(W, \phi^{\mathrm{op},*}\psi) &\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(W, \operatorname{colim}^{\psi}_{\mathcal{C}} \phi^{\mathrm{op},*}\mathsf{y}_{\mathcal{C}}) \\
&\simeq \operatorname{colim}^{\psi}_{\mathcal{C}} \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(W, \mathcal{C}(\phi, -)) \\
&\simeq \operatorname{colim}^{\psi}_{\mathcal{C}} \mathcal{C}(\operatorname{colim}^{W}_{\mathcal{K}} \phi, -) \\
&\simeq (\operatorname{colim}^{\psi}_{\mathcal{C}} \mathsf{y}_{\mathcal{C}})(\operatorname{colim}^{W}_{\mathcal{K}} \phi) \\
&\simeq \psi(\operatorname{colim}^{W}_{\mathcal{K}} \phi)
\end{aligned}$$

as required, where the third equivalence uses the definition of $\operatorname{colim}^{W}_{\mathcal{K}} \phi$. ◻

We can now simply say that a presheaf is *absolute* if it satisfies the equivalent conditions of Lemma 8.6.5.

Example 8.6.6. The representable presheaf $\mathcal{K}(-, k)$ is absolute.

Example 8.6.7. The terminal weight for Idem is absolute by Corollary 8.5.9.

Lemma 8.6.8. *Absolute objects are closed under absolute colimits. In other words, if $W \in \mathsf{PSh}(\mathcal{K})$ is an absolute weight, and $\phi\colon \mathcal{K} \to \mathcal{C}$ is a functor such that $\phi(k)$ is an absolute object of $\mathcal{C}$, then the colimit $\operatorname{colim}^{W}_{\mathcal{K}} \phi$ is absolute, if it exists.*

Proof. We have

$$\mathcal{C}(\operatorname{colim}^{W}_{\mathcal{K}} \phi, -) \simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(W, \mathcal{C}(\phi, -)).$$

Here $\mathcal{C}(\phi, -)\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{K})$ preserves colimits since these are computed pointwise and the values of ϕ are absolute, hence so does the composite with $\mathsf{Map}_{\mathsf{PSh}(\mathcal{K})}(W, -)$ since W is an absolute presheaf by Lemma 8.6.5. ◻

Proposition 8.6.9. *A weight $W \in \mathsf{PSh}(\mathcal{K})$ is absolute if and only if it is a retract of a representable presheaf.*

Observation 8.6.10. Suppose $\mathcal{C}$ is an ordinary category, and $F\colon \mathcal{K} \to \mathcal{C}$ is a functor, where $\mathcal{K}$ is an ∞-category. Then F factors as $\mathcal{K} \to h\mathcal{K} \xrightarrow{F'} \mathcal{C}$. We claim that the colimit of F is the same as that if F', if either exist. Indeed, since $\mathsf{Fun}(\mathcal{K}, \mathcal{C}) \simeq \mathsf{Fun}(h\mathcal{K}, \mathcal{C})$, the colimit of F represents the presheaf

$$\mathsf{Map}_{\mathsf{Fun}(\mathcal{K},\mathcal{C})}(F, \mathrm{const}_{(-)}) \simeq \mathsf{Map}_{\mathsf{Fun}(h\mathcal{K},\mathcal{C})}(F', \mathrm{const}_{(-)}).$$

Proof of Proposition 8.6.9. Suppose first that ϕ is an absolute presheaf. We know that $\phi \simeq \operatorname{colim}_{\mathcal{E}} \mathsf{y} \circ p$ where $p\colon \mathcal{E} \to \mathcal{K}$ is the right fibration for ϕ, so that

$$\mathsf{Map}(\phi, \phi) \simeq \operatorname{colim}_{\mathcal{E}} \mathsf{Map}(\phi, \mathsf{y} \circ p).$$

Here we have

$$\pi_0(\mathrm{colim}_{\mathcal{E}}\,\mathsf{Map}(\phi, \mathsf{y}\circ p)) \simeq \mathrm{colim}_{h\mathcal{E}}\,\pi_0\mathsf{Map}(\phi, \mathsf{y}\circ p),$$

since π_0 is a left adjoint and **Set** is an ordinary category. The right-hand side is a colimit of sets, so we can conclude that there exists some $e \in \mathcal{E}$ and $[f] \in \pi_0\mathsf{Map}(\phi, \mathsf{y}(p(e)))$ that maps to the component of $[\mathrm{id}_\phi]$ on the left-hand side. This means that the composite

$$\phi \xrightarrow{f} \mathsf{y}(p(e)) \to \phi$$

is equivalent to id_ϕ, i.e. ϕ is a retract of the representable presheaf $\mathsf{y}(p(e))$, as required.

Conversely, a retract of a representable is absolute since representable presheaves are absolute (Example 8.6.6), retracts are colimits over Idem (Corollary 8.5.8), these are absolute (Example 8.6.7), and absolute colimits of absolute presheaves are again absolute (Lemma 8.6.8). $\square$

Corollary 8.6.11. *An ∞-category has all absolute colimits if and only if it is idempotent-complete.*

Definition 8.6.12. For a small ∞-category $\mathcal{C}$, let $\mathcal{C}^{\mathrm{idem}}$ denote the full subcategory of $\mathsf{PSh}(\mathcal{C})$ spanned by the absolute presheaves, that is by the retracts of idempotents in $\mathcal{C}$.

Observation 8.6.13. It follows from Lemma 8.6.8 that $\mathcal{C}^{\mathrm{idem}}$ is closed under absolute colimits in $\mathsf{PSh}(\mathcal{C})$, and so it is in particular idempotent-complete. Moreover, $\mathcal{C}^{\mathrm{idem}}$ is a small ∞-category, since the ∞-category $\mathsf{Fun}(\mathsf{Idem}, \mathcal{C})$ of idempotents in $\mathcal{C}$ is small.

Proposition 8.6.14. *Suppose $\mathcal{D}$ is an idempotent-complete ∞-category. Then any functor $\mathcal{C} \to \mathcal{D}$ admits a left Kan extension to $\mathcal{C}^{\mathrm{idem}}$ and any functor $\mathcal{C}^{\mathrm{idem}} \to \mathcal{D}$ is left Kan extended from $\mathcal{C}$. In particular, any functor $\mathcal{C} \to \mathcal{D}$ factors uniquely through $\mathcal{C}^{\mathrm{idem}}$.*

Proof. For $\phi \in \mathcal{C}^{\mathrm{idem}}$ and $F \colon \mathcal{C} \to \mathcal{D}$, the colimit of

$$\mathcal{C}_{/\phi} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is equivalently the colimit of F weighted by ϕ, which exists in $\mathcal{D}$ since ϕ is absolute. Hence F has a left Kan extension to $\mathcal{C}^{\mathrm{idem}}$. Conversely, given a functor $G \colon \mathcal{C}^{\mathrm{idem}} \to \mathcal{D}$, the canonical map

$$\mathrm{colim}^{\phi}_{\mathcal{C}}\, G \to G(\mathrm{colim}^{\phi}_{\mathcal{C}}\, \mathsf{y}) \simeq G(\phi)$$

is an equivalence, since this is an absolute colimit. Thus G is left Kan extended from $\mathcal{C}$. $\square$

Definition 8.6.15. Proposition 8.6.14 gives for any functor $f\colon \mathcal{C} \to \mathcal{D}$ a unique commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathcal{C}^{\mathrm{idem}} \arrow[r, "f^{\mathrm{idem}}"'] & \mathcal{D}^{\mathrm{idem}}. \end{tikzcd}$$

We say that f is a *Morita equivalence* if f^{idem} is an equivalence of ∞-categories.

Observation 8.6.16. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to G. If $\mathcal{D}$ admits small colimits and G preserves these, then F preserves absolute objects, since for $x \in \mathcal{C}$ absolute we have

$$\mathcal{D}(F(x), \operatorname{colim} \phi) \simeq \mathcal{C}(x, \operatorname{colim} G\phi) \simeq \operatorname{colim} \mathcal{D}(F(x), \phi).$$

In particular, for any functor $f\colon \mathcal{C} \to \mathcal{D}$, the left Kan extension $(f^{\mathrm{op}})_!\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ preserves absolute objects, since its right adjoint $(f^{\mathrm{op}})^*$ preserves colimits. Since we moreover have a commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathsf{PSh}(\mathcal{C}) \arrow[r, "(f^{\mathrm{op}})_!"'] & \mathsf{PSh}(\mathcal{D}) \end{tikzcd}$$

from Proposition 8.4.4, it follows that f^{idem} must (by uniqueness) be the restriction of $(f^{\mathrm{op}})_!$ to absolute presheaves.

Proposition 8.6.17. *Suppose $\mathcal{M}$ is a cocomplete ∞-category and $\mathcal{C}$ is a small ∞-category. Then the unique colimit-preserving functor $F\colon \mathsf{PSh}(\mathcal{C}) \to \mathcal{M}$ extending a functor $i\colon \mathcal{C} \to \mathcal{M}$ is an equivalence if and only if the following conditions hold:*

(1) i is fully faithful.

(2) The objects $i(c)$ are absolute for all $c \in \mathcal{C}$.

(3) Every object of $\mathcal{M}$ is the colimit of a small diagram in $\mathcal{C}$.

Proof. It is clear that these conditions are necessary, so it remains to show that they imply that F is an equivalence. The last assumption implies that F is essentially surjective, so it remains to show that it is fully faithful. For this we compute

$$
\begin{aligned}
\mathcal{M}(F(\phi), F(\psi)) &\simeq \mathcal{M}(F(\operatorname{colim}^{\phi}_{\mathcal{C}} \mathsf{y}), F(\operatorname{colim}^{\psi}_{\mathcal{C}} \mathsf{y})) \\
&\simeq \lim\nolimits^{\phi}_{\mathcal{C}^{\mathrm{op}}} \mathcal{M}(i, \operatorname{colim}^{\psi}_{\mathcal{C}} i) \\
&\simeq \lim\nolimits^{\phi}_{\mathcal{C}^{\mathrm{op}}} \operatorname{colim}^{\psi}_{\mathcal{C}} \mathcal{C}(-,-) \\
&\simeq \lim\nolimits^{\phi}_{\mathcal{C}^{\mathrm{op}}} \operatorname{colim}^{\psi}_{\mathcal{C}} \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathsf{y}, \mathsf{y}) \\
&\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\operatorname{colim}^{\phi}_{\mathcal{C}} \mathsf{y}, \operatorname{colim}^{\psi}_{\mathcal{C}} \mathsf{y}) \\
&\simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\phi, \psi),
\end{aligned}
$$

using that i is fully faithful and takes values in absolute objects in $\mathcal{M}$. □

Corollary 8.6.18. *For any ∞-category $\mathcal{C}$, the inclusion $i\colon \mathcal{C} \hookrightarrow \mathcal{C}^{\mathrm{idem}}$ induces an equivalence $(i^{\mathrm{op}})^*\colon \mathsf{PSh}(\mathcal{C}^{\mathrm{idem}}) \xrightarrow{\sim} \mathsf{PSh}(\mathcal{C})$.*

Proof. The fully faithful inclusion $\mathcal{C}^{\mathrm{idem}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$ satisfies the assumptions of Proposition 8.6.17 and so extends to an equivalence $\mathsf{PSh}(\mathcal{C}^{\mathrm{idem}}) \xrightarrow{\sim} \mathsf{PSh}(\mathcal{C})$. Its composition with $(i^{\mathrm{op}})_!$ is moreover the identity, since this is the unique a colimit-preserving functor that extends the Yoneda embedding. It follows that $(i^{\mathrm{op}})_!$ is an equivalence, with inverse its right adjoint $(i^{\mathrm{op}})^*$. □

Corollary 8.6.19. *A functor $f\colon \mathcal{C} \to \mathcal{D}$ is a Morita equivalence if and only if $(f^{\mathrm{op}})^*\colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$ is an equivalence.*

Proof. Suppose f is a Morita equivalence. Then the commutative square from Definition 8.6.15 induces a commutative square

$$
\begin{tikzcd}
\mathsf{PSh}(\mathcal{D}^{\mathrm{idem}}) \arrow[r, "(f^{\mathrm{idem,op}})^*"] \arrow[d, "\sim"'] & \mathsf{PSh}(\mathcal{C}^{\mathrm{idem}}) \arrow[d, "\sim"] \\
\mathsf{PSh}(\mathcal{D}) \arrow[r, "(f^{\mathrm{op}})^*"'] & \mathsf{PSh}(\mathcal{C}).
\end{tikzcd}
$$

It f^{idem} is an equivalence, it follows that so is $(f^{\mathrm{op}})^*$. Conversely, if $(f^{\mathrm{op}})^*$ is an equivalence, then so is its left adjoint $(f^{\mathrm{op}})_!$, and therefore so is its restriction to the full subcategories of absolute objects, which is f^{idem} by Observation 8.6.16. □

Corollary 8.6.20. *An ∞-category $\mathcal{M}$ is equivalent to a presheaf ∞-category if and only if $\mathcal{M}$ is cocomplete and there exists a small full subcategory of absolute objects that generates $\mathcal{M}$ under colimits.* □

8.7 (⋆) Kan extensions and cofinality via weighted (co)limits

We can also use weighted colimits together with the desription of natural transformations in terms of ends (§7.5) to prove the pointwise formula for Kan extensions:

Proposition 8.7.1. *Given functors $\phi\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{E}$, if the weighted colimits $\operatorname{colim}_{\mathcal{C}}^{\mathcal{D}(\phi(-),d)} F$ exist in $\mathcal{E}$ for all $d \in \mathcal{D}$, then the functor*

$$\phi_! F := \operatorname{colim}_{\mathcal{C}}^{\mathcal{D}(\phi(-),-)} F$$

is a left Kan extension of F along ϕ. Dually, if the weighted limits $\lim_{\mathcal{C}}^{\mathcal{D}(d,\phi(-))} F$ exist in $\mathcal{E}$ for all $d \in \mathcal{D}$, then the functor

$$\phi_* F := \lim_{\mathcal{C}}^{\mathcal{D}(-,\phi(-))} F$$

is a right Kan extension of F along ϕ.

Proof. Let G be a functor $\mathcal{D} \to \mathcal{E}$. We prove the statement for left Kan extensions by the computation[1]

$$\begin{aligned}
\mathsf{Map}_{\mathsf{Fun}(\mathcal{D},\mathcal{E})}(\phi_! F, G) &\simeq \lim_{(x,y)\in\mathcal{D}^{\mathrm{op}}\times\mathcal{D}}^{\mathcal{D}(-,-)} \mathcal{E}(\operatorname{colim}_{\mathcal{C}}^{\mathcal{D}(\phi(-),x))} F, G(y)) \\
&\simeq \lim_{(x,y)\in\mathcal{D}^{\mathrm{op}}\times\mathcal{D}}^{\mathcal{D}(-,-)} \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\mathcal{D}(\phi(-),x), \mathcal{E}(F, G(y))) \\
&\simeq \mathsf{Map}_{\mathsf{Fun}(\mathcal{D},\mathsf{PSh}(C))}(\mathcal{D}(\phi,-), \mathcal{E}(F,G)) \\
&\simeq \mathsf{Map}_{\mathsf{Fun}(\mathcal{C}^{\mathrm{op}},\mathsf{Fun}(\mathcal{D},\mathsf{Spc}))}(\mathcal{D}(\phi,-), \mathcal{E}(F,G)) \\
&\simeq \lim_{(c,c')\in\mathcal{C}\times\mathcal{C}^{\mathrm{op}}}^{\mathcal{C}^{\mathrm{op}}(-,-)} \mathsf{Map}_{\mathsf{Fun}(\mathcal{D},\mathsf{Spc})}(\mathcal{D}(\phi(c),-), \mathcal{E}(F(c'),G)) \\
&\simeq \lim_{(c,c')\in\mathcal{C}\times\mathcal{C}^{\mathrm{op}}}^{\mathcal{C}^{\mathrm{op}}(-,-)} \mathcal{E}(F(c'), G(\phi(c))) \\
&\simeq \mathsf{Map}_{\mathsf{Fun}(\mathcal{C},\mathcal{E})}(F, G\circ\phi),
\end{aligned}$$

where the penultimate equivalence uses the Yoneda Lemma. □

We can also (independently of Proposition 8.7.1) use weighted colimits to understand cofinality, as a special case of the following computation of (co)limits weighted by a left Kan extension:

Proposition 8.7.2. *Given $\phi\colon \mathcal{C} \to \mathcal{D}$, $W\colon \mathcal{C} \to \mathsf{Spc}$ and $F\colon \mathcal{D} \to \mathcal{E}$, we have a natural equivalence*

$$\lim_{\mathcal{D}}^{\phi_! W} F \simeq \lim_{\mathcal{C}}^{W} F \circ \phi,$$

if either limit exists. Dually, for $V\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Spc}$ we have

$$\operatorname{colim}_{\mathcal{D}}^{(\phi^{\mathrm{op}})_! V} F \simeq \operatorname{colim}_{\mathcal{C}}^{V} F \circ \phi.$$

[1] We leave it as an exercise for the reader to check if we have actually shown that all of these equivalences are sufficiently natural to justify the conclusion…

Proof. We have

$$\begin{aligned}\mathcal{E}(e, \lim_{\mathcal{D}}^{\phi_! W} F) &\simeq \mathsf{Map}_{\mathsf{Fun}(\mathcal{D},\mathsf{Spc})}(\phi_! W, \mathcal{E}(e, F)) \\ &\simeq \mathsf{Map}_{\mathsf{Fun}(\mathcal{C},\mathsf{Spc})}(W, \mathcal{E}(e, F \circ \phi)) \\ &\simeq \mathcal{E}(e, \lim_{\mathcal{C}}^{W} F \circ \phi).\end{aligned}$$

The case of colimits is proved similarly. □

Corollary 8.7.3. *A functor $\phi \colon \mathcal{C} \to \mathcal{D}$ induces an equivalence on limits*

$$\lim_{\mathcal{D}} F \xrightarrow{\sim} \lim_{\mathcal{C}} F \circ \phi$$

for any functor $F \colon \mathcal{D} \to \mathcal{E}$ such that either limit exists, if and only if ϕ is coinitial in the sense that $\|\mathcal{C}_{/d}\| \simeq *$ for all $d \in \mathcal{D}$.

Proof. It suffices to consider functors to Gpd_∞, so we may assume all small limits exist in $\mathcal{D}$. For a functor $F \colon \mathcal{D} \to \mathcal{E}$, Proposition 8.7.2 then gives an equivalence

$$\lim_{\mathcal{C}} F \circ \phi \simeq \lim_{\mathcal{C}}^{\mathrm{const}_*} F \circ \phi \simeq \lim_{\mathcal{D}}^{\phi_! \mathrm{const}_*} F,$$

where

$$\phi_!(\mathrm{const}_*)(d) \simeq \operatorname{colim}_{\mathcal{C}_{/d}} \mathrm{const}_* \simeq \|\mathcal{C}_{/d}\|.$$

If the right-hand side is contractible here we thus get

$$\lim_{\mathcal{C}} F \circ \phi \simeq \lim_{\mathcal{D}} F,$$

as required. The converse follows as in the proof of Theorem 6.5.13. □

8.8 (⋆) Full faithfulness via Kan extensions

In this section we'll discuss a trick, due to [HRS25], for checking full faithfulness using left Kan extensions and the Yoneda embedding:

Proposition 8.8.1 (Haine–Ramzi–Steinebrunner)**.** *A functor $F \colon \mathcal{C} \to \mathcal{D}$ is fully faithful if and only if $F_!^{\mathrm{op}} \colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ is fully faithful.*

Proof. We know that if F is fully faithful then so is F^{op}, and left Kan extension along a fully faithful functor is fully faithful (Lemma 8.3.4). On the other hand, we have a commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook, "y_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, hook, "y_{\mathcal{D}}"] \\ \mathsf{PSh}(\mathcal{C}) \arrow[r, "F_!^{\mathrm{op}}"'] & \mathsf{PSh}(\mathcal{D}), \end{tikzcd}$$

so if $F_!^{\mathrm{op}}$ is fully faithful then so is F by the dual of Lemma 2.4.5. □

We'll discuss two applications of this: we'll show that the maps $\mathcal{K}, \mathcal{L} \to \mathcal{K} \star \mathcal{L}$ into a join are fully faithful, and that fully faithful functors are closed under cobase change. For the first result we need the following input:

Lemma 8.8.2. *If $p\colon \mathcal{E} \to \mathcal{B}$ is a cartesian fibration with fibrewise initial objects, then p has a fully faithful left adjoint, given by the inclusion of these in $\mathcal{E}$.*

Proof. Let $\mathcal{E}_0$ be the full subcategory of $\mathcal{E}$ on the fibrewise initial objects. We know that p restricts to an equivalence $p_0\colon \mathcal{E}_0 \xrightarrow{\sim} \mathcal{B}$, and we claim that the composite $s\colon \mathcal{B} \xrightarrow{p_0^{-1}} \mathcal{E}_0 \hookrightarrow \mathcal{E}$ of is a left adjoint to p, with unit transformation the equivalence $ps \simeq \mathrm{id}_{\mathcal{B}}$. Indeed, we have that

$$\mathcal{E}(s(b), e) \to \mathcal{B}(ps(b), pe) \simeq \mathcal{B}(b, pe)$$

is an equivalence by Observation 5.5.2. □

Corollary 8.8.3. *For any ∞-categories $\mathcal{A}$ and $\mathcal{B}$, the canonical functors $\mathcal{A}, \mathcal{B} \hookrightarrow \mathcal{A} \star \mathcal{B}$ are fully faithful.*

Proof. It suffices to prove that one of the two inclusions is fully faithful, since we have $(\mathcal{A} \star \mathcal{B})^{\mathrm{op}} \simeq \mathcal{B}^{\mathrm{op}} \star \mathcal{A}^{\mathrm{op}}$, and fully faithful functors are preserved by taking opposites. Now observe that, by definition of the join, we have for any ∞-categories $\mathcal{L}$ and $\mathcal{K}$ a pullback square

$$\begin{tikzcd}
\mathsf{Fun}(\mathcal{K} \star \mathcal{L}, \mathcal{C}) \arrow[r] \arrow[d, "(\ell^*, r^*)"] & \mathsf{Fun}(\mathcal{K} \times \mathcal{L}, \mathsf{Ar}(\mathcal{C})) \arrow[d] \\
\mathsf{Fun}(\mathcal{K}, \mathcal{C}) \times \mathsf{Fun}(\mathcal{L}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\mathcal{K} \times \mathcal{L}, \mathcal{C}) \times \mathsf{Fun}(\mathcal{K} \times \mathcal{L}, \mathcal{C}),
\end{tikzcd}$$

where ℓ and r denote the canonical functors $\mathcal{K}, \mathcal{L} \to \mathcal{K} \star \mathcal{L}$. Thus the left vertical functor (ℓ^*, r^*) is a bifibration, and in particular the restriction $\ell^*\colon \mathsf{Fun}(\mathcal{K} \star \mathcal{L}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{K}, \mathcal{C})$ is a cartesian fibration with fibre $\mathsf{Fun}(\mathcal{L}, \mathcal{C}_{\phi/})$ at ϕ. By (the dual of) Corollary 5.5.10 this has an initial object if ϕ has a colimit in $\mathcal{C}$ (given by the constant functor that selects the colimit cone). We conclude from Lemma 8.8.2 that ℓ^* has a fully faithful left adjoint if $\mathcal{C}$ has all colimits of shape $\mathcal{K}$. This applies in particular to $\mathcal{C}$ being Gpd_∞. Taking $\mathcal{K} = \mathcal{B}^{\mathrm{op}}$ and $\mathcal{L} = \mathcal{A}^{\mathrm{op}}$ we can then conclude from Proposition 8.8.1 that $\mathcal{B} \to \mathcal{A} \star \mathcal{B}$ is fully faithful. □

For our second application we need the following input:

Proposition 8.8.4. *Suppose we have a pullback square of ∞-categories*

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "G'"] \arrow[d, "p"'] & \mathcal{D}' \arrow[d, "q"] \\
\mathcal{C} \arrow[r, "G"'] & \mathcal{D}
\end{tikzcd}$$

where G has a fully faithful left adjoint. Then G' also has a fully faithful left adjoint.

Proof. Let F be the left adjoint to G; since F is fully faithful, the counit is a natural equivalence $\mathrm{id}_{\mathcal{D}} \simeq GF$ (Proposition 6.3.10). We therefore have a homotopy in the outer square in

$$\begin{tikzcd}
\mathcal{D}' \arrow[drr, bend left, "\mathrm{id}_{\mathcal{D}'}"] \arrow[dr, dashed, "F'"] \arrow[ddr, bend right, "Fq"'] & & \\
& \mathcal{C}' \arrow[r, "G'"] \arrow[d, "p"'] & \mathcal{D}' \arrow[d, "q"] \\
& \mathcal{C} \arrow[r, "G"'] & \mathcal{D},
\end{tikzcd}$$

so that there is a unique filler F'. For $x \in \mathcal{D}'$ and $y \in \mathcal{C}'$ we then have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}'(F'x, y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}'(G'F'x, G'y) \arrow[r, "\sim"] \arrow[d] & \mathcal{D}'(x, G'y) \arrow[d] \\
\mathcal{C}(Fqx, py) \arrow[r, "\sim"'] & \mathcal{D}(GFqx, Gpy) \arrow[r, "\sim"] & \mathcal{D}(qx, Gpy)
\end{tikzcd}$$

where the left square is a pullback since $\mathcal{C}'$ is a pullback (Lemma 7.4.2). Thus the top left morphism is also an equivalence, which shows that the equivalence $\mathrm{id}_{\mathcal{D}'} \simeq G'F'$ is the counit of an adjunction $F' \dashv G'$ where F' is hence fully faithful (by Proposition 6.3.10 again). $\square$

Corollary 8.8.5 (Haine–Ramzi–Steinebrunner). *Suppose*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{B} \arrow[d, "q"] \\
\mathcal{C} \arrow[r, "g"'] & \mathcal{D}
\end{tikzcd}$$

is a pushout square of ∞-categories such that f is fully faithful. Then g is also fully faithful.

Proof. By Proposition 8.8.1 it suffices to show that $(g^{\mathrm{op}})_! \colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ is fully faithful. Taking presheaves on the pushout square we get a pullback square (Proposition 7.4.3)

$$\begin{tikzcd}
\mathsf{PSh}(\mathcal{D}) \arrow[r, "(g^{op})^*"] \arrow[d, "(q^{op})^*"'] & \mathsf{PSh}(\mathcal{C}) \arrow[d, "(p^{op})^*"] \\
\mathsf{PSh}(\mathcal{B}) \arrow[r, "(f^{op})^*"'] & \mathsf{PSh}(\mathcal{A}).
\end{tikzcd}$$

Since f is fully faithful, the functor $(f^{\mathrm{op}})^*$ has a fully faithful left adjoint (Proposition 8.8.1), and so the same is true of $(g^{\mathrm{op}})^{*)}$ by Proposition 8.8.4. Since left adjoints are unique, this shows that $(g^{\mathrm{op}})_!$ is indeed fully faithful. $\square$

Chapter 9

Finite and filtered ∞-categories

9.1 Interlude on set theory

Up to now we have been pretty cavalier about size issues for (∞-)categories. Since we're about to talk about various finiteness conditions, this seems a good point to try to make things a little more precise.

Notation 9.1.1. Recall that a *cardinal* is an isomorphism class of sets; for a set S we denote its cardinal by $|S|$.

Definition 9.1.2. For a cardinal κ, a set S is *κ-finite* if $|S| < \kappa$.

Definition 9.1.3. A cardinal κ is *regular* if

- κ is infinite,
- given a family of sets $S_i, i \in I$ such that $|I| < \kappa$ and $|S_i| < \kappa$ for all i, then $\coprod_{i \in I} S_i$ also has cardinality $< \kappa$.

Example 9.1.4. The countable cardinal $\omega = \aleph_0$ is regular. The ω-finite sets are just the finite ones.

Definition 9.1.5. A cardinal κ is a *strong limit* if $\lambda < \kappa$ implies $2^{\lambda} < \kappa$.

Definition 9.1.6. A cardinal is *inaccessible* if it is uncountable, regular, and a strong limit.

Definition 9.1.7. A *(Grothendieck) universe* is a set U such that

- for $S \in U$ and $T \in S$, we have $T \in U$,
- for $S \in U$, we have $P(S) \in U$ (where $P(S)$ is the power set $\mathbf{2}^S$),
- $\mathbb{N} \in U$,
- for $I \in U$ and $f : I \to U$, we have $\bigcup_{i \in I} f(i) \in U$.

Exercise 9.1. If U is a Grothendieck universe, then $|U|$ is an inaccessible cardinal.

Definition 9.1.8. The *von Neumann hierarchy* is the family of sets V_α defined by

$$V_\alpha = \bigcup_{\beta<\alpha} P(V_\beta),$$

starting with $V_0 = \emptyset$.

Observation 9.1.9. A cardinal κ is inaccessible if and only if V_κ is a Grothendieck universe — thus Grothendieck universes exist if and only if inaccessible cardinals do.

Definition 9.1.10. If U is a Grothendieck universe, then the elements of U are called the U*-small* sets.

Remark 9.1.11. If U is a universe, then the U-small sets give a model of the ZFC axioms.

We have implicitly assumed the existence of at least one universe, so that it makes sense to talk about a large (∞-)category of (U-)small (∞-)categories. In category theory it is usually convenient to assume at least a weak version of Grothendieck's *universe axiom*, which says that for any set there exists a universe that contains it.

For more background on set theory and categories we recommend the expository paper [Shu08] by Shulman. There is also a nice series of blog posts by Leinster on large cardinals, starting with [Lei21].

9.2 Finite ∞-categories

For ordinary categories, it is common to say that a category $\mathcal{C}$ is (essentially) finite if it contains finitely many isomorphism classes of objects and each set of morphisms between these is finite. This notion of finiteness does *not* extend to ∞-categories, however. In fact, there is no intrinsic characterization of finiteness for ∞-categories — instead, we have to define this notion indirectly:

Definition 9.2.1. For a regular cardinal κ, we define the full subcategory

$$\mathsf{Cat}_\infty^{\kappa\text{-fin}} \subseteq \mathsf{Cat}_\infty$$

of κ*-finite* ∞-categories to be spanned by the smallest collection of objects such that

- $\emptyset$, $[0]$, $[1]$ are κ-finite,
- $\mathsf{Cat}_\infty^{\kappa\text{-fin}}$ is closed under pushouts,
- $\mathsf{Cat}_\infty^{\kappa\text{-fin}}$ is closed under coproducts indexed by κ-finite sets.

We will refer to ω-finite ∞-categories as simply *finite*; in this case, the closure under finite coproducts is automatic once we have $\emptyset$ and pushouts.

Remark 9.2.2. We can similarly define the κ-*finite* ∞-groupoids by omitting $[1]$ in this definition. In terms of topology, the finite ∞-groupoids then correspond to the *finite CW-complexes*.

Remark 9.2.3. In terms of quasicategories, we can characterize the κ-finite ∞-categories as those that can be modelled by a (non-fibrant) simplicial set that has a κ-finite set of non-degenerate simplices.

Proposition 9.2.4.

(1) An ∞-category $\mathcal{C}$ has κ-finite colimits if and only if it has an initial object, pushouts, and (for $\kappa > \omega$) coproducts indexed by κ-finite sets.

(2) Suppose $\mathcal{C}$ is an ∞-category with κ-finite colimits. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves κ-finite colimits if and only if F preserves the initial object, pushouts, and (for $\kappa > \omega$) κ-finite coproducts.

Proof. To prove (1), we let $\mathcal{X} \subseteq \mathsf{Cat}_\infty$ be the full subcategory on those ∞-categories $\mathcal{K}$ such that $\mathcal{C}$ has all $\mathcal{K}$-indexed colimits. We want to show that $\mathcal{X}$ contains $\mathsf{Cat}_\infty^{\kappa\text{-fin}}$. By assumption $\mathcal{X}$ contains $\emptyset$, and it also contains $[0]$ and $[1]$ since these have a terminal object. It therefore suffices to show that $\mathcal{X}$ is closed under pushouts and κ-finite coproducts. This follows from the fact that colimits over a colimit of ∞-categories decompose as in Corollary 7.4.5. Part (2) is proved in the same way. □

Corollary 9.2.5. *The ∞-category $\mathsf{Cat}_\infty^{\kappa\text{-fin}}$ is closed under κ-finite colimits.* □

Proposition 9.2.6. $\mathsf{Cat}_\infty^{\kappa\text{-fin}}$ *is closed under products.*

Proof. Let $\mathcal{X} \subseteq \mathsf{Cat}_\infty$ be the full subcategory on those ∞-categories $\mathcal{K}$ such that $\mathcal{K} \times -$ preserves κ-finite ∞-categories. (Then $\mathcal{X} \subseteq \mathsf{Cat}_\infty^{\kappa\text{-fin}}$ since $\mathcal{K} \in \mathcal{X}$ implies $\mathcal{K} \times [0] \simeq \mathcal{K}$ is κ-finite.) Clearly $\mathcal{X}$ contains $\emptyset$ and $[0]$, and since $- \times -$ preserves colimits in each variable we see that $\mathcal{X}$ is closed under pushouts and κ-finite coproducts. It therefore only remains to show that $[1] \in \mathcal{X}$. Let $\mathcal{Y} \subseteq \mathsf{Cat}_\infty$ therefore be the full subcategory of ∞-categories $\mathcal{L}$ such that $[1] \times \mathcal{L}$ is κ-finite. Again we see that $\mathcal{L}$ contains $\emptyset$ and $[0]$ and is closed under pushouts and κ-finite coproducts, so we only need to show that $[1] \times [1]$ is κ-finite. But this decomposes as a pushout $[2] \amalg_{[1]} [2]$ where $[2] \simeq [1] \amalg_{[0]} [1]$ is κ-finite. □

Exercise 9.2. Assuming that for every small ∞-category there is some (small) regular cardinal κ such that it is κ-finite, show that an ∞-category has all small colimits if and only if it has an initial object, pushouts, and coproducts indexed by all small sets.

Exercise 9.3. Show that $\mathcal{K}$ is κ-finite if and only if $\mathcal{K}^{\mathrm{op}}$ is κ-finite.

9.3 Filtered ∞-categories

Definition 9.3.1. An ∞-category $\mathcal{I}$ is *κ-filtered* for a regular cardinal κ if for every κ-finite ∞-category $\mathcal{K}$ and functor $\phi\colon \mathcal{K} \to \mathcal{I}$, the undercategory $\mathcal{I}_{\phi/}$ is weakly contractible. We usually abbreviate ω-filtered to *filtered*.

Example 9.3.2. If $\mathcal{I}$ has a terminal object, then $\mathcal{I}$ is κ-filtered for any κ, since $\mathcal{I}_{\phi/}$ also has a terminal object for any ϕ (Proposition 5.6.8), and so is weakly contractible (Observation 6.5.12).

Lemma 9.3.3. *If $\mathcal{I}$ has κ-finite colimits, then $\mathcal{I}$ is κ-filtered.*

Proof. Given $\phi\colon \mathcal{K} \to \mathcal{I}$ with $\mathcal{K}$ κ-finite, then the colimit of ϕ gives an initial object of $\mathcal{I}_{\phi/}$, so that this ∞-category is in particular weakly contractible (Observation 6.5.12). □

Observation 9.3.4. If $\lambda < \kappa$ then any κ-filtered ∞-category is also λ-filtered, since a λ-finite ∞-category as also κ-finite. In particular a κ-filtered ∞-category is in particular filtered, for any regular cardinal κ.

Observation 9.3.5. Taking ϕ to be the unique functor $\emptyset \to \mathcal{I}$, we get $\mathcal{I}_{\phi/} \simeq \mathcal{I}$, so a κ-filtered ∞-category is in particular weakly contractible.

Exercise 9.4. Suppose that $\mathcal{I}$ is a retract of $\mathcal{J}$ via $f\colon \mathcal{I} \to \mathcal{J}$ and $r\colon \mathcal{J} \to \mathcal{I}$ such that $rf \simeq \mathrm{id}_{\mathcal{I}}$. Show that for any diagram $p\colon \mathcal{K} \to \mathcal{I}$, the ∞-category $\mathcal{I}_{p/}$ is a retract of $\mathcal{J}_{fp/}$. Conclude that if $\mathcal{J}$ is κ-filtered then so is $\mathcal{I}$.

Lemma 9.3.6. *If $\mathcal{I}$ is κ-filtered, then for every functor $\phi\colon \mathcal{K} \to \mathcal{I}$ with $\mathcal{K}$ κ-finite, the undercategory $\mathcal{I}_{\phi/}$ is also κ-filtered.*

Proof. A diagram $\psi\colon \mathcal{L} \to \mathcal{I}_{\phi/}$ is the same thing as a diagram $\psi'\colon \mathcal{L} \star \mathcal{K} \to \mathcal{I}$ that restricts to ϕ on $\mathcal{K}$, and $(\mathcal{I}_{\phi/})_{\psi/} \simeq \mathcal{I}_{\psi'/}$ by Proposition 5.6.1. Here $\mathcal{L} \star \mathcal{K}$ is again κ-finite since this is the pushout

$$\mathcal{L} \amalg_{\mathcal{L}\times\mathcal{K}\times\{0\}} \mathcal{L} \times \mathcal{K} \times [1] \amalg_{\mathcal{L}\times\mathcal{K}\times\{1\}} \mathcal{K},$$

where all the pieces are κ-finite by Proposition 9.2.6. The ∞-category $\mathcal{I}_{\psi'/}$ is therefore weakly contractible, as required. □

Observation 9.3.7. $\mathcal{I}$ is κ-filtered if and only if the diagonal functor $\mathcal{I} \to \mathsf{Fun}(\mathcal{K}, \mathcal{I})$ is cofinal for every κ-finite $\mathcal{K}$.

Lemma 9.3.8. *If $\mathcal{I}$ is κ-filtered, then $\mathcal{I}_{i/} \to \mathcal{I}$ is cofinal for any $i \in \mathcal{I}$.*

Proof. For $j \in \mathcal{I}$ we have $\mathcal{I}_{i/} \times_{\mathcal{I}} \mathcal{I}_{j/} \simeq \mathcal{I}_{\{i,j\}/}$ by Corollary 7.4.4, and the latter is weakly contractible since $\mathcal{I}$ is κ-filtered. □

To compare our definition of filtered ∞-categories to that used by Lurie [Lur09, Ker], which is also useful to find examples thereof, we need the following fact:

Fact 9.3.9. *Suppose an ∞-category $\mathcal{I}$ has the property that any functor $\mathcal{K} \to \mathcal{I}$ from a finite ∞-category $\mathcal{K}$ extends to a cocone $\mathcal{K}^{\triangleright} \to \mathcal{I}$. Then $\mathcal{I}$ is weakly contractible.*

Remark 9.3.10. The reason Fact 9.3.9 is true is that every morphism $S^n \to \|\mathcal{I}\|$ lifts to some functor $\mathcal{K} \to \mathcal{I}$ with $\mathcal{K}$ finite — the extension to $\mathcal{K}^{\triangleright}$ then provides a nullhomotopy of the map from S^n. If we model $\mathcal{I}$ by a quasicategory, then we can see this by describing $\|\mathcal{I}\|$ using Kan's Ex^{∞}-functor.

Proposition 9.3.11. *Given Fact 9.3.9, an ∞-category $\mathcal{I}$ is κ-filtered if and only if every diagram $\mathcal{K} \to \mathcal{I}$ with $\mathcal{K}$ κ-finite extends to a cocone $\mathcal{K}^{\triangleright} \to \mathcal{I}$, i.e. the undercategory $\mathcal{I}_{\phi/}$ is non-empty.*

Proof. We must show that this apparently weaker condition implies that $\mathcal{I}$ is κ-filtered. The argument from Lemma 9.3.6 also implies that this condition is inherited by slices, so this reduces to showing that if the condition holds for $\mathcal{I}$ then $\mathcal{I}$ must be weakly contractible — this follows from Fact 9.3.9. □

Corollary 9.3.12. *A partially ordered set $(S, <)$ gives a κ-filtered ∞-category if and only if every κ-finite set of elements in S has an upper bound.*

Proof. If $i\colon T \to S$ is the inclusion of a κ-finite set of elements, then an upper bound for this subset is precisely an object of $S_{i/}$, so this must exist of S is κ-filtered. Conversely, a diagram $p\colon \mathcal{K} \to S$ with $\mathcal{K}$ a κ-finite ∞-category factors uniquely first through $h\mathcal{K}$ and then through the truncation of $h\mathcal{K}$ to a poset P, which is necessarily κ-finite (as the κ-finite posets are closed under pushouts and κ-finite coproducts). An upper bound in S for the images of the elements of P then provides an extension of p to $\mathcal{K}^{\triangleright} \to S$. This implies that S is filtered by Proposition 9.3.11. □

Example 9.3.13. The partially ordered sets $\mathbb{N}$ and $\mathbb{Z}$ are filtered.

Exercise 9.5. An ordinary category $\mathcal{C}$ is usually said to be *filtered* if

- $\mathcal{C} \neq \emptyset$,
- for any two objects x, y in $\mathcal{C}$ there exists an object z with morphisms $x \to z$ and $y \to z$,
- for any two parallel morphisms $f, g\colon x \to y$ there exists a third morphism $h\colon y \to z$ such that $hf = hg$.

Using Proposition 9.3.11, show that this is equivalent to $\mathcal{C}$ being filtered as an ∞-category.

Theorem 9.3.14 (Lurie). *An ∞-category $\mathcal{I}$ is κ-filtered if and only if $\mathcal{I}$-indexed colimits commute with κ-finite limits in Gpd_{∞}. That is, if $\mathcal{K}$ is κ-finite then for every functor $F\colon \mathcal{K} \times \mathcal{I} \to \mathsf{Gpd}_{\infty}$, the canonical map*

$$\mathrm{colim}_{k \in \mathcal{K}} \lim_{i \in \mathcal{I}} F(k, -) \to \lim_{i \in \mathcal{I}} \mathrm{colim}_{\mathcal{K}} F(-, i)$$

is an equivalence. Equivalently, the colimit functor

$$\operatorname{colim}_{\mathcal{I}} \colon \mathsf{Fun}(\mathcal{I}, \mathsf{Gpd}_\infty) \to \mathsf{Gpd}_\infty$$

preserves κ-finite limits.

One direction is easy:

Lemma 9.3.15. *If $\mathcal{I}$-indexed colimits commute with κ-filtered limits for some ∞-category $\mathcal{I}$, then $\mathcal{I}$ is κ-filtered.*

Proof. For $\phi \colon \mathcal{K} \to \mathcal{I}$ with $\mathcal{K}$ κ-finite, we know from Corollary 7.1.7 that $\mathcal{I}_{\phi/} \to \mathcal{I}$ is the left fibration for the functor

$$x \mapsto \lim_{\mathcal{K}} \mathcal{I}(\phi, x).$$

Therefore $\|\mathcal{I}_{\phi/}\|$ is the colimit $\operatorname{colim}_{x \in \mathcal{I}} \lim_{\mathcal{K}} \mathcal{I}(\phi, x)$. We can rewrite this as

$$\lim_{k \in \mathcal{K}} \operatorname{colim}_{\mathcal{I}} \mathcal{I}(\phi(k), -) \simeq \lim_{k \in \mathcal{K}} \|\mathcal{I}_{\phi(k)/}\| \simeq \lim_{\mathcal{K}} * \simeq *,$$

where we have used that $\mathcal{I}_{\phi(k)/}$ is weakly contractible (since it has an initial object, Observation 6.5.12). Thus $\mathcal{I}_{\phi/}$ is weakly contractible, as required. □

The converse direction, that colimits over κ-filtered ∞-categories commute with κ-finite limits in Gpd_∞, is much more involved, and we defer the proof to §9.9.

9.4 Compact objects

Definition 9.4.1. Let $\mathcal{C}$ be an ∞-category with κ-filtered colimits. An object $c \in \mathcal{C}$ is *κ-compact* if the functor

$$\mathcal{C}(c, -) \colon \mathcal{C} \to \mathsf{Gpd}_\infty$$

preserves κ-filtered colimits. The full subcategory of $\mathcal{C}$ on its κ-compact objects will be denoted $\mathcal{C}^\kappa$. We refer to ω-compact objects as just *compact*.

Proposition 9.4.2. *κ-compact objects of an ∞-category $\mathcal{C}$ are closed under κ-finite colimits and retracts in $\mathcal{C}$.*

Proof. First consider a diagram $p \colon \mathcal{K} \to \mathcal{C}$ with $\mathcal{K}$ κ-finite. We then have

$$\mathcal{C}(\operatorname{colim}_{\mathcal{K}} p, -) \simeq \lim_{\mathcal{K}^{\mathrm{op}}} \mathcal{C}(p, -)$$

This preserves κ-filtered colimits by Theorem 9.3.14, since $\mathcal{K}^{\mathrm{op}}$ is κ-finite (Exercise 9.3). On the other hand, if c is a retract of c' and the latter is κ-compact, then $\mathcal{C}(c, -)$ is a retract of $\mathcal{C}(c', -)$, and therefore preserves κ-filtered colimits since a retract of an equivalence is an equivalence. □

Remark 9.4.3. If κ is uncountable, then Proposition 8.5.4 shows that splittings of idempotents can be computed by a κ-finite colimit, so that the closure under retracts is actually redundant. This is not true for $\kappa = \omega$, however.

Fact 9.4.4. *$[1]$ is a compact object of Cat_∞.*

Exercise 9.6. Show that if $\mathcal{C} \simeq \operatorname{colim}_{\mathcal{K}} F$ is a filtered colimit in Cat_∞, then $\mathcal{C}^\simeq \simeq \operatorname{colim}_{\mathcal{K}} F^\simeq$, and mapping ∞-groupoids in $\mathcal{C}$ are filtered colimits of those in $F(k)$.

Exercise 9.7. Use Fact 9.4.4 and the fact that $[1]$ detects equivalences to show that filtered colimits commute with finite limits in Cat_∞.

Corollary 9.4.5. *κ-finite ∞-categories are κ-compact objects in Cat_∞.*

Proof. Both sides contain $\emptyset, [0], [1]$ and are closed under κ-finite colimits. □

We would like to characterize the κ-compact objects in ∞-categories of presheaves. For this we need to use the following:

Fact 9.4.6. *Every small ∞-category is the colimit of a small κ-filtered diagram of κ-finite ∞-categories.*

Exercise 9.8. Show that an ∞-category is cocomplete if and only if it has κ-filtered colimits and κ-finite colimits for some regular cardinal κ.

Proposition 9.4.7. *A presheaf $\phi \in \mathsf{PSh}(\mathcal{C})$ is κ-compact if and only if ϕ is a retract of a κ-finite colimit of representables.*

Proof. We first suppose ϕ is κ-compact. Let $p\colon \mathcal{E} \to \mathcal{C}$ be the right fibration for ϕ, so that $\phi \simeq \operatorname{colim}_{\mathcal{E}} \mathsf{y} \circ p$. By Fact 9.4.6 we can write $\mathcal{E}$ as $\operatorname{colim}_{\mathcal{K}} F$ where $\mathcal{K}$ is κ-filtered and $F\colon \mathcal{K} \to \mathsf{Cat}_\infty$ is a diagram of κ-finite ∞-categories. By Corollary 7.4.5 we then have

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\phi,\phi) \simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\phi, \operatorname{colim}_{\mathcal{K}} \operatorname{colim}_{F(-)} \mathsf{y}\circ p) \simeq \operatorname{colim}_{\mathcal{K}} \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\phi, \operatorname{colim}_{F(-)} \mathsf{y}\circ p).$$

Arguing as in the proof of Proposition 8.6.9 it follows that there exists some $k \in \mathcal{K}$ such that ϕ is a retract of $\operatorname{colim}_{F(k)} \mathsf{y} \circ p$, which is a κ-finite colimit of representables.

For the converse, suppose that ϕ is a retract of a κ-finite colimit of representables. Representable presheaves are κ-compact (since they are even absolute, Example 8.6.6), and κ-compact objects are closed under κ-finite colimits and retracts by Proposition 9.4.2, so ϕ is indeed κ-compact. □

9.5 Flat presheaves and filtered cocompletions

In this section we will show that any small ∞-category has a free cocompletion under κ-filtered colimits.

Definition 9.5.1. A presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ is κ*-flat* if for the associated right fibration $\mathcal{E} \to \mathcal{C}$, the ∞-category $\mathcal{E}$ is κ-filtered. We write $\mathrm{Ind}_\kappa(\mathcal{C})$ for the full subcategory of $\mathsf{PSh}(\mathcal{C})$ spanned by the κ-flat functors; for $\kappa = \omega$ we just write $\mathrm{Ind}(\mathcal{C})$.

Exercise 9.9. Show that a presheaf W is κ-flat if and only if W-weighted colimits commute with κ-finite limits in Gpd_∞.

Example 9.5.2. Representable presheaves are κ-flat for any κ by Example 9.3.2, since the corresponding right fibration is of the form $\mathcal{I}_{/x} \to \mathcal{I}$, where $\mathcal{I}_{/x}$ has a terminal object. Moreover, any retract of a representable presheaf is κ-flat since κ-filtered ∞-categories are closed under retracts (Exercise 9.4).

Observation 9.5.3. If $\lambda < \kappa$ then $\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathrm{Ind}_\lambda(\mathcal{C})$, since every κ-filtered ∞-category is also λ-filtered. On the other hand, by Example 9.5.2 we always have $\mathcal{C}^{\mathrm{idem}} \subseteq \mathrm{Ind}_\kappa(\mathcal{C})$. [1]

Our goal is to show that $\mathrm{Ind}_\kappa(\mathcal{C})$ is the free cocompletion of $\mathcal{C}$ under κ-filtered colimits.

Remark 9.5.4. The notation $\mathrm{Ind}(\mathcal{C})$ comes from the name *ind-object* of $\mathcal{C}$ for a formal colimit of a filtered diagram in $\mathcal{C}$. Here "ind-object" comes from the somewhat old-fashioned term "inductive limit" for a filtered colimit. (Confusingly, "inductive limit" is also an old name for "colimit".)

Proposition 9.5.5. *A κ-filtered colimit of κ-filtered ∞-categories is again κ-filtered.*

Proof. Suppose $\mathcal{I} = \mathrm{colim}_{j\in\mathcal{J}} \mathcal{I}_j$ where $\mathcal{J}$ and each $\mathcal{I}_j$ is κ-filtered. Using Theorem 9.3.14, we must show that $\mathcal{I}$-indexed colimits commute with κ-finite colimits in Gpd_∞. But we have $\mathrm{colim}_{\mathcal{I}} \simeq \mathrm{colim}_{\mathcal{J}} \mathrm{colim}_{\mathcal{I}_j}$ by Corollary 7.4.5, so this is true as all of these colimits commute with κ-finite limits. □

Exercise 9.10. We can also prove Proposition 9.5.5 without using Theorem 9.3.14: Consider $\phi\colon \mathcal{K} \to \mathcal{I}$ with $\mathcal{K}$ κ-finite. Then $\mathcal{K}$ is a κ-compact object of Cat_∞ by Corollary 9.4.5, so that

$$\mathsf{Map}(\mathcal{K}, \mathcal{I}) \simeq \mathrm{colim}_{j\in\mathcal{J}} \mathsf{Map}(\mathcal{K}, \mathcal{I}_j).$$

It follows (by taking π_0, as in the proof of Proposition 8.6.9) that ϕ factors through $\phi'\colon \mathcal{K} \to \mathcal{I}_j$ for some index j. Since $\mathcal{J}_{j/} \to \mathcal{J}$ is cofinal by Lemma 9.3.8, we can identify $\mathcal{I}$ as $\mathrm{colim}_{j'\in\mathcal{J}_{j/}} \mathcal{I}_{j'}$. Show that we then get an equivalence

$$\mathcal{I}_{\phi/} \simeq \mathrm{colim}_{j'\in\mathcal{J}_{j/}} (\mathcal{I}_{j'})_{\phi'_{j'}/}$$

where $\phi'_{j'}$ is the composite $\mathcal{K} \xrightarrow{\phi'} \mathcal{I}_j \to \mathcal{I}_{j'}$, and conclude from this that $\mathcal{I}$ is filtered.

Exercise 9.11. Show that the full subcategory of right fibrations over $\mathcal{C}$ is closed under filtered colimits in $\mathsf{Cat}_{\infty/\mathcal{C}}$.

[1] In fact, if we make κ sufficiently large relative to $\mathcal{C}$ then $\mathrm{Ind}_\kappa(\mathcal{C})$ reduces to just the idempotent-completion $\mathcal{C}^{\mathrm{idem}}$.

Corollary 9.5.6. $\mathrm{Ind}_\kappa(\mathcal{C})$ *is closed under κ-filtered colimits in* $\mathsf{PSh}(\mathcal{C})$*.*

Proof. κ-filtered colimits in $\mathsf{PSh}(\mathcal{C})$ are equivalently κ-filtered colimits in $\mathsf{RFib}(\mathcal{C})$, and these are computed in Cat_∞ by Exercise 9.11 and (the dual of) Corollary 5.6.11. It therefore follows from Proposition 9.5.5 that κ-flat presheaves are closed under κ-filtered colimits. □

Corollary 9.5.7. *A presheaf is κ-flat if and only if it is a κ-filtered colimit of representables.*

Proof. Suppose $\mathcal{F} \to \mathcal{C}$ is the right fibration for a presheaf F. Then (Observation 7.3.6) F is the colimit of the composite

$$\mathcal{F} \to \mathcal{C} \xrightarrow{\mathsf{y}} \mathsf{PSh}(\mathcal{C}).$$

If F is κ-flat this shows that it is a κ-filtered colimit of representables. Conversely, representable presheaves are κ-flat by Example 9.5.2, and κ-flat presheaves are closed under κ-filtered colimits by Corollary 9.5.6. □

Proposition 9.5.8. *Suppose $\mathcal{C}$ has κ-finite colimits. Then a presheaf F on $\mathcal{C}$ is κ-flat if and only if it preserves κ-finite limits.*

Proof. The representable presheaves preserve κ-finite limits, and presheaves that do so are closed under κ-filtered colimits (since κ-filtered colimits commute with κ-finite limits in Gpd_∞). Corollary 9.5.7 therefore implies that all κ-flat presheaves preserve κ-finite colimits. Conversely, if F preserves κ-finite limits then the total space of the corresponding right fibration admits κ-finite colimits by Corollary 7.6.8, and so is κ-filtered by Lemma 9.3.3. □

Proposition 9.5.9. *Suppose $\mathcal{C}$ is a small ∞-category and $\mathcal{D}$ is an ∞-category with κ-filtered colimits.*

(i) Every functor $\mathcal{C} \to \mathcal{D}$ admits a left Kan extension to a functor $\mathrm{Ind}_\kappa \mathcal{C} \to \mathcal{D}$, and this preserves κ-filtered colimits.

(ii) Every functor $\mathrm{Ind}_\kappa \mathcal{C} \to \mathcal{D}$ that preserves κ-filtered colimits is left Kan extended from its restriction to $\mathcal{C}$.

Proof. Let i denote the fully faithful inclusion $\mathcal{C} \hookrightarrow \mathrm{Ind}_\kappa \mathcal{C}$. The Kan extension of a functor $F \colon \mathcal{C} \to \mathcal{D}$ along i exists if for every $\phi \in \mathrm{Ind}_\kappa \mathcal{C}$ the colimit of the composite

$$\mathcal{C}_{/\phi} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$$

exists in $\mathcal{D}$. Here $\mathcal{C}_{/\phi} \to \mathcal{C}$ is the right fibration for the presheaf ϕ; as ϕ is κ-flat, this is a κ-filtered colimit, and so by assumption this colimit indeed exists in $\mathcal{D}$. Now we check that $i_!F$ preserves κ-filtered colimits: using Lemma 8.4.2 we have

$$i_!F(\mathrm{colim}_{\mathcal{J}}\, \phi) \simeq \mathrm{colim}_{\mathcal{C}}^{\mathrm{colim}_{\mathcal{J}} \phi} F \simeq \mathrm{colim}_{\mathcal{J}}\, \mathrm{colim}_{\mathcal{C}}^{\phi} F \simeq \mathrm{colim}_{\mathcal{J}}\, i_!F(\phi),$$

since all of these colimits are κ-filtered and so exist in $\mathcal{D}$.

Now consider a functor $G\colon \operatorname{Ind}_\kappa \mathcal{C} \to \mathcal{D}$ that preserves κ-filtered colimits. Then

$$(i_! i^* G)(\phi) \to G(\phi)$$

is the canonical map

$$\operatorname{colim}^{\phi}_{\mathcal{C}} G \circ i \to G(\operatorname{colim}^{\phi}_{\mathcal{C}} i(-)),$$

which is an equivalence since G preserves κ-filtered colimits and ϕ is κ-flat. $\square$

Corollary 9.5.10. *Suppose $\mathcal{C}$ is a small ∞-category and $\mathcal{D}$ is an ∞-category with κ-filtered colimits. Then the restriction*

$$\mathsf{Fun}(\operatorname{Ind}_\kappa \mathcal{C}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

has a fully faithful left adjoint with image the functors that preserve κ-filtered colimits. $\square$

Observation 9.5.11. For a functor of small ∞-categories $f\colon \mathcal{C} \to \mathcal{D}$, Proposition 9.5.9 implies that there is a unique commutative square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d] \\
\operatorname{Ind}_\kappa \mathcal{C} \arrow[r, "\operatorname{Ind}_\kappa f"'] & \operatorname{Ind}_\kappa \mathcal{D}
\end{tikzcd}$$

where $\operatorname{Ind}_\kappa f$ preserves κ-filtered colimits. We claim that $\operatorname{Ind}_\kappa f$ is the restriction of the left Kan extension functor $f_!^{\mathrm{op}}$. We have a commutative square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "y_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "y_{\mathcal{D}}"] \\
\mathsf{PSh}(\mathcal{C}) \arrow[r, "f_!^{\mathrm{op}}"'] & \mathsf{PSh}(\mathcal{D}).
\end{tikzcd}$$

from Proposition 8.4.4, which shows in particular that $f_!^{\mathrm{op}}$ preserves representable presheaves. Since it also preserves κ-filtered colimits (being a left adjoint) it follows from Corollary 9.5.7 that $f_!^{\mathrm{op}}$ restricts to a functor $\operatorname{Ind}_\kappa \mathcal{C} \to \operatorname{Ind}_\kappa \mathcal{D}$. Moreover, this restriction preserves κ-filtered colimits, since these are computed in presheaves by Corollary 9.5.6. By uniqueness we must therefore have

$$\operatorname{Ind}_\kappa f \simeq f_!^{\mathrm{op}}|_{\operatorname{Ind}_\kappa \mathcal{C}}$$

as claimed.

9.6 Recognizing filtered cocompletions

In this section we will characterize the ∞-categories that are of the form $\mathrm{Ind}_\kappa \mathcal{C}$ for some small ∞-category $\mathcal{C}$ and regular cardinal κ, and also show that the functor $\mathrm{Ind}_\kappa f$ induced by a functor f of small ∞-categories is an equivalence if and only if f is a Morita equivalence. The starting point for this is the following variant of Proposition 8.6.17:

Proposition 9.6.1. *Suppose $\mathcal{M}$ is an ∞-category with κ-filtered colimits and $\mathcal{C}$ is a small ∞-category. Then the unique κ-filtered-colimit-preserving functor $F\colon \mathrm{Ind}_\kappa \mathcal{C} \to \mathcal{M}$ extending a functor $i\colon \mathcal{C} \to \mathcal{M}$ is an equivalence if and only if the following conditions hold:*

(1) i is fully faithful.

(2) The objects $i(c)$ are κ-compact for all $c \in \mathcal{C}$.

(3) Every object of $\mathcal{M}$ is the colimit of a small κ-filtered diagram in $\mathcal{C}$.

Proof. This follows by exactly the same argument as in the proof of Proposition 8.6.17. □

Variant 9.6.2. The same proof shows that F is fully faithful if and only if i is fully faithful and takes values in κ-compact objects.

Corollary 9.6.3. *For any small ∞-category $\mathcal{C}$, the inclusion $i\colon \mathcal{C} \hookrightarrow \mathcal{C}^{\mathrm{idem}}$ induces an equivalence $i_!^{\mathrm{op}}\colon \mathrm{Ind}_\kappa \mathcal{C} \xrightarrow{\sim} \mathrm{Ind}_\kappa \mathcal{C}^{\mathrm{idem}}$.*

Proof. The inclusion $\mathcal{C}^{\mathrm{idem}} \hookrightarrow \mathrm{Ind}_\kappa \mathcal{C}$ satisfies the conditions of Proposition 9.6.1 and so extends to an equivalence $\alpha\colon \mathrm{Ind}_\kappa \mathcal{C}^{\mathrm{idem}} \to \mathrm{Ind}_\kappa \mathcal{C}$. The composition of α with $i_!^{\mathrm{op}}$ is moreover the identity, since this restricts to the composite $\mathcal{C} \to \mathcal{C}^{\mathrm{idem}} \to \mathrm{Ind}_\kappa \mathcal{C}$ whose unique κ-filtered-colimit-preserving extension is $\mathrm{id}_{\mathrm{Ind}_\kappa \mathcal{C}}$. Thus $i_!^{\mathrm{op}} \simeq \mathrm{Ind}_\kappa i$ is also an equivalence. □

Proposition 9.6.4. *For any small ∞-category $\mathcal{C}$, the full subcategory $(\mathrm{Ind}_\kappa \mathcal{C})^\kappa$ of κ-compact objects is an idempotent-completion of $\mathcal{C}$.*

Proof. We know $\mathcal{C}^{\mathrm{idem}} \subseteq \mathrm{Ind}_\kappa \mathcal{C}$ from Example 9.5.2. Since κ-filtered colimits are computed in $\mathsf{PSh}(\mathcal{C})$ (Corollary 9.5.6), the objects of $\mathcal{C}^{\mathrm{idem}}$ are κ-compact. It remains to show that any κ-compact object ϕ of $\mathrm{Ind}_\kappa \mathcal{C}$ is a retract of a representable presheaf. Let $p\colon \mathcal{E} \to \mathcal{C}$ be the right fibration for ϕ; then we know $\phi \simeq \mathrm{colim}_{\mathcal{E}}\, \mathsf{y} \circ p$, where $\mathcal{E}$ is a κ-filtered ∞-category. It follows that

$$\mathsf{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(\phi, \phi) \simeq \mathrm{colim}_{\mathcal{E}}\, \mathsf{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(\phi, \mathsf{y} \circ p).$$

Arguing as in the proof of Proposition 8.6.9 it follows that id_ϕ has to be the image of some map $\phi \to \mathsf{y} \circ p(e)$, i.e. that ϕ is a retract of some $\mathsf{y}(p(e))$, as required. □

Corollary 9.6.5. *For a functor $f\colon \mathcal{C} \to \mathcal{D}$ of small ∞-categories, the induced functor $\mathrm{Ind}_\kappa f\colon \mathrm{Ind}_\kappa \mathcal{C} \to \mathrm{Ind}_\kappa \mathcal{D}$ is an equivalence if and only if f is a Morita equivalence.*

Proof. Suppose $\mathrm{Ind}_\kappa f$ is an equivalence. Then it restricts to an equivalence on the full subcategories of κ-compact objects, and so is a Morita equivalence by Proposition 9.6.4. Conversely, if f is a Morita equivalence we get a commutative square

$$\begin{tikzcd} \mathrm{Ind}_\kappa \mathcal{C} \arrow[r, "\mathrm{Ind}_\kappa f"] \arrow[d, "\sim"'] & \mathrm{Ind}_\kappa \mathcal{D} \arrow[d, "\sim"] \\ \mathrm{Ind}_\kappa \mathcal{C}^{\mathrm{idem}} \arrow[r, "\mathrm{Ind}_\kappa f^{\mathrm{idem}}"'] & \mathrm{Ind}_\kappa \mathcal{D}^{\mathrm{idem}} \end{tikzcd}$$

where the vertical maps are equivalences by Corollary 9.6.3. Thus if f^{idem} is an equivalence, so is $\mathrm{Ind}_\kappa f$. □

Corollary 9.6.6. *The following are equivalent for an ∞-category $\mathcal{M}$:*

(1) There exists an equivalence $\mathcal{M} \simeq \mathrm{Ind}_\kappa \mathcal{C}$ for some small ∞-category $\mathcal{C}$.

(2) $\mathcal{M}$ has κ-filtered colimits and there exists a small full subcategory of κ-compact objects that generates $\mathcal{M}$ under κ-filtered colimits.

(3) $\mathcal{M}$ has κ-filtered colimits, the full subcategory $\mathcal{M}^\kappa$ of κ-compact objects is small, and this generates $\mathcal{M}$ under κ-filtered colimits.

Proof. The equivalence of (1) and (2) is immediate from Proposition 9.6.1, while (3) immediately implies (2). It only remains to observe that (1) implies that $\mathcal{M}^\kappa$ is small, since by Proposition 9.6.4 it is equivalent to $\mathcal{C}^{\mathrm{idem}}$, which is small (Observation 8.6.13). □

Observation 9.6.7. If $\mathcal{M}$ satisfies the equivalent conditions of Corollary 9.6.6, then the inclusion $\mathcal{M}^\kappa \hookrightarrow \mathcal{M}$ extends to an equivalence

$$\mathrm{Ind}_\kappa \mathcal{M}^\kappa \xrightarrow{\sim} \mathcal{M}.$$

9.7 Accessible ∞-categories

Definition 9.7.1. An ∞-category $\mathcal{C}$ is *κ-accessible* if it satisifies the equivalent criteria of Corollary 9.6.6. We say that $\mathcal{C}$ is *accessible* if it is κ-accessible for *some* regular cardinal κ.

Definition 9.7.2. Suppose $\mathcal{C}$ is an accessible ∞-category. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *accessible* if it preserves κ-filtered colimits for some κ (and so also for all $\lambda > \kappa$).

Example 9.7.3. Cat_∞ is κ-accessible for any regular cardinal κ, since κ-finite ∞-categories are compact (Corollary 9.4.5) and these generate Cat_∞ under κ-filtered colimits by Fact 9.4.6.

Example 9.7.4. For any small ∞-category $\mathcal{C}$, the presheaf ∞-category $\mathsf{PSh}(\mathcal{C})$ is κ-accessible for any κ. Indeed, by Proposition 9.4.7 the full subcategory $\mathsf{PSh}(\mathcal{C})^\kappa$ of κ-compact objects consists of retracts of colimits of κ-finite diagrams in $\mathcal{C}$, and so is small as the set of equivalence classes of κ-finite diagrams in $\mathcal{C}$ is small. Moreover, this generates $\mathsf{PSh}(\mathcal{C})$ under κ-filtered colimits, since every presheaf is a small colimit of representables, and this can be rewritten as a κ-filtered colimit of κ-finite colimits of representables using Fact 9.4.6.

Lemma 9.7.5. *Any accessible ∞-category $\mathcal{C}$ is locally small.*

Proof. Suppose $\mathcal{C}$ is κ-accessible for some regular cardinal κ; then $\mathcal{C}^\kappa$ is a small ∞-category. Given objects x and y, we can write them as κ-filtered colimits $x \simeq \operatorname{colim}_{k\in\mathcal{K}} x_k$, $y \simeq \operatorname{colim}_{\ell\in\mathcal{L}} y_\ell$ of κ-compact objects. We then have

$$\mathcal{C}(x,y) \simeq \lim_{k\in\mathcal{K}^{\mathrm{op}}} \mathcal{C}(x_k, y) \simeq \lim_{k\in\mathcal{K}^{\mathrm{op}}} \operatorname{colim}_{\ell\in\mathcal{L}} \mathcal{C}^\kappa(x_k, y_\ell).$$

This is a small ∞-groupoid since $\mathcal{C}^\kappa$ is small, and small ∞-groupoids are closed under small limits and colimits. □

Somewhat surprisingly, we have:

Theorem 9.7.6. *A small ∞-category $\mathcal{C}$ is accessible if and only if $\mathcal{C}$ is idempotent-complete.*

We refer the reader to [Lur09, §5.4.3] for a proof (for now).

Warning 9.7.7. If $\mathcal{C}$ is a κ-accessible ∞-category and $\lambda > \kappa$, then it is *not* necessarily true that $\mathcal{C}$ is λ-accessible. However, we *can* always increase the index of accessiblity if we make λ sufficiently large in the following technical sense:

Definition 9.7.8. For regular cardinals κ and λ, we write $\kappa \ll \lambda$ if for any $\kappa_0 < \kappa$ and $\lambda_0 < \lambda$ we have $\lambda_0^{\kappa_0} < \lambda$.

Example 9.7.9. We have $\omega \ll \kappa$ for every regular cardinal κ.

Observation 9.7.10. For any regular cardinal κ, there exist arbitrarily large regular cardinals λ with $\kappa \ll \lambda$. For example, we can take λ to be the successor of any cardinal of the form τ^κ.

Remark 9.7.11. The notation $\kappa \ll \lambda$ is slightly confusing, as we for instance have $\omega \ll \omega$. If κ is an uncountable regular cardinal, then $\kappa \ll \kappa$ if and only if κ is inaccessible.

Fact 9.7.12. *Suppose $\lambda \gg \kappa$. Then any κ-filtered ∞-category is a λ-filtered colimit of λ-finite κ-filtered ∞-categories.*

Proposition 9.7.13. *Suppose $\mathcal{C}$ is a κ-accessible ∞-category and λ is a regular cardinal such that $\lambda > \kappa$ and $\lambda \gg \kappa$. Then $\mathcal{C}$ is also λ-accessible.*

Proof. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by the colimits of all λ-finite κ-filtered diagrams in $\mathcal{C}^{\kappa}$. Since the set of equivalence classes of such diagrams is small, $\mathcal{C}'$ is a small ∞-category. Moreover, it consists of λ-compact objects (since κ-compact objects are in particular λ-compact and λ-compact objects are closed under λ-finite colimits by Proposition 9.4.2). It remains to show that $\mathcal{C}'$ generates $\mathcal{C}$ under λ-filtered colimits. For an object $x \in \mathcal{C}$, we know there exists a diagram $F \colon \mathcal{K} \to \mathcal{C}^{\kappa}$, with $\mathcal{K}$ κ-filtered, whose colimit is x. By Fact 9.7.12 we can write $\mathcal{K}$ as a λ-filtered colimit of λ-finite κ-filtered ∞-categories, say as $\mathcal{K} \simeq \operatorname{colim}_{\ell \in \mathcal{L}} \mathcal{K}_{\ell}$. Then Corollary 7.4.5 implies that

$$x \simeq \operatorname{colim}_{\mathcal{K}} F \simeq \operatorname{colim}_{\ell \in \mathcal{L}} \operatorname{colim}_{\mathcal{K}_{\ell}} F|_{\mathcal{K}_{\ell}}.$$

Here $\operatorname{colim}_{\mathcal{K}_{\ell}} F|_{\mathcal{K}_{\ell}}$ is an object of $\mathcal{C}'$, so x is a λ-filtered colimit of objects of $\mathcal{C}'$, as required. □

The main point of accessibility is that this condition is preserved under many categorical constructions (but possibly with a larger index of accessibility). We end this section by stating some important results of this form — since the proofs get rather technical, we will not discuss them (for now?) and refer the reader to [Lur09, §5.4] for details.

Theorem 9.7.14. *If $\mathcal{C}$ is an accessible ∞-category and $\mathcal{K}$ is a small ∞-category, then* $\mathsf{Fun}(\mathcal{K}, \mathcal{C})$ *is accessible.*

Theorem 9.7.15. *Given a diagram $p \colon \mathcal{K} \to \widehat{\mathsf{Cat}}_{\infty}$ of (large) ∞-categories where $p(k)$ is an accessible ∞-category for every $k \in \mathcal{K}$ and $p(f)$ is an accessible functor for every morphism f in $\mathcal{K}$, then:*

(1) The ∞-category $\lim_{\mathcal{K}} p$ is accessible.

(2) A functor $\mathcal{C} \to \lim_{\mathcal{K}} p$ where $\mathcal{C}$ is accessible, is accessible if and only if the composite $\mathcal{C} \to p(k)$ is accessible for every $k \in \mathcal{K}$.

9.8 (⋆) Confluent ∞-categories

Everything in this section is due to Christian Sattler and David Wärn [SW25].

Definition 9.8.1. An ∞-category $\mathcal{I}$ is *confluent* if and only if $\mathcal{I}_{\phi/}$ is weakly contractible for any cospan $\phi \colon \Lambda^2_0 \to \mathcal{I}$.

Observation 9.8.2. Since Λ^2_0 is the pushout $\{0 < 1\} \amalg_{\{0\}} \{0 < 2\}$, for any cospan $\phi \colon \Lambda^2_0 \to \mathcal{I}$ we get a decomposition

$$\mathcal{I}_{\phi/} \simeq \mathcal{I}_{\phi_{01}/} \times_{\mathcal{I}_{\phi(0)/}} \mathcal{I}_{\phi_{02}/},$$

where $\phi_{0i} := \phi(0 \to i)$. Here $\mathcal{I}_{\phi_{0i}/} \to \mathcal{I}_{\phi(i)/}$ is an equivalence, so we can view this as the pullback

$$\mathcal{I}_{\phi/} \simeq \mathcal{I}_{\phi(1)/} \times_{\mathcal{I}_{\phi(0)/}} \mathcal{I}_{\phi(2)/},$$

where the maps are given by composition with ϕ_{01} and ϕ_{02}. Thus $\mathcal{I}$ is confluent if and only if for all pairs of morphisms $x \to y$, $x \to z$ in $\mathcal{I}$, the induced pullback

$$\mathcal{I}_{y/} \times_{\mathcal{I}_{x/}} \mathcal{I}_{z/}$$

is weakly contractible.

Example 9.8.3. Any ∞-category with pushouts is confluent, since $\mathcal{I}_{\phi/}$ then has an initial object.

Lemma 9.8.4. *$\mathcal{I}$ is confluent if and only if for every $f \colon x \to y$ in $\mathcal{I}$, the functor $f^* \colon \mathcal{I}_{y/} \to \mathcal{I}_{x/}$ is cofinal.*

Proof. f^* is cofinal when for every $g \colon x \to z$, the ∞-category

$$(\mathcal{I}_{y/})_{g/} := \mathcal{I}_{y/} \times_{\mathcal{I}_{x/}} (\mathcal{I}_{x/})_{g/}$$

is weakly contractible. Here $(\mathcal{I}_{x/})_{g/} \simeq \mathcal{I}_{z/}$, so this gives precisely the pullbacks that are required for $\mathcal{I}$ to be confluent in Observation 9.8.2. □

The following is a key closure property of confluent ∞-categories:

Lemma 9.8.5. *Suppose $p \colon \mathcal{E} \to \mathcal{B}$ is a left fibration and $\mathcal{B}$ is confluent. Then $\mathcal{E}$ is also confluent.*

Proof. Given morphisms $f \colon e \to e'$ and $g \colon e \to e''$ in $\mathcal{E}$, we must show that the pullback $\mathcal{E}_{e'/} \times_{\mathcal{E}_{e/}} \mathcal{E}_{e''/}$ is weakly contractible. But this is equivalent to $\mathcal{B}_{pe'/} \times_{\mathcal{B}_{pe/}} \mathcal{B}_{pe''/}$ since p is a left fibration, and this is weakly contractible when $\mathcal{B}$ is confluent. □

Our main aim is to prove the following:

Theorem 9.8.6 (Sattler–Wärn)**.** *An ∞-category $\mathcal{I}$ is confluent if and only if $\mathcal{I}$-indexed colimits in Gpd_∞ commute with pullbacks.*

Lemma 9.8.7. *Suppose $p \colon \mathcal{E} \to \mathcal{B}$ is a left fibration and $\mathcal{B}$ is confluent. Then $L^r_{\mathcal{B}}(p) := \mathfrak{F}_{\mathrm{cart}}(p)^r$ is a Kan fibration.*

Proof. Here $\mathfrak{F}_{\mathrm{cart}}(p)$ is the cartesian fibration for the functor $b \mapsto \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{b/}$ with functoriality given by composition. We must show that for $f \colon b' \to b$, the functor

$$f^* \colon \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{b/} \to \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{b'/}$$

localizes to an equivalence. But this is the pullback along the left fibration p of the functor $f^* \colon \mathcal{B}_{b/} \to \mathcal{B}_{b'/}$, which is cofinal by Lemma 9.8.4. This functor is therefore itself cofinal by Corollary 6.5.17 and so gives an equivalence on localizations by Proposition 6.5.16. □

Lemma 9.8.8. *Suppose we have a pullback square of ∞-categories*

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "q'"] \arrow[d, "p'"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{F} \arrow[r, "q"] & \mathcal{B}
\end{tikzcd}$$

where p and q are left fibrations and $\mathcal{B}$ is confluent. Then the commutative square of ∞-groupoids

$$\begin{tikzcd}
\|\mathcal{Q}\| \arrow[r, "\|q'\|"] \arrow[d, "\|p'\|"'] & \|\mathcal{E}\| \arrow[d, "\|p\|"] \\
\|\mathcal{F}\| \arrow[r, "\|q\|"] & \|\mathcal{B}\|.
\end{tikzcd}$$

is also a pullback.

Proof. From Corollary 6.4.6 we get a pullback square

$$\begin{tikzcd}
L^r_{\mathcal{E}}(q') \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
L^r_{\mathcal{B}}(q) \arrow[r] & \mathcal{B}.
\end{tikzcd}$$

Since contravariant equivalences give equivalences on localizations by Proposition 6.4.7 it suffices to show that this square gives a pullback on localizations. But here the horizontal morphisms are Kan fibrations by Lemma 9.8.7, so this follows from Corollary 7.7.7. □

Proof of Theorem 9.8.6. If colimits over $\mathcal{I}$ commute with pullbacks, then for $x \to y$ and $x \to z$ in $\mathcal{I}$ we have

$$\begin{aligned}
\|\mathcal{I}_{y/} \times_{\mathcal{I}_{x/}} \mathcal{I}_{z/}\| &\simeq \operatorname{colim}_{i \in \mathcal{I}} \mathcal{I}(y, i) \times_{\mathcal{I}(x,i)} \mathcal{I}(z, i) \\
&\simeq \left(\operatorname{colim}_{i \in \mathcal{I}} \mathcal{I}(y, i)\right) \times_{\operatorname{colim}_{i \in \mathcal{I}} \mathcal{I}(x,i)} \left(\operatorname{colim}_{i \in \mathcal{I}} \mathcal{I}(z, i)\right) \\
&\simeq \|\mathcal{I}_{y/}\| \times_{\|\mathcal{I}_{x/}\|} \|\mathcal{I}_{z/}\| \\
&\simeq *,
\end{aligned}$$

so that $\mathcal{I}$ is confluent. For the converse we want to show that the colimit functor $\mathsf{Fun}(\mathcal{I}, \mathsf{Gpd}_\infty) \to \mathsf{Gpd}_\infty$ preserves pullbacks. In terms of fibrations, this means that the functor

$$\mathsf{LFib}(\mathcal{I}) \xrightarrow{\|-\|} \mathsf{Gpd}_\infty$$

preserves pullbacks. Since pullbacks in $\mathsf{LFib}(\mathcal{I})$ are computed in Cat_∞, this follows from Lemma 9.8.8 as the source of any left fibration over $\mathcal{I}$ is confluent by Lemma 9.8.5 and any morphism between left fibrations over $\mathcal{I}$ is itself a left fibration (Observation 3.2.7). □

9.9 (⋆) Filtered colimits and finite limits

Our goal in this section is to prove Theorem 9.3.14 as a consequence of Theorem 9.8.6. We start by relating filtered and confluent ∞-categories:

Proposition 9.9.1. *An ∞-category is filtered if and only if it is confluent and weakly contractible.*

Proof. Suppose $\mathcal{I}$ is confluent and weakly contractible, and let $\mathcal{X} \subset \mathsf{Cat}_\infty$ be the full subcategory of ∞-categories $\mathcal{K}$ such that $\mathcal{I}_{\phi/}$ is weakly contractible for every functor $\phi \colon \mathcal{K} \to \mathcal{I}$. Since $\mathcal{I}$ is weakly contractible, we have $\emptyset \in \mathcal{X}$, and we also have $[0], [1] \in \mathcal{X}$ as any slice for these has an initial object. To show that all finite ∞-categories lie in $\mathcal{X}$ it then suffices to show that $\mathcal{X}$ is closed under pushouts. Given a pushout

$$\begin{tikzcd} \mathcal{K}_0 \arrow[r] \arrow[d] & \mathcal{K}_1 \arrow[d] \\ \mathcal{K}_2 \arrow[r] & \mathcal{K} \end{tikzcd}$$

and a functor $\phi \colon \mathcal{K} \to \mathcal{I}$, let $\phi_i := \phi|_{\mathcal{K}_i}$; we have an equivalence

$$\mathcal{I}_{\phi/} \simeq \mathcal{I}_{\phi_1/} \times_{\mathcal{I}_{\phi_0/}} \mathcal{I}_{\phi_2/},$$

which gives an equivalence

$$\|\mathcal{I}_{\phi/}\| \simeq \|\mathcal{I}_{\phi_1/}\| \times_{\|\mathcal{I}_{\phi_0/}\|} \|\mathcal{I}_{\phi_2/}\|,$$

since $\mathcal{I}$ is confluent and the right-hand side can be viewed as a pullback of colimits over $\mathcal{I}$. If $\mathcal{K}_i$ is in $\mathcal{X}$ for $i = 0, 1, 2$ it follows that $\mathcal{I}_{\phi/}$ is weakly contractible, so that $\mathcal{K}$ also lies in $\mathcal{X}$ as required. □

Combining this with Lemma 9.8.5, we get:

Corollary 9.9.2. *Suppose $\mathcal{I}$ is filtered and $\mathcal{E} \to \mathcal{I}$ is a left fibration. Then $\mathcal{E}$ is filtered if and only if it is weakly contractible.*

We can now prove the finite case of Theorem 9.3.14:

Corollary 9.9.3. *Suppose $\mathcal{I}$ is a filtered ∞-category. Then $\mathcal{I}$-indexed colimits in* Gpd_∞ *commute with finite limits.*

Proof. Let $\mathcal{X} \subseteq \mathsf{Cat}_\infty$ be the full subcategory of ∞-categories $\mathcal{K}$ such that $\mathcal{I}$-indexed colimits commute with $\mathcal{K}$-indexed limits. Then $\mathcal{X}$ contains $\emptyset$ as $\mathcal{I}$ is weakly contractible; it also contains limits over $[0]$ and $[1]$ since these contain an initial object. To see that $\mathcal{X}$ contains all finite ∞-categories it then suffices to show that it is closed under pushouts. This holds because $\mathcal{I}$-indexed colimits in Gpd_∞ commute with pullbacks by Theorem 9.8.6, since $\mathcal{I}$ is confluent, and a limit over a pushout in Cat_∞ decomposes as a pullback of limits over the components of the pushout by the dual of Corollary 7.4.5. □

Proposition 9.9.4. *Suppose $\mathcal{I}$ is a κ-filtered ∞-category and $\mathcal{E}_s \to \mathcal{I}$ is a collection of left fibrations indexed by a κ-finite set S. If each $\mathcal{E}_s$ is filtered, then so is*

$$\mathcal{I} \times_{\prod_I \mathcal{I}} \prod_{s \in SS} \mathcal{E}_s.$$

Proof. Let $\mathcal{Q} := \mathcal{I} \times_{\prod_I \mathcal{I}} \prod_I \mathcal{E}_i$. We use the criterion of Proposition 9.3.11. Given $\phi \colon \mathcal{K} \to \mathcal{Q}$ with $\mathcal{K}$ finite, we want to show there exists an extension to $\mathcal{K}^{\triangleright}$. By assumption each $\mathcal{E}_s$ is filtered, so the composite $\mathcal{K} \to \mathcal{E}_s$ admits an extension to $\mathcal{K}^{\triangleright}$ for every s. We obtain a diagram of shape $\coprod_s \mathcal{K}^{\triangleright} \to \mathcal{I}$. Here the source is a κ-finite ∞-category, so this extends over a right cone as $\mathcal{I}$ is κ-filtered. Now we use that $\mathcal{Q} \to \mathcal{I}$ is a left fibration to construct a compatible extension over $\mathcal{K}^{\triangleright}$ for every s. □

Proposition 9.9.5. *Suppose $\mathcal{I}$ is a κ-filtered ∞-category. Then $\mathcal{I}$-indexed colimits in* Gpd_∞ *commute with κ-finite limits.*

Proof. We want to show that the functor $\operatorname{colim}_{\mathcal{I}} \colon \mathsf{Fun}(\mathcal{I}, \mathsf{Gpd}_\infty) \to \mathsf{Gpd}_\infty$ preserves κ-finite limits. Since $\mathcal{I}$ is in particular filtered, we know from Corollary 9.9.3 that $\operatorname{colim}_{\mathcal{I}}$ preserves finite limits. By Proposition 9.2.4 it therefore suffices to show that it also preserves κ-finite products. We thus consider a collection of functors $\phi_s \colon \mathcal{I} \to \mathsf{Gpd}_\infty$ indexed by a κ-finite set S; we need to show that the canonical map

$$\operatorname{colim}_{\mathcal{I}} \prod_{s \in S} \phi_s \to \prod_{s \in S} \operatorname{colim}_{\mathcal{I}} \phi_s$$

is an equivalence. For this we apply the criterion of Corollary 7.7.4, which tells us that it suffices to prove that given $x_s \in \operatorname{colim}_{\mathcal{I}} \phi_s$ for each s, the colimit of $\prod_{s \in S} (\phi_s)_{x_s}$ is contractible. Translating this in terms of fibrations, we need to show that given a collection of left fibrations $\mathcal{E}_s \to \mathcal{I}$ indexed by $s \in S$ such that each $\mathcal{E}_s$ is weakly contractible, the ∞-category $\mathcal{I} \times_{\prod_S \mathcal{I}} \prod_{s \in S} \mathcal{E}_s$ is also weakly contractible. This follows from Proposition 9.9.4 since each $\mathcal{E}_s$ is filtered by Corollary 9.9.2. □

Chapter 10

Presentable ∞-categories

10.1 Presentable ∞-categories

Definition 10.1.1. An ∞-category is (κ-)*presentable* if it is (κ-)accessible and cocomplete.

Example 10.1.2. Cat_∞ and $\mathsf{PSh}(\mathcal{C})$ for any small ∞-category $\mathcal{C}$ are presentable.

We want to characterize the presentable ∞-categories as those of the form $\mathrm{Ind}_\kappa \mathcal{C}$ where $\mathcal{C}$ is a small ∞-category with κ-finite colimits. For this we need to know that ∞-categories of this form are always presentable, i.e. that they are cocomplete. This requires some preliminary discussion.

Proposition 10.1.3. *Suppose $\mathcal{C}$ is a small idempotent-complete ∞-category. Then the following are equivalent for a regular cardinal κ:*

(1) *$\mathcal{C}$ has κ-finite colimits.*

(2) *The inclusion $\mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})^\kappa$ has a left adjoint.*

Observation 10.1.4. Suppose $i\colon \mathcal{C} \hookrightarrow \mathcal{D}$ is a fully faithful functor with left adjoint L. Given a diagram $p\colon \mathcal{K} \to \mathcal{C}$ such that $i \circ p$ has a colimit in $\mathcal{D}$, we claim that p also has a colimit in $\mathcal{C}$, namely $L(\mathrm{colim}_{\mathcal{K}}\, ip)$. Indeed, this is the colimit of the diagram Lip since the left adjoint L preserves colimits, but $Li \simeq \mathrm{id}_{\mathcal{C}}$ since i is fully faithful (Corollary 6.3.11), so $Lip \simeq p$.

Proof of Proposition 10.1.3. Suppose first that $\mathcal{C}$ has κ-finite colimits. We must show that the copresheaf $\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\phi, \mathsf{y}(-))$ on $\mathcal{C}$ is corepresentable for every $\phi \in \mathsf{PSh}(\mathcal{C})^\kappa$. By Proposition 9.4.7, there exists a functor $p\colon \mathcal{K} \to \mathcal{C}$ with $\mathcal{K}$ κ-finite so that ϕ is a retract of $\psi := \mathrm{colim}\, \mathsf{y} \circ p$. We first observe that

$$\mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\psi, \mathsf{y}(-)) \simeq \lim_{\mathcal{K}^{\mathrm{op}}} \mathcal{C}(p, -)$$

is corepresented by the colimit $\mathrm{colim}\, p$ in $\mathcal{C}$. Since $\mathcal{C}$ is by assumption idempotent-complete, it then follows that the original copresheaf is corepresented by a retract of $\mathrm{colim}\, p$.

Now suppose (2) holds, so that the inclusion $i\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})^{\kappa}$ has a left adjoint L. For a diagram $p\colon \mathcal{K} \to \mathcal{C}$ with $\mathcal{K}$ κ-finite, we know that $i \circ p$ has a colimit in $\mathsf{PSh}(\mathcal{C})^{\kappa}$, and so Observation 10.1.4 shows that p has a colimit in $\mathcal{C}$, namely $L(\operatorname{colim} ip)$. □

Observation 10.1.5. Suppose we have an adjunction $f \dashv g$ for functors

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g$$

among small ∞-categories. Then we get an adjunction $g^{\mathrm{op}} \dashv f^{\mathrm{op}}$ on opposite ∞-categories (Exercise 6.3) and so an adjunction $f^{\mathrm{op},*} \dashv g^{\mathrm{op},*}$ on presheaves by Lemma 6.3.8. Since adjoints are unique, this means that $f^{\mathrm{op},*}$ is equivalent to $g_!^{\mathrm{op}}$, so that we also have

$$f_!^{\mathrm{op}} \dashv g_!^{\mathrm{op}}.$$

Here both functors restrict to $\operatorname{Ind}_{\kappa}(-)$, and so we have an adjunction

$$\operatorname{Ind}_{\kappa} f : \operatorname{Ind}_{\kappa} \mathcal{C} \rightleftarrows \operatorname{Ind}_{\kappa} \mathcal{D} : \operatorname{Ind}_{\kappa} g$$

on κ-filtered cocompletions.

Corollary 10.1.6. *Suppose $\mathcal{C}$ is a small ∞-category with κ-finite colimits. Then:*

(1) The inclusion $\operatorname{Ind}_{\kappa} \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$ has a left adjoint.

(2) $\operatorname{Ind}_{\kappa} \mathcal{C}$ is cocomplete.

(3) $\operatorname{Ind}_{\kappa} \mathcal{C}$ is a κ-presentable ∞-category.

Proof. By Proposition 10.1.3 the inclusion $i\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})^{\kappa}$ has a left adjoint L. From Observation 10.1.5 we get an induced adjunction

$$\operatorname{Ind}_{\kappa} L : \mathsf{PSh}(\mathcal{C}) \rightleftarrows \operatorname{Ind}_{\kappa} \mathcal{C} : \operatorname{Ind}_{\kappa} i,$$

since $\mathsf{PSh}(\mathcal{C}) \simeq \operatorname{Ind}_{\kappa} \mathsf{PSh}(\mathcal{C})^{\kappa}$ by Example 9.7.4 and Observation 9.6.7. Here we need to identify $\operatorname{Ind}_{\kappa} i$ with the fully faithful inclusion of $\operatorname{Ind}_{\kappa} \mathcal{C}$ in $\mathsf{PSh}(\mathcal{C})$. To do so we observe that $\operatorname{Ind}_{\kappa} i$ is the unique κ-filtered-colimit-preserving functor extending the composite

$$\mathcal{C} \xrightarrow{i} \mathsf{PSh}(\mathcal{C})^{\kappa} \hookrightarrow \mathsf{PSh}(\mathcal{C}),$$

which is the Yoneda embedding. The cocompleteness of $\operatorname{Ind}_{\kappa} \mathcal{C}$ now follows from Observation 10.1.4; since we know $\operatorname{Ind}_{\kappa} \mathcal{C}$ is accessible, it is therefore presentable. □

Proposition 10.1.7. *The following are equivalent for an ∞-category $\mathcal{C}$:*

(1) $\mathcal{C}$ is presentable.

(2) $\mathcal{C}$ is accessible, and for every regular cardinal κ the full subcategory $\mathcal{C}^{\kappa}$ has κ-finite colimits.

(3) There exists a regular cardinal κ such that $\mathcal{C}$ is κ-accessible and $\mathcal{C}^{\kappa}$ admits κ-finite colimits.

(4) $\mathcal{C}$ is equivalent to $\operatorname{Ind}_{\kappa} \mathcal{D}$ where $\mathcal{D}$ is a small ∞-category with κ-finite colimits.

(5) $\mathcal{C}$ is cocomplete and there exists a small full subcategory of κ-compact objects for some regular cardinal κ that generates $\mathcal{C}$ under small colimits.

Proof. (1) implies (2) since κ-compact objects are always closed under any κ-finite colimits that exist by Proposition 9.4.2, while (2) clearly implies (3) and this in turn implies (4) by Observation 9.6.7. If (4) holds, we know that $\mathcal{C}$ is presentable by Corollary 10.1.6, so the first four conditions are equivalent. To see they are also equivalent to (5), we first observe that this clearly holds when $\mathcal{C}$ is κ-presentable for some given κ, since then every object is a small κ-filtered colimit of κ-compact objects. Conversely, suppose (5) holds, with $\mathcal{C}_0 \subseteq \mathcal{C}$ the full subcategory of κ-compact objects in question. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ spanned by the colimits of κ-finite diagrams in $\mathcal{C}_0$. Then $\mathcal{C}'$ is a small ∞-category and also consists of κ-compact objects (Proposition 9.4.2). Moreover, any object of $\mathcal{C}$ is the colimit of a small diagram in $\mathcal{C}_0$ and so of a κ-filtered diagram in $\mathcal{C}'$, so that $\mathcal{C}$ is κ-accessible by Corollary 9.6.6. □

Remark 10.1.8. We will see another important characterization of presentable ∞-categories, as the accessible localizations of presheaf ∞-categories, below in §10.4.

Corollary 10.1.9. *If $\mathcal{C}$ is a presentable ∞-category, then $\mathcal{C}$ admits small limits.*

Proof. By Proposition 10.1.7 we have an equivalence $\mathcal{C} \simeq \operatorname{Ind}_{\kappa} \mathcal{D}$, where $\mathcal{D}$ is a small ∞-category with κ-finite colimits. We can therefore identify $\operatorname{Ind}_{\kappa} \mathcal{D} \subset \mathsf{PSh}(\mathcal{D})$ as the full subcategory of presheaves that preserve κ-finite limits by Proposition 9.5.8. But this full subcategory is closed under limits in $\mathsf{PSh}(\mathcal{D})$, since these are computed pointwise (Corollary 5.5.10) and limits commute. □

Observation 10.1.10. Suppose $\mathcal{C}$ is a κ-presentable ∞-category and $\lambda > \kappa$ is another regular cardinal. Then $\mathcal{C}$ is also λ-presentable, since it is generated under small colimits by κ-compact objects, which are also λ-compact (as every λ-filtered ∞-category is also κ-filtered). Note that this contrasts with the situation for κ-accessible ∞-categories (Proposition 9.7.13) where we needed a strong set-theoretic assumption on λ for a κ-accessible ∞-category to also be λ-accessbile.

Lemma 10.1.11. *Suppose $\mathcal{C}$ is a presentable ∞-category. Then for every object $x \in \mathcal{C}$ there exists some regular cardinal λ such that x is λ-compact.*

Proof. Let κ be a regular cardinal such that $\mathcal{C}$ is κ-presentable. Then there is a κ-filtered diagram $p\colon \mathcal{K} \to \mathcal{C}^\kappa$ whose colimit in $\mathcal{C}$ is x. We can choose a regular cardinal $\lambda > \kappa$ such that $\mathcal{K}$ is λ-finite. Since κ-compact objects are in particular λ-compact, and λ-compact objects are closed under λ-finite colimits, it follows that x is λ-compact. □

Proposition 10.1.12. *Suppose $\mathcal{C}$ is a κ-presentable ∞-category and $\mathcal{D}$ is a cocomplete ∞-category. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves colimits if and only if F preserves κ-filtered colimits and $F|_{\mathcal{C}^\kappa}\colon \mathcal{C}^\kappa \to \mathcal{D}$ preserves κ-finite colimits.*

Proof. First suppose F preserves colimits. Since $\mathcal{C}^\kappa$ is closed under κ-finite colimits in $\mathcal{C}$ (Proposition 9.4.2), it follows that $F|_{\mathcal{C}^\kappa}$ preserves κ-finite colimits.

Now we consider the less trivial direction. Since $\mathcal{C}$ is κ-presentable, we have $\mathcal{C} \simeq \mathrm{Ind}_\kappa\,\mathcal{C}^\kappa$, and the inclusion $i\colon \mathrm{Ind}_\kappa\,\mathcal{C}^\kappa \hookrightarrow \mathsf{PSh}(\mathcal{C}^\kappa)$ has a left adjoint L by Corollary 10.1.6. Let $f := F|_{\mathcal{C}^\kappa}$; since F preserves κ-filtered colimits, Proposition 9.5.9 implies that F is the left Kan extension of f along the inclusion $\mathcal{C}^\kappa \to \mathrm{Ind}_\kappa\,\mathcal{C}^\kappa$. Let

$$G := i_!F\colon \mathsf{PSh}(\mathcal{C}^\kappa) \to \mathcal{D}$$

be the further left Kan extension of F; then G is also the left Kan extension of f along the Yoneda embedding, and so G is colimit-preserving by Proposition 8.4.1. Since $i^* \dashv L^*$ we get a canonical natural transformation $\beta\colon G \to F \circ L$ adjoint to the inverse of the counit $F \xrightarrow{\sim} i^*G$. We claim that if β is an equivalence, then F preserves colimits: To see this, consider a diagram $p\colon \mathcal{K} \to \mathcal{C}$, let $\bar{q}\colon \mathcal{K}^\triangleright \to \mathsf{PSh}(\mathcal{C}^\kappa)$ be a colimit cone for $i \circ p$; then we know that $L(\bar{q})$ is a colimit cone for p. Since G preserves colimits, it follows that $G(\bar{q}) \simeq FL(\bar{q})$ is a colimit cone in $\mathcal{D}$, so that F indeed preserves the colimit of p.

It remains to show that β is in fact an equivalence. Let $\mathcal{E} \subseteq \mathsf{PSh}(\mathcal{C}^\kappa)$ denote the full subcategory of presheaves ϕ such that $\beta_\phi\colon G(\phi) \to F(L\phi)$ is an equivalence in $\mathcal{D}$; we want to show that $\mathcal{E}$ is all of $\mathsf{PSh}(\mathcal{C}^\kappa)$. First observe that since F preserves κ-filtered colimits, $\mathcal{E}$ is closed under these. It therefore suffices to show that $\mathcal{E}$ contains $\mathsf{PSh}(\mathcal{C}^\kappa)^\kappa$, as this generates $\mathsf{PSh}(\mathcal{C}^\kappa)$ under such colimits.

By Proposition 10.1.3, the inclusion $j\colon \mathcal{C}^\kappa \hookrightarrow \mathsf{PSh}(\mathcal{C}^\kappa)^\kappa$ has a left adjoint ℓ, so that $L \simeq \mathrm{Ind}_\kappa\,\ell$ by the proof of Corollary 10.1.6. From the construction of this adjoint, we also see that β restricts on $\mathsf{PSh}(\mathcal{C}^\kappa)^\kappa$ to the transformation

$$\gamma\colon j_!f \to f \circ \ell$$

similarly adjoint to the inverse of the unit $f \xrightarrow{\sim} j^*j_!f$. It thus suffices to check that this is an equivalence. Let $\mathcal{E}' \subseteq \mathsf{PSh}(\mathcal{C}^\kappa)^\kappa$ be the full subcategory of objects ϕ such that γ_ϕ is an equivalence. We first observe that γ is an equivalence on objects in $\mathcal{C}^\kappa$, so $\mathcal{C}^\kappa \subseteq \mathcal{E}'$. Moreover, the functor $j_!f$ preserves κ-finite colimit as it is the restriction of the colimit-preserving functor G to the full subcategory $\mathsf{PSh}(\mathcal{C}^\kappa)^\kappa$, which is closed under such colimits. Since f by assumption preserves κ-finite colimits, as does the left adjoint ℓ, it follows that $\mathcal{E}'$ is closed under κ-finite colimits. It is also obviously closed under retracts. Therefore, as we know from Proposition 9.4.7 that every object in $\mathsf{PSh}(\mathcal{C}^\kappa)^\kappa$ is a retract of the colimit of a κ-finite diagram in $\mathcal{C}^\kappa$, we can conclude that $\mathcal{E}'$ is all of $\mathsf{PSh}(\mathcal{C}^\kappa)^\kappa$. This completes the proof. □

The condition of presentability is closed under many categorical constructions. In particular, we have:

Proposition 10.1.13. *If $\mathcal{C}$ is presentable and $\mathcal{K}$ is a small ∞-category, then $\mathsf{Fun}(\mathcal{K}, \mathcal{C})$ is presentable.*

Proof. Combine Theorem 9.7.14 with (the dual of) Corollary 5.5.10. □

Proposition 10.1.14. *Given a diagram $p\colon \mathcal{K} \to \widehat{\mathsf{Cat}}_\infty$ of (large) ∞-categories where $p(k)$ is a presentable ∞-category for every $k \in \mathcal{K}$ and $p(f)$ is a colimit-preserving functor for every morphism f in $\mathcal{K}$, then:*

(1) The ∞-category $\lim_{\mathcal{K}} p$ is presentable.

(2) A functor $\mathcal{C} \to \lim_{\mathcal{K}} p$ where $\mathcal{C}$ is presentable, is colimit-preserving if and only if the composite $\mathcal{C} \to p(k)$ preserves colimits for every $k \in \mathcal{K}$.

Proof. Combine Theorem 9.7.15 with (the dual of) Corollary 5.5.15. □

10.2 The adjoint functor theorem

Our goal in this section is to prove the *adjoint functor theorem* for presentable ∞-categories, which gives very useful characterizations of when functors between presentable ∞-categories are left or right adjoints. This boils down to identifying the representable presheaves (which is easy) and the corepresentable copresheaves (which is hard) on a presentable ∞-category. We start with the easy direction:

Proposition 10.2.1. *If $\mathcal{C}$ is a presentable ∞-category, then a presheaf $\Phi\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ is representable if and only if Φ preserves small limits.*

Observation 10.2.2. For a small ∞-category $\mathcal{C}$ and a presheaf $\phi\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Gpd}_\infty$, the right Kan extension $(\mathsf{y}^{\mathrm{op}})_*\phi\colon \mathsf{PSh}(\mathcal{C})^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ is given by

$$(\mathsf{y}^{\mathrm{op}})_*\phi(\psi) \simeq \lim_{\mathcal{C}}^{\psi} \phi \simeq \mathsf{Map}_{\mathsf{PSh}(\mathcal{C})}(\psi, \phi).$$

In other words, $\mathsf{y}^{\mathrm{op}}_*\phi$ is the presheaf on $\mathsf{PSh}(\mathcal{C})$ represented by ϕ.

Proof of Proposition 10.2.1. Any representable presheaf preserves limits, so it suffices to prove the converse. Choose κ so that $\mathcal{C}$ is κ-presentable; then Φ^{op} is left Kan extended from a functor $\phi^{\mathrm{op}}\colon \mathcal{C}^\kappa \to \mathsf{Gpd}_\infty^{\mathrm{op}}$ that preserves κ-finite colimits by Proposition 10.1.12. Let $\Phi'\colon \mathsf{PSh}(\mathcal{C}^\kappa)^{\mathrm{op}} \to \mathsf{Gpd}_\infty$ be the right Kan extension of Φ along the inclusion $(\mathrm{Ind}_\kappa \mathcal{C}^\kappa)^{\mathrm{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C}^\kappa)^{\mathrm{op}}$; then Φ' is also the right Kan extension of ϕ along y^{op} and $\Phi \simeq \Phi'|_{\mathrm{Ind}_\kappa \mathcal{C}^\kappa}$. By Observation 10.2.2 we know that Φ' is the presheaf on $\mathsf{PSh}(\mathcal{C}^\kappa)$ represented by ϕ. But we also know from Proposition 9.5.8 that ϕ lies in the full subcategory $\mathrm{Ind}_\kappa \mathcal{C}^\kappa$, since it preserves κ-finite limits. Hence the restriction Φ of Φ' is indeed the presheaf represented by ϕ. □

Corollary 10.2.3 (Adjoint Functor Theorem, easy half)**.** *Suppose $\mathcal{C}$ is a presentable ∞-category and $\mathcal{D}$ is a locally small ∞-category. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves small colimits.*

Proof. We know from Corollary 6.3.7 that F is a left adjoint if and only if the presheaf $\mathcal{D}(F(-), d)$ on $\mathcal{C}$ is representable for every $d \in \mathcal{D}$. By Proposition 10.2.1 this is equivalent to these presheaves preserving small limits, which happens for all d precisely if F preserves colimits. □

The other half of the presentable adjoint functor theorem will follow from the following characterization of corepresentable copresheaves:

Proposition 10.2.4. *Suppose $\mathcal{C}$ is a presentable ∞-category. Then a copresheaf $\Phi\colon \mathcal{C} \to \mathsf{Gpd}_\infty$ is corepresentable if and only if Φ is accessible and preserves small limits.*

For the proof we need an indirect criterion for the existence of initial objects[1]. To prove this we will in turn make use of the following somewhat non-obvious criterion for initial objects:

Proposition 10.2.5. *An object x of an ∞-category $\mathcal{C}$ is inital if and only if x is the limit of the functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$.*

Proof. Using Proposition 5.6.6, we see that it suffices to show that a cone $\mathcal{C}^{\triangleleft} \to \mathcal{C}$ that restricts to $\mathrm{id}_{\mathcal{C}}$ is terminal in $\mathcal{C}_{/\mathrm{id}}$ if and only if it takes $-\infty \to x$ to id_x.

We first check that this condition must hold for a terminal cone. Observe that given two cones $\alpha, \beta\colon \mathcal{C}^{\triangleleft} \to \mathcal{C}$ on $\mathrm{id}_{\mathcal{C}}$, we get a morphism of cones $\alpha \to \beta$ by taking the composite

$$[1] \star \mathcal{C} \simeq [0] \star \mathcal{C}^{\triangleleft} \xrightarrow{\mathrm{id}_{[0]} \star \beta} \mathcal{C}^{\triangleleft} \xrightarrow{\alpha} \mathcal{C};$$

on the cone points, the morphism $\alpha(-\infty) \to \beta(-\infty)$ is the component of α at the object $\beta(-\infty) \in \mathcal{C}$. In particular, from a cone α we get a canonical endomorphism of cones $\alpha \to \alpha$, given on the cone point by the image of $-\infty \to \alpha(-\infty)$ under α. If α is a limit cone, then this endomorphism must be equivalent to id_α, since a limit cone is precisely a terminal object in the ∞-category of cones. In particular its component at $-\infty$ is an identity map, which means that α takes $-\infty \to \alpha(-\infty)$ to $\mathrm{id}_{\alpha(-\infty)}$.

[1] I learned this from the paper [NRS20] of Nguyen, Raptis and Schrade on adjoint functor theorems.

Conversely, suppose γ is a cone such that $\gamma(-\infty \to x) \simeq \mathrm{id}_x$; we want to prove that it is a terminal cone. Given a map of cones $\phi\colon \beta \to \gamma$, we have a natural commutative triangle

$$\begin{tikzcd} \beta(-\infty) \arrow[rr, "\phi_{-\infty}"] \arrow[dr, "\beta_x"'] & & x \arrow[dl, "\mathrm{id}_x"] \\ & x & \end{tikzcd}$$

so that $\phi_{-\infty} \simeq \beta_x$. More precisely, if we define $\sigma\colon \mathcal{C}_{/\mathrm{id}} \to \mathcal{C}_{/x}$ to take β to $\beta_x\colon \beta(-\infty) \to x$, then we have a commutative diagram

$$\begin{tikzcd} (\mathcal{C}_{/\mathrm{id}})_{/\gamma} \arrow[rr, "\mathrm{ev}_{-\infty}"] \arrow[dr] & & \mathcal{C}_{/x} \\ & (\mathcal{C}_{/\mathrm{id}}). \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Since $\mathcal{C}_{/\mathrm{id}} \to \mathcal{C}$ is a right fibration, the top horizontal functor here is an equivalence by Lemma 3.3.10. Inverting it, σ thus provides a section s of the forgetful functor $(\mathcal{C}_{/\mathrm{id}})_{/\gamma} \to \mathcal{C}_{/\mathrm{id}}$. Now (the dual of) Proposition 5.6.6 implies that this exhibits γ as a terminal object precisely if s takes γ to id_γ. But under the equivalence $\mathrm{ev}_{-\infty}$ this corresponds to the condition that σ takes γ to id_x, which is true by assumption. □

Corollary 10.2.6. *Let $\mathcal{C}$ be a locally small ∞-category with small limits. Suppose there exists a full subcategory $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ such that $\mathcal{C}_0$ is small and for every object $x \in \mathcal{C}$ there exists an object $y \in \mathcal{C}_0$ such that $\mathcal{C}(y, x)$ is non-empty. Then $\mathcal{C}$ has an initial object.*

Proof. We want to use Proposition 10.2.5 to conclude that $\mathcal{C}$ has an initial object. Since $\mathcal{C}_0$ is small, the inclusion i has a limit, so it suffices to show that i is coinitial, since then its limit is also the limit of $\mathrm{id}_\mathcal{C}$ by (the dual of) Theorem 6.5.13.

We thus need to show that for every $x \in \mathcal{C}$, the ∞-category $\mathcal{C}_{0/x}$ is weakly contractible. Consider a functor $F\colon \mathcal{K} \to \mathcal{C}_{0/x}$ where $\mathcal{K}$ is small. The composite $F'\colon \mathcal{K} \xrightarrow{F} \mathcal{C}_{0/x} \to \mathcal{C}_{/x}$ admits a limit cone $\bar{F}'\colon \mathcal{K}^{\triangleleft} \to \mathcal{C}_{/x}$, since $\mathcal{C}$ is complete. By assumption, there exists a map $y \to \bar{F}'(-\infty)$ with $y \in \mathcal{C}_0$. Composing $\bar{F}'$ with this, we therefore obtain a cone $\bar{F}\colon \mathcal{K}^{\triangleleft} \to \mathcal{C}_{0/x}$ extending F. This implies that $\mathcal{C}_{0/x}$ is weakly contractible by Fact 9.3.9. □

Proof of Proposition 10.2.4. Since $\mathcal{C}$ is presentable, all corepresentable copresheaves on $\mathcal{C}$ preserve limits and are accessible, since we know from Lemma 10.1.11 that for every object there is some κ for which it is κ-compact. It remains to prove the converse, for which we consider the left fibration $p\colon \mathcal{E} \to \mathcal{C}$ corresponding to Φ; we want to prove that $\mathcal{E}$ has an initial object.

The ∞-category $\mathcal{E}$ is locally small, and it has small limits by the dual of Corollary 7.6.8. Choose κ so that $\mathcal{C}$ is κ-presentable and Φ is κ-accessible, and define the full subcategory $\mathcal{E}^\kappa \subseteq \mathcal{E}$ by the pullback

$$\begin{tikzcd}
\mathcal{E}^\kappa \arrow[r, "j"] \arrow[d, "q"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{C}^\kappa \arrow[r, "i"'] & \mathcal{C};
\end{tikzcd}$$

we will apply the criterion of Corollary 10.2.6 to $\mathcal{E}^\kappa$ to conclude that $\mathcal{E}$ has an initial object.

Given an object $e \in \mathcal{E}$, we must show that there exists a morphism from some object of $\mathcal{E}^\kappa$ to e. We can write the image $c = p(e)$ as a κ-filtered colimit

$$c \simeq \operatorname{colim}_{i\in\mathcal{I}} c_i$$

of κ-compact objects c_i. Since Φ is κ-accessible, it follows that the object

$$e \in \mathcal{E}_c \simeq \Phi(c) \simeq \operatorname{colim}_{i\in\mathcal{I}} \Phi(c_i)$$

must lie in the image of $\Phi(c_i)$ for some i. In terms of the fibration $\mathcal{E}$, this means that there exists a morphism $e' \to e$ lying over $c_i \to c$ where c_i is κ-compact; thus e' lies in $\mathcal{E}^\kappa$, as required. □

Corollary 10.2.7 (Adjoint Functor Theorem, second half)**.** *Suppose $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories. Then a functor $G\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint if and only if it is accessible and preserves small limits.*

Proof. We know G is a right adjoint if and only if the copresheaf $\mathcal{D}(d, G(-))$ is corepresentable for every $d \in \mathcal{D}$. By Proposition 10.2.4 this is true if and only if this copresheaf is accessible and preserves limits for every $d \in \mathcal{D}$. The latter condition holds if and only if G preserves limits, so it remains to relate the two accessibility conditions.

Suppose first that G is accessible and preserves small limits. Then for a given $d \in \mathcal{D}$ we can find a regular cardinal κ so that d is κ-compact and G is κ-accessible; then $\mathcal{D}(d, G(-))$ is again κ-accessible. Conversely, if these copresheaves are accessible for every d, first choose κ so that $\mathcal{D}$ is κ-presentable. Since $\mathcal{D}^\kappa$ is small, we can choose $\lambda \geq \kappa$ so that $\mathcal{D}(d, G(-))$ is λ-accessible for all $d \in \mathcal{D}^\kappa$. Since the functors $\mathcal{D}(d, -)$ with $d \in \mathcal{D}^\kappa$ then jointly detect equivalences in $\mathcal{D}$, it follows that G must be a λ-accessible functor. □

10.3 Factorization systems

Definition 10.3.1. A *factorization system* on an ∞-category $\mathcal{C}$ consists of a pair of wide subcategories $(\mathcal{L}, \mathcal{R})$ of $\mathcal{C}$ such that

- $\mathcal{L} \perp \mathcal{R}$, i.e. every morphism in $\mathcal{L}$ is left orthogonal to every morphism in $\mathcal{R}$,
- every morphism f in $\mathcal{C}$ admits a factorization $f \simeq r\ell$ with r in $\mathcal{R}$ and ℓ in $\mathcal{L}$.

Remark 10.3.2. Sometimes factorization systems are called *orthogonal* factorization systems. This is probably Bourbaki's fault.

Observation 10.3.3. If $(\mathcal{L}, \mathcal{R})$ is a factorization system on $\mathcal{C}$, then $(\mathcal{R}^{\mathrm{op}}, \mathcal{L}^{\mathrm{op}})$ is a factorization system on $\mathcal{C}^{\mathrm{op}}$.

Exercise 10.1. Show that for any ∞-category $\mathcal{C}$, we have factorization systems $(\mathcal{C}^{\simeq}, \mathcal{C})$ and $(\mathcal{C}, \mathcal{C}^{\simeq})$.

Example 10.3.4. Epimorphisms and monomorphisms form a factorization system on Gpd_∞ by Lemma 2.4.3 and Observation 2.2.8.

Example 10.3.5. Essentially surjective and fully faithful functors form a factorization system on Cat_∞ by Lemma 2.8.3 and Observation 2.8.5.

Example 10.3.6. Cofinal functors and right fibrations form a factorization system on Cat_∞ by Observation 6.5.5 and Corollary 6.5.6. Dually, coinitial functors and left fibrations also form a factorization system.

Notation 10.3.7. Given an ∞-groupoid $S \subseteq \mathcal{C}[1]$ of morphisms in an ∞-category $\mathcal{C}$, we write

- $\mathrm{LO}(S)$ for the ∞-groupoid of morphisms that are left orthogonal to S,
- $\mathrm{RO}(S)$ for the ∞-groupoid of morphisms that are right orthogonal to S.

Proposition 10.3.8. *Suppose $(\mathcal{L}, \mathcal{R})$ is a factorization system on an ∞-category $\mathcal{C}$. Then $\mathcal{L} = \mathrm{LO}(\mathcal{R})$ and $\mathcal{R} = \mathrm{RO}(\mathcal{L})$.*

Proof. By assumption $\mathcal{L} \subseteq \mathrm{LO}(\mathcal{R})$, so it remains to show that any morphism $f\colon X \to Z$ in $\mathrm{LO}(\mathcal{R})$ lies in $\mathcal{L}$. We can factor f as $X \xrightarrow{\ell} Y \xrightarrow{r} Z$ where ℓ is in $\mathcal{L}$ and r is in $\mathcal{R}$. But then r is also in $\mathrm{LO}(\mathcal{R})$ by Lemma 2.4.5 and so is left orthogonal to itself. This means that r is an equivalence by Exercise 2.11. □

Corollary 10.3.9. *If $(\mathcal{L}, \mathcal{R})$ is a factorization system on $\mathcal{C}$, then the morphisms in both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.*

Proof. This follows from Proposition 10.3.8 together with Lemma 2.4.9 and its dual. □

Remark 10.3.10. This definition of factorization systems is taken from [ABFJ22]. Note that it is a bit different from that given in [Lur09], which does not a priori ask for the classes of morphisms in a factorization system to be closed under composition (or to contain all equivalences), but instead demands closure under retracts. These definitions agree since our definition also implies closure under retracts (Corollary 10.3.9) while Lurie's definition implies closure under composition and equivalences by [Lur09, Corollary 5.2.8.13].

Proposition 10.3.11. *Suppose $(\mathcal{L}, \mathcal{R})$ is a factorization system on an ∞-category $\mathcal{C}$, and consider a functor $p\colon \mathcal{E} \to \mathcal{C}$.*

(i) Suppose $\mathcal{E}$ admits p-cocartesian lifts of morphisms in $\mathcal{L}$. Let $\mathcal{L}'$ be the subcategory of $\mathcal{E}$ containing the cocartesian morphisms over $\mathcal{L}$, and let $\mathcal{R}'$ be the subcategory containing all morphisms that lie over $\mathcal{R}$. Then $(\mathcal{L}', \mathcal{R}')$ is a factorization system on $\mathcal{E}$.

(ii) Suppose $\mathcal{E}$ admits p-cartesian lifts of morphisms in $\mathcal{R}$. Let $\mathcal{R}''$ be the subcategory of $\mathcal{E}$ containing the cartesian morphisms over $\mathcal{R}$, and let $\mathcal{L}''$ be the subcategory containing all morphisms that lie over $\mathcal{L}$. Then $(\mathcal{L}'', \mathcal{R}'')$ is a factorization system on $\mathcal{E}$.

Proof. We prove (i); part (ii) is dual. To start, we show any morphism $f\colon x \to y$ in $\mathcal{E}$ admits the required factorization: We can factor $p(f)$ as $p(x) \xrightarrow{\ell} c \xrightarrow{r} p(y)$ with ℓ in $\mathcal{L}$ and r in $\mathcal{R}$ and choose a p-cocartesian lift $\bar{\ell}\colon x \to \ell_! x$ of ℓ at x. Then f factors uniquely as $x \xrightarrow{\bar{\ell}} \ell_! x \xrightarrow{\bar{r}} y$ where $p(\bar{r}) \simeq r$ and so $\bar{r}$ lies in $\mathcal{R}$, as required.

Now we show that if $\bar{\ell}\colon x \to y$ is a cocartesian lift of $\ell\colon a \to b$ in $\mathcal{L}$ and $\bar{r}\colon z \to w$ is a lift of $r\colon c \to d$ in $\mathcal{R}$, then $\bar{\ell}$ is left orthogonal to $\bar{r}$. Consider the commutative cube

$$
\begin{tikzcd}
\mathcal{E}(y,z) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{E}(y,w) \arrow[dr] \arrow[dd] & \\
& \mathcal{C}(b,c) \arrow[rr] \arrow[dd] & & \mathcal{C}(b,d) \arrow[dd] \\
\mathcal{E}(x,z) \arrow[rr] \arrow[dr] & & \mathcal{E}(x,w) \arrow[dr] & \\
& \mathcal{C}(a,c) \arrow[rr] & & \mathcal{C}(a,d)
\end{tikzcd}
$$

Here the left and right faces are pullbacks since $\bar{\ell}$ is p-cocartesian and the front face is a pullback since ℓ is left orthogonal to r. It follows that the back face is also a pullback, as required. □

Example 10.3.12. Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration. Then $\mathcal{E}$ has a factorization system with left class the cocartesian morphisms and right class the morphisms that lie over equivalences in $\mathcal{B}$. Dually, if p is a cartesian fibration, then $\mathcal{E}$ has a factorization system with left class the morphisms that lie over equivalences and right class the cartesian morphisms.

Proposition 10.3.13. *Suppose $(\mathcal{L}, \mathcal{R})$ is a factorization system on an ∞-category $\mathcal{C}$, and let $\mathsf{Ar}_{\mathcal{L}}(\mathcal{C})$ and $\mathsf{Ar}_{\mathcal{R}}(\mathcal{C})$ denote the full subcategories of $\mathsf{Ar}(\mathcal{C})$ spanned by the morphisms in $\mathcal{L}$ and $\mathcal{R}$, respectively.*

(1) $\mathrm{ev}_1\colon \mathsf{Ar}_{\mathcal{R}}(\mathcal{C}) \to \mathcal{C}$ *is a cocartesian fibration; the cocartesian morphisms are those whose image under* ev_0 *lies in* $\mathcal{L}$*.*

(2) The inclusion $\mathsf{Ar}_{\mathcal{R}}(\mathcal{C}) \hookrightarrow \mathsf{Ar}(\mathcal{C})$ *has a left adjoint (which takes an object of* $\mathsf{Ar}(\mathcal{C})$ *to the* $\mathcal{R}$*-part of its factorization).*

(3) $\mathrm{ev}_0\colon \mathsf{Ar}_{\mathcal{L}}(\mathcal{C}) \to \mathcal{C}$ *is a cartesian fibration; the cartesian morphisms are those whose image under* ev_1 *lies in* $\mathcal{R}$*.*

(4) The inclusion $\mathsf{Ar}_{\mathcal{L}}(\mathcal{C}) \hookrightarrow \mathsf{Ar}(\mathcal{C})$ *has a right adjoint (which takes an object of* $\mathsf{Ar}(\mathcal{C})$ *to the* $\mathcal{L}$*-part of its factorization).*

Proof. We prove the first two points; the last two are dual. Consider a morphism α in $\mathsf{Ar}(\mathcal{C})$, given by a commutative square

$$\begin{tikzcd} x \arrow[r, "\ell"] \arrow[d, "s"'] & x' \arrow[d, "t"] \\ y \arrow[r, "f"'] & y' \end{tikzcd}$$

where ℓ lies in $\mathcal{L}$. We claim that for any morphism $r\colon z \to w$ in $\mathcal{R}$, the commutative square

$$\begin{tikzcd} \mathsf{Ar}(\mathcal{C})(t,r) \arrow[r, "\alpha^*"] \arrow[d] & \mathsf{Ar}(\mathcal{C})(s,r) \arrow[d] \\ \mathcal{C}(y',w) \arrow[r, "f^*"'] & \mathcal{C}(y,w) \end{tikzcd}$$

is a pullback. Indeed, we have a commutative cube

$$\begin{tikzcd}[row sep=small] \mathsf{Ar}(\mathcal{C})(t,r) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Ar}(\mathcal{C})(s,r) \arrow[dr] \arrow[dd] & \\ & \mathcal{C}(x',z) \arrow[rr] \arrow[dd] & & \mathcal{C}(x,z) \arrow[dd] \\ \mathcal{C}(y',w) \arrow[rr] \arrow[dr] & & \mathcal{C}(y,w) \arrow[dr] & \\ & \mathcal{C}(x',w) \arrow[rr] & & \mathcal{C}(x,w), \end{tikzcd}$$

where the left and right faces are pullbacks by Proposition 3.6.1. Moreover, the front face is a pullback since ℓ is left orthogonal to r, hence so is the back face.

If s and t lie in $\mathcal{R}$, then this implies that α is an ev_1-cocartesian morphism in $\mathsf{Ar}_{\mathcal{R}}(\mathcal{C})$. To prove (1) it then remains to check that given $s\colon x \to y$ and $f\colon y \to y'$, there exists a cocartesian lift of f at s. But we know that the composite fs factors as $r\ell$ with r in $\mathcal{R}$ and ℓ in $\mathcal{L}$, and the commutative square

$$\begin{tikzcd} x \arrow[r, "\ell"] \arrow[d, "s"'] & x' \arrow[d, "r"] \\ y \arrow[r, "f"'] & y' \end{tikzcd}$$

is then ev_1-cocartesian.

To prove (2) we must show that for $s \colon x \to y$ in $\mathsf{Ar}(\mathcal{C})$, the copresheaf $\mathsf{Ar}(\mathcal{C})(s,-)$ is corepresentable when restricted to $\mathsf{Ar}_{\mathcal{R}}(\mathcal{C})$. Choose a factorization of s as $r\ell$ with ℓ in $\mathcal{L}$ and t in $\mathcal{R}$; we have a commutative square

$$\begin{tikzcd} x \arrow[r, "\ell"] \arrow[d, "s"'] & x' \arrow[d, "t"] \\ y \arrow[r, equal] & y, \end{tikzcd}$$

and composition with this, viewed as a morphism in $\mathsf{Ar}(\mathcal{C})$, gives a pullback square

$$\begin{tikzcd} \mathsf{Ar}_{\mathcal{R}}(\mathcal{C})(t,r) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{C})(s,r) \arrow[d] \\ \mathcal{C}(y,w) \arrow[r, equal] & \mathcal{C}(y,w), \end{tikzcd}$$

so that $\mathsf{Ar}_{\mathcal{R}}(\mathcal{C})(t,-) \simeq \mathsf{Ar}(\mathcal{C})(s,-)$ on $\mathsf{Ar}_{\mathcal{R}}(\mathcal{C})$, as required. □

Proposition 10.3.14. *The following are equivalent for a pair $(\mathcal{L},\mathcal{R})$ of wide subcategories of $\mathcal{C}$:*

(1) $(\mathcal{L},\mathcal{R})$ is a factorization system on $\mathcal{C}$.

(2) If $\mathsf{Fun}_{\mathcal{L},\mathcal{R}}([2],\mathcal{C})$ denotes the full subcategory of $\mathsf{Fun}([2],\mathcal{C})$ on the composable pairs $X \xrightarrow{\ell} Y \xrightarrow{r} Z$ with ℓ in $\mathcal{L}$ and r in $\mathcal{R}$, then

$$d_1 \colon \mathsf{Fun}_{\mathcal{L},\mathcal{R}}([2],\mathcal{C}) \to \mathsf{Ar}(\mathcal{C})$$

is an equivalence.

(3) If $\mathcal{C}[2]_{\mathcal{L},\mathcal{R}}$ denotes the core of $\mathsf{Fun}_{\mathcal{L},\mathcal{R}}([2],\mathcal{C})$, then

$$d_1 \colon \mathcal{C}[2]_{\mathcal{L},\mathcal{R}} \to \mathcal{C}[1]$$

is an equivalence.

(4) For any morphism $f \colon X \to Y$, the ∞-groupoid of factorizations $f \simeq r\ell$, with ℓ in $\mathcal{L}$ and r in $\mathcal{R}$, is contractible.

Warning 10.3.15. The statement of Proposition 10.3.14 is not quite the same as that of [Lur09, Proposition 5.2.8.17], as we assume from the start that the morphisms in $\mathcal{L}$ and $\mathcal{R}$ are closed under composition. As Lurie shows, this is in fact a consequence of assumption (2), but it is unlikely to be implied by (3).

Proof of Proposition 10.3.14. We first prove that (1) implies (2). Using Proposition 10.3.13 we can identify $\mathsf{Fun}_{\mathcal{L},\mathcal{R}}([2], \mathcal{C})$ with the full subcategory of the ∞-category $\mathsf{Ar}_{\mathrm{coct}}(\mathsf{Ar}_{\mathcal{R}}(\mathcal{C}))$ spanned by commutative squares of the form

$$\begin{tikzcd} \bullet \arrow[r, "\mathcal{L}"] \arrow[d, equal] & \bullet \arrow[d, "\mathcal{R}"] \\ \bullet \arrow[r] & \bullet; \end{tikzcd}$$

under this equivalence the composition functor to $\mathsf{Ar}(\mathcal{C})$ corresponds to composition with $\mathrm{ev}_1 \colon \mathsf{Ar}_{\mathcal{R}}(\mathcal{C}) \to \mathcal{C}$. We also know from Proposition 3.6.4 that we have a pullback square

$$\begin{tikzcd} \mathsf{Ar}_{\mathrm{coct}}(\mathsf{Ar}_{\mathcal{R}}(\mathcal{C})) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{C}) \arrow[d] \\ \mathsf{Ar}_{\mathcal{R}}(\mathcal{C}) \arrow[r] & \mathcal{C}, \end{tikzcd}$$

where our full subcategory corresponds to the pullback of the full subcategory $\mathsf{Ar}_{\mathrm{eq}}(\mathcal{C}) \subseteq \mathsf{Ar}_{\mathcal{R}}(\mathcal{C})$. Since this is equivalent to $\mathcal{C}$, the composite

$$\mathsf{Fun}_{\mathcal{L},\mathcal{R}}([2], \mathcal{C}) \hookrightarrow \mathsf{Ar}_{\mathrm{coct}}(\mathsf{Ar}_{\mathcal{R}}(\mathcal{C})) \to \mathsf{Ar}(\mathcal{C})$$

is an equivalence, as required.

(3) is immediate from (2), and (4) is equivalent to (3) since a morphism of ∞-groupoids is an equivalence if and only if its fibres are contractible. We are therefore left with proving that (3) implies (1); for this we are required to show that morphisms in $\mathcal{L}$ are left orthogonal to ones in $\mathcal{R}$. Let us write $\mathcal{C}[2]_{\mathcal{L},\bullet}$ for the sub-∞-groupoid $\mathcal{C}[2]$ consisting of composable pairs whose first component lies in $\mathcal{L}$, etc. We can then consider the sub-∞-groupoid $\mathrm{Sq}_{\mathcal{L},\mathcal{R}}(\mathcal{C})$ of $\mathsf{Map}([1] \times [1], \mathcal{C})$ consisting of squares of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\ell"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

where ℓ lies in $\mathcal{L}$ and r in $\mathcal{R}$. Using the pushout decomposition of $[1] \times [1]$, this decomposes as

$$\mathrm{Sq}_{\mathcal{L},\mathcal{R}}(\mathcal{C}) \simeq \mathcal{C}[2]_{\mathcal{L},\bullet} \times_{\mathcal{C}[1]} \mathcal{C}[2]_{\bullet,\mathcal{R}}.$$

We then need to prove that the map

$$\mathcal{C}[3]_{\mathcal{L},\bullet,\mathcal{R}} \to \mathrm{Sq}_{\mathcal{L},\mathcal{R}}(\mathcal{C})$$

induced by the commutative square

$$\begin{tikzcd} \mathcal{C}[3]_{\mathcal{L},\bullet,\mathcal{R}} \arrow[r, "d_1"] \arrow[d, "d_2"'] & \mathcal{C}[2]_{\bullet,\mathcal{R}} \arrow[d, "d_1"] \\ \mathcal{C}[2]_{\mathcal{L},\bullet} \arrow[r, "d_1"'] & \mathcal{C}[1] \end{tikzcd}$$

is an equivalence.

We have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}[3] \arrow[r, "d_2"] \arrow[d, "d_1"'] & \mathcal{C}[2] \arrow[d, "d_1"] \\
\mathcal{C}[2] \arrow[r, "d_1"'] & \mathcal{C}[1] \\
\mathcal{C}[3] \arrow[r, "d_1"] \arrow[u, "d_2"] & \mathcal{C}[2]. \arrow[u, "d_1"']
\end{tikzcd}$$

Using assumption (3) and the Segal decompositions this restricts on sub-∞-groupoids to

$$\begin{tikzcd}
\mathcal{C}[3]_{\mathcal{L},\mathcal{L},\mathcal{R}} \arrow[r, "\sim"] \arrow[d, "d_1"'] & \mathcal{C}[2]_{\mathcal{L},\bullet} \arrow[d, "d_1"] \\
\mathcal{C}[2]_{\mathcal{L},\mathcal{R}} \arrow[r, "\sim"'] & \mathcal{C}[1] \\
\mathcal{C}[3]_{\mathcal{L},\mathcal{R},\mathcal{R}} \arrow[r, "\sim"] \arrow[u, "d_2"] & \mathcal{C}[2]_{\bullet,\mathcal{R}}. \arrow[u, "d_1"']
\end{tikzcd}$$

We therefore get an equivalence

$$\mathcal{C}[4]_{\mathcal{L},\mathcal{L},\mathcal{R},\mathcal{R}} \xrightarrow{\sim} \mathrm{Sq}_{\mathcal{L},\mathcal{R}}(\mathcal{C})$$

after taking pullbacks, which moreover factors as

$$\mathcal{C}[4]_{\mathcal{L},\mathcal{L},\mathcal{R},\mathcal{R}} \xrightarrow{d_2} \mathcal{C}[3]_{\mathcal{L},\bullet,\mathcal{R}} \to \mathrm{Sq}_{\mathcal{L},\mathcal{R}}(\mathcal{C}),$$

where the first map is an equivalence by assumption (3) and the second is the map we are interested in. It follows that this is indeed an equivalence, as required. □

Corollary 10.3.16. *Suppose $(\mathcal{L}, \mathcal{R})$ is a factorization system on $\mathcal{C}$. For an ∞-category $\mathcal{K}$, let $\mathcal{L}_{\mathcal{K}}$ denote the wide subcategory of $\mathsf{Fun}(\mathcal{K}, \mathcal{C})$ containing the natural transformations whose components lie in $\mathcal{L}$, and define $\mathcal{R}_{\mathcal{K}}$ similarly. Then $(\mathcal{L}_{\mathcal{K}}, \mathcal{R}_{\mathcal{K}})$ is a factorization system on $\mathsf{Fun}(\mathcal{K}, \mathcal{C})$.*

Proof. We use characterization (2) in Proposition 10.3.14. The composition functor

$$\mathsf{Fun}_{\mathcal{L}_{\mathcal{K}},\mathcal{R}_{\mathcal{K}}}([2], \mathcal{C}^{\mathcal{K}}) \to \mathsf{Ar}(\mathcal{C}^{\mathcal{K}})$$

is equivalent to the functor

$$\mathsf{Fun}(\mathcal{K}, \mathsf{Fun}_{\mathcal{L},\mathcal{R}}([2], \mathcal{C})) \to \mathsf{Fun}(\mathcal{K}, \mathsf{Ar}(\mathcal{C}))$$

given by composition in $\mathcal{C}$, which is an equivalence by assumption. □

10.4 Bousfield localizations

Definition 10.4.1. A *Bousfield localization* is a functor $L\colon \mathcal{C} \to \mathcal{C}'$ such that L has a fully faithful right adjoint.

Remark 10.4.2. We saw in Corollary 6.3.11 that a Bousfield localization is in particular a localization. Note that in [Lur09] what we call a Bousfield localization is just called a *localization*.

Definition 10.4.3. A *reflective subcategory* of an ∞-category $\mathcal{C}$ is a fully faithful functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ that has a left adjoint.

We can characterize Bousfield localizations $L \dashv i$ in terms of the endofunctor $\Lambda := iL$:

Proposition 10.4.4. *Consider an endofunctor $\Lambda\colon \mathcal{C} \to \mathcal{C}$ of an ∞-category $\mathcal{C}$, and let*

$$\mathcal{C} \xrightarrow{L} \mathcal{C}' \xrightarrow{i} \mathcal{C}$$

be its unique factorization as an essentially surjective functor L and a fully faithful functor i (Example 10.3.5). Then the following are equivalent:

(1) There exists an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

where G is fully faithful, and an equivalence $\Lambda \simeq GF$.

(2) L is left adjoint to i.

(3) There exists a natural transformation $\alpha\colon \mathrm{id}_{\mathcal{C}} \to \Lambda$ such that for every object $x \in \mathcal{C}$, the morphisms $\Lambda(\alpha_x)$ and $\alpha_{\Lambda x}$ are equivalences.

Proof. Clearly (2) implies (1). Conversely, given the adjunction $F \dashv G$ in (1), to identify F and G with L and i it suffices, since the factorization of Λ is unique, to observe that F is essentially surjective as the counit $FGx \to x$ is an equivalence for all $x \in \mathcal{D}$.

To see that (2) implies (3), we take α to be the unit of the adjunction $L \dashv i$. Then the composites

$$Lx \xrightarrow{L\alpha_x} LiLx \xrightarrow{\epsilon_{Lx}} Lx, \qquad iy \xrightarrow{\alpha_{iy}} iLiy \xrightarrow{i\epsilon_y} iy$$

are identities for $x \in \mathcal{C}$, $y \in \mathcal{C}'$, where $\epsilon\colon Li \to \mathrm{id}$ is the counit of the adjunction. Since ϵ is an equivalence by Proposition 6.3.10, we conclude that $L\alpha_x$ and α_{iy} are equivalences for all $x \in \mathcal{C}$, $y \in \mathcal{C}'$. This applies in particular to $y = Lx$, giving (3).

Finally, we show that (3) implies that α is the unit of an adjunction $L \dashv i$. We must show that the composite

$$\mathcal{C}'(Lx, y) \xrightarrow{\sim} \mathcal{C}(\Lambda x, iy) \xrightarrow{\alpha_x^*} \mathcal{C}(x, iy)$$

is an equivalence. Since L is by assumption essentially surjective, we may assume $y \simeq Lz$, so that we want to show that

$$\mathcal{C}(\Lambda x, \Lambda z) \xrightarrow{\alpha_x^*} \mathcal{C}(x, \Lambda z)$$

is an equivalence for all $x, z \in \mathcal{C}$. We claim that this map has both a left and a right inverse: On the one hand, we have from Exercise 6.2 a commutative square

$$\begin{tikzcd}
\mathcal{C}(x, \Lambda z) \arrow[r, "="] \arrow[d, "(\Lambda)"'] & \mathcal{C}(x, \Lambda z) \arrow[d, "\alpha_{\Lambda z,*}"] \\
\mathcal{C}(\Lambda x, \Lambda^2 z) \arrow[r, "\alpha_x^*"'] & \mathcal{C}(x, \Lambda^2 z),
\end{tikzcd}$$

and so a commutative diagram

$$\begin{tikzcd}
\mathcal{C}(\Lambda x, \Lambda z) \arrow[r, "\alpha_{\Lambda z,*}", "\sim"'] \arrow[d, "\alpha_x^*"'] & \mathcal{C}(\Lambda x, \Lambda^2 z) \arrow[d, "\alpha_x^*"] & \mathcal{C}(x, \Lambda z) \arrow[l, "(\Lambda)"'] \arrow[dl, "\alpha_{z,*}"] \\
\mathcal{C}(x, \Lambda z) \arrow[r, "\sim", "\alpha_{\Lambda z,*}"'] & \mathcal{C}(x, \Lambda^2 z). &
\end{tikzcd}$$

Since $\alpha_{\Lambda z}$ is an equivalence, it follows that the composite

$$\mathcal{C}(x, \Lambda z) \xrightarrow{(\Lambda)} \mathcal{C}(\Lambda x, \Lambda^2 z) \xrightarrow{(\alpha_{\Lambda z}^{-1})_*} \mathcal{C}(\Lambda x, \Lambda z) \xrightarrow{\alpha_x^*} \mathcal{C}(x, \Lambda z)$$

is the identity.For the other direction, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(\Lambda x, \Lambda z) \arrow[r, "\alpha_x^*"] \arrow[d, "(\Lambda)"'] \arrow[dd, bend right=60, "\alpha_{\Lambda z,*}"'] & \mathcal{C}(x, \Lambda z) \arrow[d, "(\Lambda)"] \\
\mathcal{C}(\Lambda^2 x, \Lambda^2 z) \arrow[r, "\sim", "(\Lambda\alpha_x)^*"'] \arrow[d, "\alpha_{\Lambda x}^*"] & \mathcal{C}(\Lambda x, \Lambda^2 z) \\
\mathcal{C}(\Lambda x, \Lambda^2 z) &
\end{tikzcd}$$

Since $\Lambda\alpha_x$ and $\alpha_{\Lambda z}$ are equivalences, the composite

$$\mathcal{C}(\Lambda x, \Lambda z) \xrightarrow{\alpha_x^*} \mathcal{C}(x, \Lambda z) \xrightarrow{(\Lambda)} \mathcal{C}(\Lambda x, \Lambda^2 z) \xrightarrow{(\Lambda\alpha_x)^{-1,*}} \mathcal{C}(\Lambda^2 x, \Lambda^2 z) \xrightarrow{\alpha_{\Lambda x}^*} \mathcal{C}(\Lambda x, \Lambda^2 z) \xrightarrow{(\alpha_{\Lambda z}^{-1})_*} \mathcal{C}(\Lambda x, \Lambda z)$$

is again the identity. This provides the other inverse to α_x^*, as required.[2] □

[2] A posteriori, the two maps $\alpha_{\Lambda x}$ and $\Lambda\alpha_x$ are actually the same, since by naturality we have $\alpha_{\Lambda x} \circ \alpha_x \simeq \Lambda(\alpha_x) \circ \alpha_x$.

Definition 10.4.5. Let S be a collection of morphisms in an ∞-category $\mathcal{C}$. We say an object $x \in \mathcal{C}$ is *S-local* if for every morphism $f: c \to d$ in S, the map

$$f^*: \mathcal{C}(d, x) \to \mathcal{C}(c, x)$$

is an equivalence of ∞-groupoids. Furthermore, we say that a morphism $f: c \to d$ is an *S-equivalence* if for every S-local object x, the map

$$f^*: \mathcal{C}(d, x) \to \mathcal{C}(c, x)$$

is an equivalence.

Proposition 10.4.6. *Suppose $L: \mathcal{C} \to \mathcal{C}'$ is a Bousfield localization with fully faithful right adjoint i, and let S be the collection of all morphisms in $\mathcal{C}$ that are taken to equivalences by L. Then:*

(1) An object of $\mathcal{C}$ is S-local if and only if it is in the image of i.

(2) Every S-equivalence belongs to S.

Proof. We first observe that every object in the image of i is S-local, since for $f: c \to d$ in S and $x \in \mathcal{C}'$ we have a commutative square

$$\begin{tikzcd} \mathcal{C}(d, ix) \arrow[r, "\sim"] \arrow[d, "f^*"'] & \mathcal{C}'(Ld, x) \arrow[d, "(Lf)^*"] \\ \mathcal{C}(c, ix) \arrow[r, "\sim"'] & \mathcal{C}'(Lc, x) \end{tikzcd}$$

and Lf is an equivalence. To prove (2), suppose $f: c \to d$ is an S-equivalence. Then the same commutative square shows that $(Lf)^*: \mathcal{C}'(Ld, x) \to \mathcal{C}'(Lc, x)$ is an equivalence for all $x \in \mathcal{C}'$. Thus Lf is an equivalence in $\mathcal{C}'$, i.e. f is in S, as required.

It remains to show that the converse direction of (1). For any object $y \in \mathcal{C}$ we know from Proposition 10.4.4 that the unit map $\eta_y: y \to \Lambda y$ lies in S, where $\Lambda := iL$. If y is S-local, composition with η_y thus gives an equivalence

$$\mathcal{C}(\Lambda y, y) \xrightarrow{\sim} \mathcal{C}(y, y).$$

Hence there exists a map $\phi: \Lambda y \to y$ so that $\phi \circ \eta_y \simeq \mathrm{id}_y$; note that then $L(\phi)$ must be an equivalence, so $\phi \in S$. We also have a naturality square

$$\begin{tikzcd} \Lambda y \arrow[r, "\phi"] \arrow[d, "\eta_{\Lambda y}"'] & y \arrow[d, "\eta_y"] \\ \Lambda^2 y \arrow[r, "\Lambda\phi"'] & \Lambda y, \end{tikzcd}$$

which tells us that $\eta_{\Lambda y}^{-1}(\Lambda\phi)^{-1}\eta_y \circ \phi \simeq \mathrm{id}$, so that ϕ has inverses on both sides. Then ϕ is an equivalence (as is η_y), and in particular y is in the image of Λ, as required. $\square$

Observation 10.4.7. This proof shows that in the above situation the following are equivalent for an object $x \in \mathcal{C}$:

(1) x is S-local.

(2) x is in the image of i.

(3) The unit map $\eta_x : x \to iLx$ is an equivalence.

(4) The map
$$\eta_x^* : \mathcal{C}(iLx, x) \to \mathcal{C}(x, x)$$
is an equivalence.

(5) x is local with respect to the collection of all unit maps $c \to iLc$ with $c \in \mathcal{C}$.

10.5 Presentable factorization systems

Definition 10.5.1. A collection of morphisms S in a cocomplete ∞-category $\mathcal{C}$ is *saturated* if

- S is closed under colimits in $\mathsf{Ar}(\mathcal{C})$,
- S contains all equivalences,
- S is closed under composition.

Observation 10.5.2. Suppose S is a saturated class. Then

- S is closed under cobase change, since given a pushout
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$
with $A \to A'$ in S, the map $B \to B'$ is also the pushout of
$$\begin{tikzcd} A \arrow[d] & A \arrow[l, "="'] \arrow[r] \arrow[d, "="'] & B \arrow[d, "="] \\ A' & A \arrow[l] \arrow[r] & B \end{tikzcd}$$
in $\mathsf{Ar}(\mathcal{C})$. (This observation is taken from [ABFJ22].)
- S is closed under retracts, since these are colimits of diagrams $\mathsf{Idem} \to \mathsf{Ar}(\mathcal{C})$.

Proposition 10.5.3. *For any collection of morphisms S,* $\mathrm{LO}(S)$ *is saturated.*

Proof. We know from Lemma 2.4.7 that $\mathrm{LO}(S)$ is closed under cobase change and from Lemma 2.4.5 that it is closed under composition, while it is obvious that it contains all equivalences. Consider a diagram $p\colon \mathcal{K} \to \mathsf{Ar}(\mathcal{C})$ that takes values in $\mathcal{C}$ and has colimit $f\colon \operatorname{colim}_{k\in\mathcal{K}} A_k \to \operatorname{colim}_{k\in\mathcal{K}} B_k$. If $s\colon X \to Y$ is in S and each map $A_k \to B_k$ is in $\mathrm{LO}(S)$, we want to show that the commutative square

$$\begin{tikzcd} \mathcal{C}(\operatorname{colim}_k B_k, X) \arrow[r] \arrow[d] & \mathcal{C}(\operatorname{colim}_k A_k, X) \arrow[d] \\ \mathcal{C}(\operatorname{colim}_k B_k, Y) \arrow[r] & \mathcal{C}(\operatorname{colim}_k A_k, Y) \end{tikzcd}$$

is a pullback. But this square is equivalent to

$$\begin{tikzcd} \lim_k \mathcal{C}(B_k, X) \arrow[r] \arrow[d] & \lim_k \mathcal{C}(A_k, X) \arrow[d] \\ \lim_k \mathcal{C}(B_k, Y) \arrow[r] & \lim_k \mathcal{C}(A_k, Y), \end{tikzcd}$$

which is a pullback since limits commute. □

Corollary 10.5.4. *If $(\mathcal{L}, \mathcal{R})$ is a factorization system on $\mathcal{C}$, then the class of morphisms in $\mathcal{L}$ is saturated.* □

Definition 10.5.5. The *saturation* $\overline{S}$ of a collection of morphisms S is the smallest saturated class of morphisms that contains it. The saturation always exists (assuming the Axiom of Choice) since an arbitrary intersection of saturated classes is again saturated. We say that a saturated class S is of *small generation* if there is a small set S_0 such that $S = \overline{S_0}$.

Exercise 10.2. Show that object $X \in \mathcal{C}$ is S-local if and only if $X \to *$ is right orthogonal to S.

Theorem 10.5.6. *Let $\mathcal{C}$ be a presentable ∞-category and S a small collection of morphisms in $\mathcal{C}$. Then $(\overline{S}, \mathrm{RO}(S))$ is a factorization system on $\mathcal{C}$.*

Corollary 10.5.7. *Let $\mathcal{C}$ be a presentable ∞-category and suppose S is a saturated class of small generation. Then S satisfies the 3-for-2 property: if gf and f lie in S then g lies in S.* □

Corollary 10.5.8. *Let $\mathcal{C}$ be a presentable ∞-category and S a small collection of morphisms in $\mathcal{C}$. Then $\mathrm{LO}(\mathrm{RO}(S))$ is the saturation of S.*

Proof. We know that $\mathrm{LO}(\mathrm{RO}(S))$ is a saturated class that contains S. Hence $\overline{S} \subseteq \mathrm{LO}(\mathrm{RO}(S))$. On the other hand, if $f\colon X \to Z$ is a morphism in $\mathrm{LO}(\mathrm{RO}(S))$ then it factors as $X \xrightarrow{g} Y \xrightarrow{h} Z$ where g is in $\overline{S}$ and h is in $\mathrm{RO}(S)$. But by Lemma 2.4.5 the map h is also in $\mathrm{LO}(\mathrm{RO}(S))$. Thus h is left orthogonal to itself, and so is an equivalence (Exercise 2.11). Hence f lies in $\overline{S}$, as required. □

Exercise 10.3. Use Proposition 3.6.1 to show that if $f\colon x \to y$ is a morphism in an ∞-category $\mathcal{C}$ such that x and y are both κ-compact, then f is κ-compact as an object of $\mathsf{Ar}(\mathcal{C})$.

Proposition 10.5.9. *Suppose S is a saturated class of morphisms in a presentable ∞-category $\mathcal{C}$, and let $\mathsf{Ar}_S(\mathcal{C})$ denote the full subcategory of $\mathsf{Ar}(\mathcal{C})$ spanned by the morphisms in S. Then S is of small generation if and only if $\mathsf{Ar}_S(\mathcal{C})$ is presentable.*

Proof. We first observe that, since S is saturated, the full subcategory $\mathsf{Ar}_S(\mathcal{C})$ is closed under colimits in $\mathsf{Ar}(\mathcal{C})$, so $\mathsf{Ar}_S(\mathcal{C})$ is cocomplete and colimits therein are computed in $\mathsf{Ar}(\mathcal{C})$.

Now suppose $\mathsf{Ar}_S(\mathcal{C})$ is presentable, so that there exists a small set S_0 of objects that generates $\mathsf{Ar}_S(\mathcal{C})$ under small colimits. Since colimits are computed in $\mathsf{Ar}(\mathcal{C})$, this means that every morphism in S is a small colimit in $\mathsf{Ar}(\mathcal{C})$ of elements of S_0. Hence S must clearly be the saturation of S_0.

Now suppose S is the saturation of a small subset S_0. Choose a regular cardinal κ such that $\mathcal{C}$ is κ-presentable and S_0 consists of morphisms between κ-compact objects. Let $\mathcal{E} \subseteq \mathsf{Ar}_S(\mathcal{C})$ be the full subcategory spanned by morphisms between κ-compact objects. Then $\mathcal{E}$ consists of κ-compact objects of $\mathsf{Ar}(\mathcal{C})$ (Exercise 10.3) and so of κ-compact objects in $\mathsf{Ar}_S(\mathcal{C})$. It then suffices to show that every object of $\mathsf{Ar}_S(\mathcal{C})$ is a κ-filtered colimit of objects in $\mathcal{E}$. To see this, we take T to be the collection of morphisms in $\mathcal{C}$ that arise as such κ-filtered colimits. By assumption we have $S_0 \subseteq T \subseteq S$, so it suffices to show that T is saturated.

We note that $\mathcal{E}$ is closed under κ-finite colimits in $\mathsf{Ar}(\mathcal{C})$, so that it follows from Proposition 10.1.12 that T is closed under colimits in $\mathsf{Ar}(\mathcal{C})$. We see as in Observation 10.5.2 that T is then closed under cobase change. Moreover, if $\emptyset$ is the initial object of $\mathcal{C}$, then $\mathrm{id}_\emptyset$ is contained in $\mathcal{E}$, and hence T contains all equivalences, as they are obtained from $\mathrm{id}_\emptyset$ by cobase change.

It thus only remains to show that T is closed under composition. By Variant 9.6.2 the unique extension of the inclusion $\mathcal{E} \hookrightarrow \mathsf{Ar}(\mathcal{C})$ to $\mathrm{Ind}_\kappa(\mathcal{E}) \to \mathsf{Ar}(\mathcal{C})$ is fully faithful, and its image is precisely $\mathsf{Ar}_T(\mathcal{C})$. Consider the full subcategory $\mathsf{Fun}_T([2], \mathcal{C})$ of $\mathsf{Fun}([2], \mathcal{C})$ spanned by the composable pairs of morphisms in T. Then

$$\mathsf{Fun}_T([2], \mathcal{C}) \simeq \mathsf{Ar}_T(\mathcal{C}) \times_{\mathcal{C}} \mathsf{Ar}_T(\mathcal{C}),$$

which implies that this is a presentable ∞-category. We claim that every object of $\mathsf{Fun}_T([2], \mathcal{C})$ is a κ-filtered colimit of composable pairs of morphisms in $\mathcal{E}$ — this follows from the proof that accessible ∞-categories are closed under pullbacks in [Lur09, Proposition 5.4.6.6] (which we will not go into for now), since the source and target functors to $\mathcal{C}$ take $\mathcal{E}$ to $\mathcal{C}^\kappa$. Now the morphisms in $\mathcal{E}$ are closed under composition since S is, so since T is closed under colimits this implies that it is indeed closed under composition. □

Proposition 10.5.10. *Let x be an object of a cocomplete ∞-category $\mathcal{C}$ and consider a diagram $p\colon \mathcal{K} \to \mathcal{C}_{x/}$. Let q be the composite of p with the forgetful functor to $\mathcal{C}$.*

Then the colimit f of p in $\mathcal{C}_{x/}$ fits in a pushout square

$$\begin{tikzcd} \operatorname{colim}_{\mathcal{K}} x \arrow[r] \arrow[d] & x \arrow[d, "f"] \\ \operatorname{colim}_{\mathcal{K}} q \arrow[r] & y. \end{tikzcd}$$

Proof. Let f be defined by the given pushout; we want to show that this is the colimit of p. For $g\colon x \to z$ in $\mathcal{C}_{x/}$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{C}_{x/}(f,g) \arrow[r] \arrow[d] & \mathcal{C}(y,z) \arrow[r] \arrow[d, "f^*"] & \mathcal{C}(\operatorname{colim}_{\mathcal{K}} q, z) \arrow[r] \arrow[d] & \lim_{\mathcal{K}^{\mathrm{op}}} \mathcal{C}(q,z) \arrow[d] \\ \{g\} \arrow[r] & \mathcal{C}(x,z) \arrow[r] & \mathcal{C}(\operatorname{colim}_{\mathcal{K}} x, z) \arrow[r] & \lim_{\mathcal{K}^{\mathrm{op}}} \mathcal{C}(x,z), \end{tikzcd}$$

where all three squares are pullbacks. Since limits commute, this identifies $\mathcal{C}_{x/}(f,g)$ as $\lim_{\mathcal{K}^{\mathrm{op}}} \mathcal{C}_{x/}(p,g)$, as required. □

Corollary 10.5.11. *Let S be a saturated class of morphisms in a cocomplete ∞-category $\mathcal{C}$, and let $\mathcal{C}_{x/}^{(S)}$ denote the full subcategory of $\mathcal{C}_{x/}$ spanned by morphisms in S. Then $\mathcal{C}_{x/}^{(S)}$ is closed under colimits in $\mathcal{C}_{x/}$, and so is in particular cocomplete.*

Proof. Given a diagram in $\mathcal{C}_{x/}^{(S)}$, its colimit in $\mathcal{C}_{x/}$ is by Proposition 10.5.10 a cobase change of its colimit in $\mathsf{Ar}(\mathcal{C})$. Since S is saturated, it is by definition closed under colimits in $\mathsf{Ar}(\mathcal{C})$, and it is also closed under cobase change by Observation 10.5.2. □

Proposition 10.5.12. *Suppose S is a saturated class of morphisms in a presentable ∞-category $\mathcal{C}$ such that S is of small generation. Then for every $x \in \mathcal{C}$ there exists a morphism $f\colon x \to x'$ such that f is in S and x' is S-local.*

Proof. Let $\mathcal{C}_{x/}^{(S)}$ denote the full subcategory of $\mathcal{C}_{x/}$ spanned by morphisms in S; this fits in a pullback square

$$\begin{tikzcd} \mathcal{C}_{x/}^{(S)} \arrow[r] \arrow[d] & \mathsf{Ar}_S(\mathcal{C}) \arrow[d, "\mathrm{ev}_0"] \\ \{x\} \arrow[r] & \mathcal{C}. \end{tikzcd}$$

Here $\mathsf{Ar}_S(\mathcal{C})$ is accessible by Proposition 10.5.9, and the two functors to $\mathcal{C}$ are accessible. Hence the pullback $\mathcal{C}_{x/}^{(S)}$ is also an accessible ∞-category by Theorem 9.7.15. Moreover, $\mathcal{C}_{x/}^{(S)}$ is cocomplete by Corollary 10.5.11, so it is a presentable ∞-category. It therefore has limits by Corollary 10.1.9, and so in particular has a terminal object $f\colon x \to x'$. We claim that then x' is S-local.

For $s\colon c \to d$ in S, we must show that

$$s^*\colon \mathcal{C}(d, x') \to \mathcal{C}(c, x')$$

is an equivalence. This holds if and only if for $g\colon c \to x'$, the fibre $\mathcal{C}_{c/}(s, g)$ at g is contractible. Now define $s' := g_! s$ by the pushout

$$\begin{tikzcd} c \arrow[r, "g"] \arrow[d, "s"'] & x' \arrow[d, "s'"] \\ d \arrow[r] & y; \end{tikzcd}$$

then $\mathcal{C}_{c/}(s, g) \simeq \mathcal{C}_{x'/}(s', \mathrm{id}_{x'})$, so it suffices to show that this is contractible. Since we have an equivalence $\mathcal{C}_{x'/} \simeq (\mathcal{C}_{x/})_{f/}$, we can identify this as the pullback

$$\begin{tikzcd} \mathcal{C}_{x'/}(s', \mathrm{id}_{x'}) \arrow[r] \arrow[d] & \mathcal{C}_{x/}(s'f, f) \arrow[d] \\ \{\mathrm{id}\} \arrow[r] & \mathcal{C}_{x/}(f, f), \end{tikzcd}$$

which is contractible by our assumption that f is terminal in $\mathcal{C}^{(S)}_{x/}$ since $s'f$ also lies in S. □

Proposition 10.5.13. *Suppose S is a saturated class in a presentable ∞-category $\mathcal{C}$. For $x \in \mathcal{C}$, let S_x be the class of morphisms in $\mathcal{C}_{/x}$ whose underlying morphism in $\mathcal{C}$ lies in S. Then S_x is a saturated class in $\mathcal{C}_{/x}$. Moreover, if S is of small generation, so is S_x.*

Proof. It is clear that S_x is saturated, since colimits in $\mathcal{C}_{/x}$ are computed in $\mathcal{C}$ by the dual of Corollary 5.6.11, and closure under composition and equivalences is obvious. Moreover, $\mathsf{Ar}_{S_x}(\mathcal{C}_{/x})$ is the pullback

$$\begin{tikzcd} \mathsf{Ar}_{S_x}(\mathcal{C}_{/x}) \arrow[r] \arrow[d] & \mathsf{Ar}_S(\mathcal{C}) \arrow[d] \\ \mathsf{Ar}(\mathcal{C}_{/x}) \arrow[r] & \mathsf{Ar}(\mathcal{C}), \end{tikzcd}$$

where $\mathsf{Ar}_S(\mathcal{C})$ is presentable by Proposition 10.5.9. Hence this is a pullback of presentable ∞-category along functors that preserve colimits, so that $\mathsf{Ar}_{S_x}(\mathcal{C}_{/x})$ is also presentable by Proposition 10.1.14. Appealing to Proposition 10.5.9 again, it follows that S_x is of small generation. □

Corollary 10.5.14. *Suppose S is a saturated class of morphisms in a presentable ∞-category $\mathcal{C}$ such that S is of small generation. Then for every morphism $f\colon x \to y$ there exists a factorization of f as $x \xrightarrow{s} x' \xrightarrow{g} y$ such that s is in S and g is right orthogonal to S.*

Proof. We can view f as an object in $\mathcal{C}_{/y}$. Combining Proposition 10.5.13 and Proposition 10.5.12, we see that there exists a morphism $f \xrightarrow{s} g$ in $\mathcal{C}_{/y}$ with s in S_y and g S_y-local. This gives us a factorization $f \simeq gs$ with s in S, so it remains to check that g is right orthogonal to S, i.e. that for $s\colon c \to d$ in S, the commutative square

$$\begin{tikzcd}
\mathcal{C}(d,x') \arrow[r] \arrow[d] & \mathcal{C}(d,y) \arrow[d] \\
\mathcal{C}(c,x') \arrow[r] & \mathcal{C}(c,y)
\end{tikzcd}$$

is a pullback. This is true if and only if the morphism on fibres at each $h\colon d \to y$ is an equivalence, and we can identify this as the map

$$\mathcal{C}_{/y}(p,g) \to \mathcal{C}_{/y}(ps,g)$$

given by composition with s. But since g is S_y-local, this is indeed an equivalence. ◻

Proof of Theorem 10.5.6. We know from Proposition 10.5.3 that the class of maps that are left orthogonal to $\mathrm{RO}(S)$ is saturated. Since it contains S, it must also contain the saturation $\overline{S}$, so this is left orthogonal to $\mathrm{RO}(S)$. It only remains to show that any morphism of $\mathcal{C}$ has the required factorization into these two classes, which we did in Corollary 10.5.14. ◻

10.6 Accessible localizations

Proposition 10.6.1. *Suppose $\mathcal{C}$ is a presentable[3] ∞-category and $L\colon \mathcal{C} \to \mathcal{C}'$ is a Bousfield localization with fully faithful right adjoint i; let S be the collection of morphisms inverted by L. Then the following are equivalent:*

(1) *$\mathcal{C}'$ is an accessible ∞-category.*

(2) *$\mathcal{C}'$ is a presentable ∞-category.*

(3) *i is an accessible functor.*

(4) *There exists a small set $S_0 \subseteq S$ such that every S_0-local object is S-local.*

(5) *There exists a small set S_0 such that $\mathcal{C}'$ is the full subcategory of S_0-local objects in $\mathcal{C}$.*

[3]The statement also holds more generally for an accessible ∞-category, but for the proof we then need to know that a right adjoint functor between accessible ∞-categories is always accessible, which we have not proved.

Exercise 10.4. Suppose we have an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

where $\mathcal{C}$ and $\mathcal{D}$ have κ-filtered colimits and G preserves these. Show that then F preserves κ-compact objects.

Proof. We know from Observation 10.1.4 that in this situation $\mathcal{C}'$ is automatically cocomplete (with colimits computed by applying L to colimits in $\mathcal{C}$), so that $\mathcal{C}'$ is accessible if and only if it is presentable. Moreover, if $\mathcal{C}'$ is presentable, then Corollary 10.2.7 implies that the right adjoint i is accessible. Conversely, if i is accessible then we can choose a regular cardinal κ so that $\mathcal{C}$ is κ-presentable and i is κ-accessible. It then follows that L preserves κ-compact objects. Given $x \in \mathcal{C}'$, we can write ix as a colimit of a κ-filtered diagram p of κ-compact objects in $\mathcal{C}$. Then Lp is a κ-filtered diagram of κ-compact objects in $\mathcal{C}'$ whose colimit is $Lix \simeq x$, and so $\mathcal{C}'$ is also κ-accessible. Moreover, if we let S_0 be the set of unit morphisms $c \to iLc$ with c κ-compact, then an object x that is local with respect to S_0 is local with respect to the unit maps $c \to iLc$ for every $c \in \mathcal{C}$, since these are colimits of those for c κ-compact. By Observation 10.4.7 this implies that x is S-local.

Next, if there exists a small set S_0 that detects the local objects, then we can choose a regular cardinal κ such that the source and target of every morphism in S_0 is κ-compact. It follows that the full subcategory of $\mathcal{C}$ spanned by the S_0-local objects is closed under κ-filtered colimits, i.e. that i is κ-accessible. Finally, the last two points are equivalent by Proposition 10.4.6. □

Definition 10.6.2. If $\mathcal{C}$ is a presentable ∞-category and $L\colon \mathcal{C} \to \mathcal{C}'$ is a Bousfield localization that satisfies the equivalent conditions of Proposition 10.6.1, we say that L is an *accessible localization*.

Corollary 10.6.3. *An ∞-category $\mathcal{C}$ is presentable if and only if there exists a small ∞-category $\mathcal{D}$ and an accessible localization* $\mathsf{PSh}(\mathcal{D}) \to \mathcal{C}$.

Proof. We know that $\mathsf{PSh}(\mathcal{D})$ is presentable (Example 10.1.2), so if such an accessible localization exists, then Proposition 10.6.1 implies that $\mathcal{C}$ is presentable. Conversely, if $\mathcal{C}$ is presentable then Proposition 10.1.7 implies that $\mathcal{C}$ is equivalent to $\mathrm{Ind}_\kappa \mathcal{D}$ where $\mathcal{D}$ is a small ∞-category with κ-finite colimits, and we furthermore know from Corollary 10.1.6 that $\mathrm{Ind}_\kappa \mathcal{D}$ is an accessible localization of $\mathsf{PSh}(\mathcal{D})$. □

Definition 10.6.4. A class S of morphisms in an ∞-category $\mathcal{C}$ is *strongly saturated* if

(1) S is saturated.

(2) S has the 2-of-3 property: if any two of the three morphisms gf, f and g lies in S, then so does the third.

Exercise 10.5. Check that the class of all equivalences in $\mathcal{C}$ is strongly saturated.

Definition 10.6.5. If S is a collection of morphisms in a cocomplete ∞-category $\mathcal{C}$, then the *strong saturation* $\widetilde{S}$ of S is the smallest strongly saturated class that contains S; this exists (assuming the Axiom of Choice) since any intersection of strongly saturated classes is again strongly saturated. We say that a strongly saturated class T is of *small generation* if there exists a small set S such that $T = \widetilde{S}$.

Observation 10.6.6. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a colimit-preserving functor, where $\mathcal{C}$ is cocomplete. Then the class of morphisms in $\mathcal{C}$ that are sent to equivalences by F is strongly saturated. More generally, if S is any strongly saturated class of morphisms in $\mathcal{D}$, then its preimage $F^{-1}S$ is also strongly saturated.

Observation 10.6.7. Let S_0 be a class of morphisms in $\mathcal{C}$, and let S be the class of S_0-equivalences. Then S is strongly saturated, since it is the intersection of the strongly saturated classes of morphisms taken to equivalences by the colimit-preserving functors $\mathcal{C}(-,x)\colon \mathcal{C} \to \mathsf{Gpd}_{\infty}^{\mathrm{op}}$, for $x \in \mathcal{C}$ S_0-local.

We can now prove an existence result for accessible localizations: For any strongly saturated class of small generation in an presentable ∞-category, there exists an accessible localization that inverts precisely this class of maps.

Proposition 10.6.8. *Suppose S is a small set of morphisms in a presentable ∞-category $\mathcal{C}$, and let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory of S-local objects. Then:*

(1) *For every $x \in \mathcal{C}$ there exists a morphism $f\colon x \to x'$ such that x' is S-local and f is an S-equivalence.*

(2) *The inclusion $i\colon \mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint L.*

(3) *The Bousfield localization $L \dashv i$ is accessible; in particular, the ∞-category $\mathcal{C}'$ is presentable.*

(4) *The following are equivalent for a morphism $f\colon x \to y$ in $\mathcal{C}$:*

  (i) f is in the strong saturation $\widetilde{S}$.
  (ii) f is an S-equivalence.
  (iii) Lf is an equivalence.

Proof. The first point follows from Proposition 10.5.12, since the S-equivalences are a (strongly) saturated class that contains S, and hence contains the saturation $\overline{S}$. Then the map

$$\mathcal{C}'(x',-) \to \mathcal{C}(x, i(-))$$

given by composition with f is an equivalence, so that the copresheaf $\mathcal{C}(x, i(-))$ is corepresentable. Since this is true for all $x \in \mathcal{C}$, we conclude that i has a left adjoint L, giving (2). That this is an accessible Bousfield localization now follows from Proposition 10.6.1.

We know from Observation 10.6.7 that the class of S-equivalences is a strongly saturated class that contains S, hence it contains $\widetilde{S}$, so that (i) implies (ii). Similarly, the class of morphisms sent to equivalences by L is strongly saturated by Observation 10.6.6, and contains S since for $g\colon c \to d$ in S and $x \in \mathcal{C}'$, we have a commutative square

$$\begin{tikzcd}
\mathcal{C}(d, ix) \arrow[r, "\sim"] \arrow[d, "g^*"'] & \mathcal{C}'(Ld, x) \arrow[d, "(Lg)^*"] \\
\mathcal{C}(c, ix) \arrow[r, "\sim"'] & \mathcal{C}'(Lc, x).
\end{tikzcd}$$

Here the left vertical map is an equivalence since ix is S-local, hence Lf must be an equivalence in $\mathcal{C}'$; thus (i) also implies (iii). Now consider the commutative square

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\eta_x"'] & y \arrow[d, "\eta_y"] \\
Lx \arrow[r, "Lf"'] & Ly.
\end{tikzcd}$$

Here we know from (1) that η_x and η_y lie in the (strong) saturation of S. We conclude that if Lf is an equivalence, then f must lie in the strong saturation $\widetilde{S}$ *and* be an S-equivalence, since the 3-for-2 property holds for both. This proves (4). $\square$

Bibliography

[ABFJ22] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal, *Left-exact localizations of ∞-topoi I: Higher sheaves*, Adv. Math. **400** (2022), Paper No. 108268, 64.

[Ber99] Clemens Berger, *Double loop spaces, braided monoidal categories and algebraic 3-type of space*, Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996), Contemp. Math., vol. 227, Amer. Math. Soc., Providence, RI, 1999, pp. 49–66, DOI 10.1090/conm/227/03252. MR1665460

[Cis19] Denis-Charles Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019. Available from https://cisinski.app.uni-regensburg.de/CatLR.pdf. MR3931682

[HRS25] Peter J. Haine, Maxime Ramzi, and Jan Steinebrunner, *Fully faithful functors and pushouts of ∞-categories* (2025), available at arXiv:2503.03916.

[Hau24] Rune Haugseng, *Higher categories* (2024), available at arXiv:2401.14311. To appear in the *Encyclopedia of Mathematical Physics* (2nd ed.).

[HHLN23a] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten, *Lax monoidal adjunctions, two-variable fibrations and the calculus of mates*, Proc. Lond. Math. Soc. (3) **127** (2023), no. 4, 889–957, available at arXiv:2011.08808.

[HHLN23b] ———, *Two-variable fibrations, factorisation systems and ∞-categories of spans*, Forum Math. Sigma **11** (2023), Paper No. e111, 70, DOI 10.1017/fms.2023.107, available at arXiv:2011.11042. MR4676218

[Lan21] Markus Land, *Introduction to infinity-categories*, Compact Textbooks in Mathematics, Birkhäuser/Springer, Cham, 2021.

[Lei21] Tom Leinster, *Large sets* (2021). Available from https://golem.ph.utexas.edu/category/2021/06/large_sets_1.html.

[Lur09] Jacob Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. Available from http://math.ias.edu/~lurie/.

[Lur17] ———, *Higher Algebra*, 2017. Available at http://math.ias.edu/~lurie/.

[Ker] ———, *Kerodon*. Available at http://kerodon.net.

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[Mar21] Louis Martini, *Yoneda's lemma for internal higher categories* (2021), available at arXiv:2103.17141.

[NRS20] Hoang Kim Nguyen, George Raptis, and Christoph Schrade, *Adjoint functor theorems for ∞-categories*, J. Lond. Math. Soc. (2) **101** (2020), no. 2, 659–681.

[RV22] Emily Riehl and Dominic Verity, *Elements of ∞-category theory*, Cambridge Studies in Advanced Mathematics, vol. 194, Cambridge University Press, Cambridge, 2022.

[SW25] Christian Sattler and David Wärn, *Note on confluent colimits* (2025). Available from `https://www.cse.chalmers.se/~sattler/docs/confluent/new-2025.txt`.

[Shu08] Michael A. Shulman, *Set theory for category theory* (2008), available at arXiv:0810.1279.

[Ste21] Wolfgang Steimle, *An additivity theorem for cobordism categories*, Algebr. Geom. Topol. **21** (2021), no. 2, 601–646.