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ALGEBRAIC TOPOLOGY I

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These are lecture notes from the course Algebraic Topology I given at NTNU in the Fall semester of 2020. The notes are intended as a supplement to the lectures and are not entirely self-contained — in particular they contain almost no pictures. Please let me know if you spot any errors!

Starred ($\star$) sections consist of material that was not covered in detail in the lectures, and only the statements of the main results are relevant for the exam.

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1
Introduction

The basic idea of algebraic topology is to define and study algebraic *invariants* of geometric objects. This means that given a topological space X we want to find some algebraic structure $F(X)$ (such as a group, a vector space, or maybe just a number) such that if two spaces X and X' are homeomorphic then $F(X)$ and $F(X')$ are isomorphic. Such invariants give us a way to extract geometric information from algebra: For example, suppose we have two spaces X and Y and that we can compute the invariants $F(X)$ and $F(Y)$; if these algebraic objects are not isomorphic, then we can conclude that the spaces X and Y cannot be homeomorphic. This allows us to turn a topological problem into an algebraic one — and if we're lucky, this can be much easier to solve.

Unsurprisingly, the more powerful an invariant is (i.e. the more information about spaces it captures), the more difficult it tends to be to compute it. In this course we will study a family of invariants, the *homology* groups $H_n(X)$, that strike a good balance between the two desirable properties of computability and power: we will see that they are computable for many spaces, yet contain enough information that we can also give some non-trivial applications to topology.

It is hard to say precisely what homology groups "mean" in general, and to give the precise definition we first need to set up quite a bit of machinery. To get a first idea of what homology is about, let us take a quick, informal look at homology in low dimensions.

One often sees the statement that the nth homology group of X "measures the n-dimensional holes of X", but it's not clear what this is really supposed to mean, if anything.

1.1 Homology of Graphs

Consider a finite *graph* Γ, by which we mean (informally) a finite collection of *vertices* V and of *edges* E, each of which links two vertices. If we demand that each edge is equipped with an *orientation*, we can describe the graph by two functions $s, t \colon E \to V$ that take each edge e to its source $s(e)$ and target $t(e)$.

A 1-dimensional *chain* on Γ is a formal linear combination $\sum_{i=1}^{n} a_i e_i$, $a_i \in \mathbb{Z}$, of edges e_i, and a 0-dimensional chain is similarly a formal linear combination of vertices. Let us write $C_i(\Gamma)$ for the abelian group of i-dimensional chains.

We can then define a homomorphism $\partial \colon C_1(\Gamma) \to C_0(\Gamma)$ by taking the edge e to its (oriented) *boundary* $t(e) - s(e)$. The 0-chains in the image of ∂ are called *boundaries* and form a subgroup $B_0(\Gamma) \subseteq C_0(\Gamma)$. A 1-chain σ is called a *cycle* if $\partial\sigma = 0$ — this means that the signed boundaries of the edges that make up σ cancel out; in particular, if we make loop on Γ by starting at a vertex and picking a sequence of oriented edges that eventually gets back to the same vertex, we get a cycle by taking the corresponding signed sum of edges. (Moreover, any cycle is a linear combination of such "loops".) The cycles form a subgroup $Z_1(\Gamma) \subseteq C_1(\Gamma)$.

The *homology groups* of Γ are then the abelian groups

$$H_0(\Gamma) = C_0(\Gamma)/B_0(\Gamma),$$

$$H_1(\Gamma) = Z_1(\Gamma).$$

Although the groups $C_i(\Gamma)$ obviously depend on the structure of Γ as a graph, the homology groups turn out to be *topological* invariants: they don't depend on how we divide the graph Γ up into vertices and edges.

This is not hard to see: You can check that if we pick an edge that connects two different vertices and contract it away (so its two end points are glued to a single vertex) the homology groups do not change. Repeating this, we can reduce any connected graph to one that has a single vertex, with some number of loops from that vertex to itself, and two graphs are topologically the same if and only if they reduce to the same one-vertex graph under this process.

In $H_0(\Gamma)$ we identify two vertices when they are connected by an edge, and more generally by a sequence of edges. In fact $H_0(\Gamma)$ is a free abelian group whose rank is the number of components of Γ, while $H_1(\Gamma)$ is a free abelian group whose rank is the number of independent loops on Γ.

1.2 *Homology of Surfaces*

Now let's try to do something similar one dimension up: Suppose we have a surface Σ equipped with a *triangulation*, which roughly speaking means we have divided the surface into triangles (which only overlap along entire edges, so each edge meets exactly two triangles). We then have a finite set F of (triangular) faces, a set E of edges, and a set V of vertices, and define i-dimensional *chains* for $i = 0, 1, 2$ to be $\mathbb{Z}$-linear combinations of vertices, edges, and faces, respectively; we write $C_i(\Sigma)$ for the abelian group of i-dimensional chains.

Suppose we additionally choose an orientation of each edge in E and an ordering of the vertices of each triangle in F (independent of the orientations of the edges). We can then define *boundary maps*

$$C_2(\Sigma) \xrightarrow{\partial} C_1(\Sigma) \xrightarrow{\partial} C_0(\Sigma)$$

as follows:

- For an edge $e \in E$ with vertices e_0 and e_1, ordered so that e points from e_0 to e_1, we set
$$\partial e = e_1 - e_0.$$
- For a triangle $\tau \in F$, let τ_i ($i = 0, 1, 2$) denote the vertices of τ in order, and if e is the edge of τ that connects the vertices τ_i and τ_j we set
$$\tau_{ij} = \begin{cases} e, & e \text{ goes from } \tau_i \text{ to } \tau_j, \\ -e, & e \text{ goes from } \tau_j \text{ to } \tau_i. \end{cases}$$

Then we have

$$\partial\tau = \tau_{01} + \tau_{12} + \tau_{20}.$$

These formulas extend uniquely to define ∂ on all chains by taking linear combinations.

An i-dimensional chain is called a *boundary* if it is in the image of ∂ and a *cycle* if it is in the kernel of ∂; we write $B_i(\Sigma), Z_i(\Sigma) \subseteq C_i(\Sigma)$ for the subgroups of boundaries and cycles (where we take $Z_0(\Sigma) = C_0(\Sigma)$ and $B_2(\Sigma) = 0$). Note that we have

$$\partial^2 = 0,$$

since for $\tau \in F$ we get

$$\partial^2\tau = \partial(\tau_{01} + \tau_{12} + \tau_{20}) = \tau_1 - \tau_0 + \tau_2 - \tau_1 + \tau_0 - \tau_2 = 0.$$

This means we have $B_1(\Sigma) \subseteq Z_1(\Sigma)$.

The ith *homology group* of Σ is defined to be

$$H_i(\Sigma) = Z_i(\Sigma)/B_i(\Sigma) = \begin{cases} C_0(\Sigma)/B_0(\Sigma), & i = 0, \\ Z_1(\Sigma)/B_1(\Sigma), & i = 1, \\ Z_2(\Sigma), & i = 2. \end{cases}$$

These abelian groups turn out to be *topological* invariants of Σ — they do not depend on the choice of triangulation.

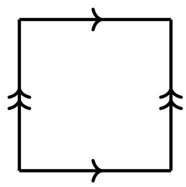
Suppose Σ is an orientable surface of genus g (i.e. a torus with g holes). We will see that the homology groups of Σ can be described as follows:

- $H_2(\Sigma) = Z_2(\Sigma)$ is free abelian of rank 1. We can make a cycle by adding up all the faces in the triangulation with signs chosen so that each edge occurs twice with opposite signs, and hence the boundary is 0; any cycle is then some multiple of this.

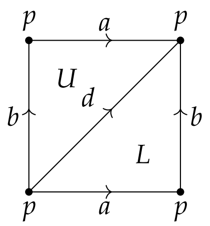
- $H_1(\Sigma)$ is a free abelian group of rank $2g$. The generating cycles can be taken to be those that "wrap around" each hole in the two possible ways.

- $H_0(\Sigma)$ is a free abelian group of rank 1. When we quotient by boundaries we again identify vertices that are connected by an edge; since Σ is connected any two vertices is connected by a sequence of edges, so we identify all vertices to a single class.

Example 1.2.1. Let's confirm these claims in the case of the torus, which we can build by taking a square and gluing opposite edges together:



If we add the diagonal, we get a triangulation of the torus with two triangular faces, three edges and one vertex:



Here we might have (depending on how we orient U and L)

$$\partial U = b + a - d = \partial L,$$

$$\partial a = \partial b = \partial d = p - p = 0.$$

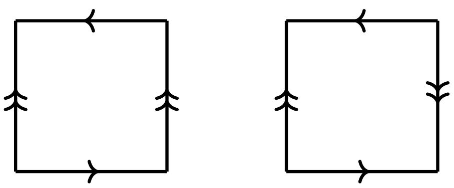
Thus we have

- $B_0 = 0$ and $H_0 = C_0$ is freely generated by p,
- $Z_1 = C_1$ and B_1 is freely generated by $(b + a - d)$, so

  $$H_1 = \mathbb{Z}\{a, b, d\}/(d = a + b) \cong \mathbb{Z}^2,$$

  generated by the images of a and b.
- $H_2 = Z_2$ is freely generated by $U - L$.

Exercise 1.1. We computed the homology of the torus by thinking of it as built from a square by gluing opposite edges, and triangulating this by cutting it into two triangles along the diagonal. Here are two other (non-orientable!) surfaces we can build by identifying opposite sides of a square, but now with a twist in either one or both directions:



The corresponding spaces are the Klein bottle and the real projective plane $\mathbb{RP}^2$, respectively. Triangulate these too by adding a diagonal and picking orientations, and compute the homology groups. [You should find that homology groups are not always *free* abelian groups.]

Exercise 1.2. The *Euler characteristic* of a triangulated surface is

$$\chi := V - E + F.$$

(i) Show that $\chi = h_0 - h_1 + h_2$ where h_i is the rank of the abelian group $H_i(\Sigma)$. Conclude that the Euler characteristic is a topological invariant. [Hint: For abelian groups $B \subseteq A$ the rank of A/B is given by $\operatorname{rk} A/B = \operatorname{rk} A - \operatorname{rk} B$. You will also need to write the boundary groups $B_i(\Sigma)$ as quotients.]

(ii) Conclude that for any way of covering the oriented surface of genus g by polygons we must have

$$V - E + F = 2 - 2g.$$

[Hint: Subdivide the polygons into triangles.]

(iii)* In particular, any convex polyhedron must satisfy *Euler's formula*,

$$V - E + F = 2.$$

Use this to classify the Platonic solids. [Hint: first observe that we have $pF = 2E = qV$ if the faces have p edges and q edges meet at each vertex, and show that $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$; since p and q are integers ≥ 3 there are not many possibilities.]

If we tried to extend our discussion so far to higher dimensions, we would end up with the theory of *simplicial homology*, which attaches homology groups to simplicial complexes — these are topological spaces that are decomposed into simplices, which are higher-dimensional analogues of triangles. This was historically the first version of homology, and goes back all the way to the original work of Poincaré around 1900. However, although it gives a correct way to compute homology groups that are topological invariants, it is painful to work with when we want to set up the theory and prove its basic formal properties. Therefore, we will instead begin by looking at the slightly more abstract, but better-behaved definition of *singular homology* (first introduced in this form by Eilenberg in 1943), which gives homology groups for a topological space without having to choose any extra structure. Thereafter, we will look at simplicial homology and its generalization to *cellular* homology groups, and see that these give a powerful combinatorial tool for computing the (singular) homology groups.

2
Some Basic Topology and Category Theory

In this chapter we first briefly review the basic notions of topological spaces and continuous maps in §2.1. When we define homology groups we will see that any continuous map $f\colon X \to Y$ induces a homomorphism of homology groups that is compatible with composition; it is convenient to phrase this in terms of *categories*: the homology groups will be *functors* from the category of topological spaces to that of abelian groups. We introduce categories and functors in §2.2 — they give a useful language that we will use throughout the course. As a first example of this, we review some basic constructions of topological spaces (products, coproducts, and quotients) and discuss how they can be interpreted categorically in §2.3–2.4.

We will eventually prove that the homology groups of a space are invariant in a stronger sense than we have considered so far: they will agree not just when the spaces X and X' are homeomorphic, but also whenever X and X' can be continuously deformed into each other — more precisely, when they are *homotopy equivalent*; we review the basic notions of homotopies and homotopy equivalences in §2.5. We then introduce a first, simple (but important) example of an algebraic invariant in §2.6: the set $\pi_0 X$ of path-components of a space X. Finally, we briefly review another important invariant, the *fundamental group* of a space, in §2.7.

2.1 Topological Spaces

Definition 2.1.1. A *topological space* is a set X equipped with a collection $\mathcal{T}_X$ of subsets of X, such that

- $\emptyset, X \in \mathcal{T}_X$,
- if $U_i \in \mathcal{T}_X$ for all $i \in I$ (where I can be any set) then $\bigcup_{i\in I} U_i \in \mathcal{T}_X$,
- if $U, U' \in \mathcal{T}_X$ then $U \cap U' \in \mathcal{T}_X$.

The collection $\mathcal{T}_X$ is called a *topology* on X and the elements of $\mathcal{T}_X$ are the *open* subsets of X. We often just say that X is a topological space without mentioning $\mathcal{T}_X$ explicitly.

Terminology 2.1.2.

- A subset $U \subseteq X$ is called *closed* if $X \setminus U$ is open.

- If x is a point of X, an *open neighbourhood* of x is an open subset U of X such that $x \in U$. (A *neighbourhood* of x is a subset $S \subseteq X$ that contains an open neighbourhood of x.)

Examples 2.1.3.

(i) A subset U of $\mathbb{R}^n$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that when $|x - y| < \epsilon$ we have $y \in U$ (i.e. U contains the open ball of radius ϵ around x).

(ii) Similarly, if (X, d) is a metric space, a subset $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that U contains the open ball of radius ϵ around x.

(iii) We can equip any set X with the *discrete topology* where all sets are open ($\mathcal{T}_X$ is the power set of X).

(iv) We can equip any set X with the *coarse* (or *indiscrete*) topology, where $\mathcal{T}_X := \{\emptyset, X\}$.

Definition 2.1.4. Let X be a topological space and $Y \subseteq X$ any subset. Then the *subspace topology* on Y is given by

$$\mathcal{T}_Y := \{V \subseteq Y : V = U \cap Y \text{ for some } U \in \mathcal{T}_X\}.$$

Example 2.1.5. The *n-sphere* S^n can be defined as the subset

$$S^n := \left\{ x \in \mathbb{R}^{n+1} : \sum_i x_i^2 = 1 \right\}$$

equipped with the subspace topology from $\mathbb{R}^{n+1}$.

Definition 2.1.6. Let X and Y be topological spaces. A *continuous map* from X to Y is a function $f\colon X \to Y$ such that if $U \subseteq Y$ is open, then $f^{-1}U \subseteq X$ is also open.

Exercise 2.1. Let X be a topological space and S a set. Show that if we equip S with the discrete topology then any function $S \to X$ is continuous, and if we equip S with the indiscrete topology then any function $X \to S$ is continuous.

Solution. First suppose S has the discrete topology. Given any function $f\colon S \to X$, for $U \subseteq X$ open we have that $f^{-1}(U)$ is open since all subsets of S are open, i.e. f is continuous.

Now suppose S has the indiscrete topology. Given any function $f\colon X \to S$, we have $f^{-1}(S) = X$ and $f^{-1}(\emptyset) = \emptyset$ which are both open subsets of X for any topology thereon. Since these are the only open subsets of S, this means f is continuous. □

Exercise 2.2. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that a function $f\colon X \to Y$ is continuous if and only if for every $x \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \epsilon$. (Note that δ may depend on x.)

Solution. For a point p in a metric space (M, d), let's write

$$B_\epsilon^M(p) := \{q \in M : d(p, q) < \epsilon\}$$

for the open ball of radius ϵ around p in M. Then we can reformulate the statement we want to prove as: f is continuous if and only if for all $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$f(B_\delta^X(x)) \subseteq B_\epsilon^Y(f(x)),$$

or equivalently $B_\delta^X(x) \subseteq B_\epsilon^Y(f(x))$. (This says precisely that if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \epsilon$.)

First suppose f is continuous. Given $x \in X$ and $\epsilon > 0$, the subset $B_\epsilon^Y(f(x)) \subseteq Y$ is open in Y, and so $f^{-1}B_\epsilon^Y(f(x))$ is an open subset of X containing x. This means by definition that there exists some $\delta > 0$ such that $B_\delta^X(x) \subseteq f^{-1}B_\epsilon^Y(f(x))$, as required.

Now we prove the converse. Let U be an open subset of Y, then we must show that $f^{-1}U$ is open in X. This means that we most show that for any $x \in f^{-1}U$ there exists some $\delta > 0$ such that $B_\delta^X(x) \subseteq f^{-1}U$, or $f(B_\delta^X(x)) \subseteq U$. Since U is open in Y, we may choose $\epsilon > 0$ such that $B_\epsilon^Y(f(x)) \subseteq U$; our assumption on f then says there exists some $\delta > 0$ such that $f(B_\delta^X(x)) \subseteq B_\epsilon^Y(f(x))$. It follows that $B_\delta^X(x) \subseteq f^{-1}U$, which shows that $f^{-1}U$ is open. □

2.2 *Categories*

Definition 2.2.1. A *category* $\mathcal{C}$ consists of

- a collection $\operatorname{ob}\mathcal{C}$ of *objects* (we write $x \in \mathcal{C}$ for $x \in \operatorname{ob}\mathcal{C}$)
- for any objects $x, y \in \mathcal{C}$ a set $\mathcal{C}(x, y)$ or $\operatorname{Hom}_{\mathcal{C}}(x, y)$ of *morphisms* from x to y (we write $f\colon x \to y$ for $f \in \mathcal{C}(x, y)$),
- for every $x \in \mathcal{C}$, an identity morphism $\operatorname{id}_x \in \mathcal{C}(x, x)$,
- for all $x, y, z \in \mathcal{C}$ a composition law

  $$\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$$

  (we write gf or $g \circ f$ for the composite of $f\colon x \to y$, $g\colon y \to z$)

such that:

- composition is associative: for $f\colon x \to y$, $g\colon y \to z$, $h\colon z \to w$, we have

  $$h(gf) = (hg)f,$$

- the identity is a unit for composition: for $f\colon x \to y$, we have

  $$f(\operatorname{id}_x) = f = (\operatorname{id}_y)f.$$

Examples 2.2.2. Most objects in mathematics form categories. For example:

- There is a category Set whose objects are sets and whose morphisms are functions.
- There is a category Grp whose objects are groups and whose morphisms are homomorphisms.
- There is a category Ab whose objects are abelian groups and whose morphisms are homomorphisms.

- There is a category $\mathsf{Vect}_{\mathbb{R}}$ whose objects are vector spaces over $\mathbb{R}$ and whose morphisms are linear maps.

Definition 2.2.3. The category Top has topological spaces as objects and continuous maps as morphisms.

Composition is just composition of functions: if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, so is gf: for $U \subseteq Z$ open, $(gf)^{-1}(U) = g^{-1}(f^{-1}U)$ is also open.

Notation 2.2.4. If X and Y are topological spaces we'll also write $C(X,Y)$ for the set $\mathsf{Top}(X,Y)$ or $\mathrm{Hom}_{\mathsf{Top}}(X,Y)$ of continuous maps from X to Y.

Definition 2.2.5. Let $\mathcal{C}$ be a category. A morphism $f\colon x \to y$ in $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g\colon y \to x$ such that $gf = \mathrm{id}_x$ and $fg = \mathrm{id}_y$.

Definition 2.2.6. A continuous map $f\colon X \to Y$ is a *homeomorphism* if it's an isomorphism in Top, i.e. if there exists a continous map $g\colon Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Warning 2.2.7. In other words, f is a homeomorphism if it is a continous bijection such that f^{-1} is also continous. However, it is in general *not* enough for f to be a continuous bijection.

But a continous bijection $f\colon X \to Y$ *is* a homeomorphism if X is compact and Y is Hausdorff.

Exercise 2.3. Prove the following basic properties of isomorphisms in a category $\mathcal{C}$:

(i) If $f\colon x \to y$ and $g\colon y \to z$ are isomorphisms, so is $gf\colon x \to z$.

(ii) Given $f\colon x \to y$, if there exist $g, h\colon y \to x$ such that
$$gf = \mathrm{id}_x, \quad fh = \mathrm{id}_y,$$
then f is an isomorphism.

(iii) If f is an isomorphism, its inverse is unique.

(iv) If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and $f\colon x \to y$ is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.

(v) Being isomorphic is an equivalence relation on objects of $\mathcal{C}$.

Solution.

(i) If $f^{-1}\colon y \to x$ and $g^{-1}\colon z \to y$ are inverses of f and g, respectively, then we have
$$(gf)(f^{-1}g^{-1}) = g(ff^{-1})g^{-1} = g\mathrm{id}_y g^{-1} = gg^{-1} = \mathrm{id}_z,$$
$$(f^{-1}g^{-1})(gf) = f^{-1}(g^{-1}g)f = f^{-1}\mathrm{id}_y f = f^{-1}f = \mathrm{id}_x,$$
which shows that $f^{-1}g^{-1}$ is inverse to gf.

(ii) It suffices to show that we also have $fg = \mathrm{id}_y$, since then g is an inverse of f. Now observe that we have
$$fg = (fg)(fh) = f(gf)h = f\mathrm{id}_x h = fh = \mathrm{id}_y.$$

(iii) If g and h are two inverses of f, then it follows from (ii) that $g = h$.

(iv) Let $f^{-1}\colon y \to x$ be the inverse of f. Then
$$F(f)F(f^{-1}) = F(ff^{-1}) = F(\mathrm{id}_y) = \mathrm{id}_{F(y)}$$
and similarly $F(f^{-1})F(f) = \mathrm{id}_{F(x)}$, so that $F(f^{-1})$ is inverse to $F(f)$.

(v) Let us write $x \cong y$ for "there exists an isomorphism $f\colon x \to y$". Then

- $x \cong x$ since id_x is always an isomorphism,

- $x \cong y$ implies $y \cong x$ since if $f\colon x \to y$ is an isomorphism, so is $f^{-1}\colon y \to x$,
- $x \cong y$ and $y \cong z$ implies $x \cong z$ since if $f\colon x \to y$ and $g\colon y \to z$ are isomorphism, part (i) shows that $gf\colon x \to z$ is an isomorphism. □

An important concept in category theory is that of a "*universal property*" of an object. We will not make this precise here, but the basic idea is that a universal property characterizes an object uniquely up to isomorphism in terms of the ways of mapping other objects into (or out of) the given object. Here is an example:

Exercise 2.4. Let X be a topological space and $U \subseteq X$ a subset. Show that the subspace topology on U has the following universal property: if T is a topological space, then a continuous map from T to U is a map of sets $T \to U$ such that the composite $T \to U \hookrightarrow X$ is continuous.

A category is itself a sort of algebraic structure, and as with any such there is a corresponding notion of structure-preserving morphisms (or "homomorphism of categories"), which are called *functors*:

Definition 2.2.8. Let $\mathcal{C}$, $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is the assignment of

- an object $F(c) \in \mathcal{D}$ to every $c \in \mathcal{C}$,
- a morphism $F(f)\colon F(c) \to F(c')$ to every $f\colon c \to c'$ in $\mathcal{C}$,

compatibly with composition and identities, i.e.

- $F(gf) = F(g)F(f)$ for all composable morphisms f, g in $\mathcal{C}$,
- $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for all $c \in \mathcal{C}$.

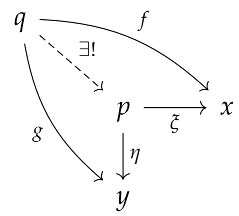
Example 2.2.9. Let $\mathcal{C}$ be a category and c an object of $\mathcal{C}$. We can define a functor $\mathcal{C}(c, -)\colon \mathcal{C} \to \mathsf{Set}$ by sending $x \in \mathcal{C}$ to the set $\mathcal{C}(c, x)$ of morphisms $c \to x$, with $f\colon x \to y$ going to the map $\mathcal{C}(c, x) \to \mathcal{C}(c, y)$ given by composition with f.

Exercise 2.5. Show that there is a functor $\mathsf{Top} \to \mathsf{Set}$ that takes a topological space to its underlying set (the "forgetful" functor) and two functors $\mathsf{Set} \to \mathsf{Top}$ that take a set to itself equipped with the discrete and indiscrete topologies, respectively.

2.3 *Products and Coproducts*

Category theory allows us to view constructions involving different sorts of mathematical objects as instances of a single concept valid in any category. Let us look at a first example of this, the categorical definition of *products*:

Definition 2.3.1. A *product* of two objects x, y in a category $\mathcal{C}$ is another object p together with morphisms ("projections") $p \xrightarrow{\xi} x$, $p \xrightarrow{\eta} y$ such that given any object q and morphisms $f\colon q \to x$, $g\colon q \to y$ there exists a unique morphism $\phi\colon q \to p$ such that $f = \xi\phi$, $g = \eta\phi$. In other words, there exists a unique morphism such that the following diagram commutes:



We often denote the product of x and y by $x \times y$.

The product of two objects, if it exists, is defined by a universal property; it follows that it is "unique up to unique isomorphism" in the following sense:

Lemma 2.3.2. *Let x, y be objects of a category $\mathcal{C}$. Suppose $(p, \xi\colon p \to x, \eta\colon p \to y)$ and $(p', \xi'\colon p' \to x, \eta'\colon p' \to y)$ are two products of x and y. Then there exists a* unique *isomorphism between p and p' that is compatible with the projections to x and y.*

Proof. Since p is a product, there is a unique morphism $\phi\colon p' \to p$ such that $\xi\phi = \xi'$ and $\eta\phi = \eta'$. And since p' is a product, there is also a unique morphism $\psi\colon p \to p'$ such that $\xi'\psi = \xi$ and $\eta'\psi = \eta$. The composite $\phi\psi\colon p \to p$ satisifes $\xi\phi\psi = \xi'\psi = \xi$ and $\eta\phi\psi = \eta$ — this implies $\phi\psi = \mathrm{id}_p$ since there is a *unique* morphism with this property. Similarly we have $\psi\phi = \mathrm{id}_{p'}$, which means that ϕ is an isomorphism with inverse ψ. □

Example 2.3.3. In the category Set, the product of two sets X, Y is the usual cartesian product, i.e.

$$X \times Y := \{(x, y) : x \in X, y \in Y\}, \tag{2.1}$$

with its projections to X and Y. To see this, we must check the universal property holds: given functions $f\colon S \to X$, $g\colon S \to Y$ there is indeed a unique function $\phi\colon S \to X \times Y$ that agrees with f and g on the projections, as this forces $\phi(s) = (f(s), g(s))$.

Remark 2.3.4. Thinking of constructions such as products in terms of universal properties can often make things simpler for us: For instance, I might prefer to define the ordered pair (x, y) as the set $\{\{a\}, \{a, b\}\}$ while you might prefer to define it as $\{b, \{a, b\}\}$. Then for sets X, Y our definitions (2.1) of their cartesian product would not be *equal*, but by Lemma 2.3.2 we know that there is a canonical isomorphism between them. Moreover, in practice we will never use any property of the cartesian product other than those that follow from its universal property, which means we don't really need to choose a preferred product at all (provided we know at least one choice does exist).

Exercise 2.6. Let x, y, z be objects of a category $\mathcal{C}$. Show that there is a canonical isomorphism

$$x \times (y \times z) \cong (x \times y) \times z,$$

provided these products exist.

Exercise 2.7. Show that the cartesian product of (abelian) groups is also the categorical product in Grp and Ab, when equipped with the canonical group structure.

Solution. If H and K are groups, the canonical group structure on $H \times K$ has multiplication $(h,k)(h',k') = (hh',kk')$, so the two projections $\pi_H\colon H \times K \to H$, $\pi_K\colon H \times K \to K$ are clearly group homomorphisms. Given group homomorphisms $\eta\colon G \to H$ and $\kappa\colon G \to K$, we must show there exists a unique group homomorphism $\phi\colon G \to H \times K$ such that $\pi_H\phi = \eta$ and $\pi_K\phi = \kappa$. We already know there exists a unique morphism of *sets* ϕ with these properties, given by $\phi(g) = (\eta(g), \kappa(g))$, so we only need to check this is in fact a group homomorphism. Indeed, we have

$$\phi(g)\phi(g') = (\eta(g),\kappa(g))(\eta(g'),\kappa(g')) = (\eta(g)\eta(g'),\kappa(g)\kappa(g')) = (\eta(gg'),\kappa(gg')) = \phi(gg')$$

since η and κ are homomorphisms. The same is true if the groups happen to be abelian. □

Now we consider products of topological spaces:

Definition 2.3.5. Let X and Y be topological spaces. The *product topology* on $X \times Y$ has as open sets those $W \subseteq X \times Y$ such that for all $w = (x,y) \in W$ there exist $U \subseteq X$, $V \subseteq Y$ open with $x \in U$, $y \in V$ and $U \times V \subseteq W$.

The product topology is the weakest (smallest) topology on $X \times Y$ such that all products of open sets are open.

Example 2.3.6. The usual topology on $\mathbb{R}^n$ is the product topology on $\mathbb{R} \times \cdots \times \mathbb{R}$. (A subset can be covered with open balls if and only if it can be covered with open cubes.)

Proposition 2.3.7. *Let X and Y be topological spaces. The projections $X \times Y \to X, Y$ are continuous for the product topology, and they exhibit $X \times Y$ as the categorical product in* Top.

Proof. Let $\xi\colon X \times Y \to X, \eta\colon X \times Y \to Y$ denote the projections. To see that ξ is continuous we must check that for $U \subseteq X$ open the preimage $\xi^{-1}U = U \times Y$ is open, which is clear from the definition since it is a product of open sets; similarly, η is continuous. Given functions $f\colon T \to X$, $g\colon T \to Y$, we know there exists a unique function $\phi\colon T \to X \times Y$ such that $\xi\phi = f, \eta\phi = g$. To see that the product topology makes $X \times Y$ the categorical product, we must show that if f and g are continuous then the unique map ϕ is also continuous. For $U \subseteq X, V \subseteq Y$ open we have that

$$\begin{aligned}
\phi^{-1}(U \times V) &= \phi^{-1}(U \times Y \cap X \times V) \\
&= \phi^{-1}(U \times Y) \cap \phi^{-1}(X \times V) \\
&= \phi^{-1}\xi^{-1}(U) \cap \phi^{-1}\eta^{-1}(V) \\
&= f^{-1}(U) \cap g^{-1}(V).
\end{aligned}$$

If f and g are continuous it follows that $\phi^{-1}(U \times V)$ is open in T. Any open set W in $X \times Y$ is by definition an (infinite) union of products of open sets in X and Y, so $\phi^{-1}(W)$ is a union of preimages of such, and hence also open. □

Exercise 2.8 $(*)$**.** Given a set I and a collection x_i $(i \in I)$ of objects of a category $\mathcal{C}$, their product (if it exists) is an object $\prod_{i \in I} x_i$ together with projections

$\pi_i\colon \prod_{i\in I} x_i \to x_i$ satisfying the following universal property: given an object y and morphisms $f_i\colon y \to x_i$ for $i \in I$, there exists a unique morphism $f\colon y \to \prod_{i\in I} x_i$ such that $\pi_i f = f_i$. Show that I-indexed cartesian products are categorical products in the category Set, and also in the categories Ab, Grp, Top when equipped with canonical (abelian) group structures and topologies. (What is an I-indexed product when I is empty?)

We can often "dualize" categorical notions by reversing the arrows. For the product, this gives the dual notion of *coproducts*:

Definition 2.3.8. Suppose x, y are objects of a category $\mathcal{C}$. The *coproduct* of x and y, if it exists, is an object $x \amalg y$ together with morphisms ("inclusions") $i\colon x \to x \amalg y, j\colon y \to x \amalg y$ such that given morphisms $f\colon x \to z$, $g\colon y \to z$ there exists a unique morphism $\phi\colon x \amalg y \to z$ such that $\phi \circ i = f, \phi \circ j = g$.

Example 2.3.9. The coproduct in Set of two sets I, J is the *disjoint union* $I \amalg J$. Though it's intuitively obvious what this means, in the usual formulation of set theory we have to define this in some awkward way to ensure I and J don't have elements in common, for example as $I \times \{0\} \cup J \times \{1\}$, with the inclusion $I, J \hookrightarrow I \amalg J$ given by $i \in I \mapsto (i, 0)$ and $j \in J \mapsto (j, 1)$, respectively.

Exercise 2.9. Show that the coproduct in Top of topological spaces X, Y is the disjoint union $X \amalg Y$ of sets, with a subset $U \subseteq X \amalg Y$ defined to be open if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y.

Example 2.3.10. The coproduct in Ab of two abelian groups A, B is the cartesian product $A \times B$ with its canonical (componentwise) group structure, often denote $A \oplus B$ when thought of as the coproduct, with the inclusions $A, B \hookrightarrow A \times B$ given by

$$a \in A \mapsto (a, 0), \;\; b \in B \mapsto (0, b).$$

A homomorphism $\phi\colon A \times B \to C$ is indeed uniquely determined by its restrictions along these inclusions, since we have

$$\phi(a, b) = \phi((a, 0) + (0, b)) = \phi(a, 0) + \phi(0, b).$$

Exercise 2.10 $(*)$**.** What is the coproduct of two copies of $\mathbb{Z}$ in Grp?

Exercise 2.11 $(*)$**.** Define I-indexed coproducts for any indexing set I, as in Exercise 2.8. Describe these in the categories Set and Top.

Exercise 2.12 $(*)$**.** If $\mathcal{C}$ is a category, we define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ to be the category with the same objects as $\mathcal{C}$, but with the direction of morphisms reversed — thus $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) := \mathrm{Hom}_{\mathcal{C}}(y, x)$. Check that a coproduct in $\mathcal{C}$ is the same thing as a product in $\mathcal{C}^{\mathrm{op}}$.

Exercise 2.13 $(*)$**.** Suppose a topological space X can be written as a union of subsets X_i ($i \in I$) such that the subsets X_i are open and disjoint. Show that $X \cong \coprod_{i\in I} X_i$ (i.e. the topology on X is the coproduct topology).

2.4 *Quotients*

In this section we will review *quotients* of topological spaces. We start by briefly recalling quotients of sets by (equivalence) relations:

Definition 2.4.1. Let I be a set. A *relation* on I is a subset $R \subseteq I \times I$; for $i, j \in I$ we write $i \sim_R j$ to mean "$(i, j) \in R$". An *equivalence relation* is a relation R that is

- reflexive: $i \sim_R i$ for all $i \in R$,
- symmetric: $i \sim_R j$ implies $j \sim_R i$,
- transitive: $i \sim_R j$ and $j \sim_R k$ implies $i \sim_R k$.

If R is an equivalence relation, then the *equivalence class* of $i \in I$ is the set $[i]_R := \{i' \in I : i' \sim_R i\}$. We write $I/\sim_R$ or I/R for the set of equivalence classes; then there is a canonical function $I \to I/R$ that takes i to $[i]_R$.

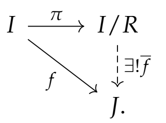
Definition 2.4.2. If R is a relation on I, the equivalence relation *generated* by R is the smallest equivalence relation $\overline{R} \subseteq I \times I$ such that $R \subseteq \overline{R}$. In this case we write $I/\sim_R$ or I/R for $I/\overline{R}$.

More explicitly, $i \sim_{\overline{R}} j$ if and only if there exists a finite sequence $i_0, \dots, i_n \in I$ $(n \geq 0)$ such that $i = i_0$, $j = i_n$ and either $i_{t-1} \sim_R i_t$ or $i_t \sim_R i_{t-1}$ for each $t = 1, \dots, n$.

Exercise 2.14. Suppose R is a relation on a set I. Show that the quotient $I/R = I/\overline{R}$ together with the quotient map

$$\pi \colon I \to I/R, \quad \pi(i) = [i]_{\overline{R}}$$

has the following universal property: any function $f \colon I \to J$ for which $i \sim_R j$ implies $f(i) = f(j)$ factors uniquely through π,



Now let's upgrade this to topology:

Definition 2.4.3. Let X be a topological space, and let R be a relation on its underlying set. Write π for the quotient map $X \to X/R = X/\overline{R}$. The *quotient topology* on X/R has as open sets those $U \subseteq X/R$ such that $\pi^{-1}U$ is open in X.

Lemma 2.4.4. *Let X be a topological space and R a relation on X. A continuous map $f \colon X \to Y$ such that*

$$x \sim_R x' \implies f(x) = f(x')$$

factors uniquely through π, i.e. there exists a unique continuous *map $\bar{f} \colon X/R \to Y$ such that $\bar{f}\pi = f$.*

Proof. By Exercise 2.14 there exists a unique function of sets $\bar{f} \colon X/R \to Y$ such that $\bar{f}\pi = f$, given by $\bar{f}([x]_{\overline{R}}) = f(x)$. Now we observe that the quotient topology has the property that f is continuous if and only if $\bar{f}$ is continuous: if $U \subseteq Y$ is open, then $\bar{f}^{-1}U$ is by definition open in X/R if and only if $\pi^{-1}\bar{f}^{-1}U = f^{-1}U$ is open in X. □

Examples 2.4.5. Many examples of topological spaces can be defined as quotients:

(i) Let S be the square $\{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$ with the subspace topology from $\mathbb{R}^2$. Then we can define the torus by identifying opposite sides of the square, i.e. taking the quotient

$$S/((x, 0) \sim (x, 1), (0, y) \sim (1, y)).$$

(ii) We can define the real n-dimensional projective space $\mathbb{RP}^n$ by identifying antipodal points on the n-sphere, i.e. as the quotient $S^n/(x \sim -x)$.

(iii) Similarly, we can define the *complex* n-dimensional projective space $\mathbb{CP}^n$ by viewing S^{2n+1} as a subset of $\mathbb{C}^{n+1}$ and forming the quotient $S^{2n+1}/(x \sim \lambda x : \lambda \in \mathbb{C}, |\lambda| = 1)$.

(iv) Let $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ denote the n-dimensional disc of radius 1, and $\partial D^n := \{x : |x| = 1\}$ its boundary. Then $S^n \cong D^n/\partial D^n$.

Exercise 2.15. Let $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the closed n-disk and

$$\partial D^n := S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$$

be the $(n-1)$-sphere, both equipped with the subspace topology from $\mathbb{R}^n$.

(i) Find explicit homeomorphisms $D^1/\partial D^1 \cong S^1$ and $D^2/\partial D^2 \cong S^2$. [Feel free to use that these are compact Hausdorff spaces, so that a continuous bijection is necessarily a homeomorphism.]

(ii) Show that the following three descriptions of the torus are homeomorphic:

$$T_1 := \{(x,y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}/((x,0) \sim (x,1), (0,y) \sim (1,y))$$
$$T_2 := S^1 \times S^1$$
$$T_3 := \{(R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)\} \subseteq \mathbb{R}^3 \quad (R > r)$$

(iii)* Find an explicit homeomorphism $D^n/\partial D^n \cong S^n$.

2.5 *Homotopies and Homotopy Equivalences*

Definition 2.5.1. A *homotopy* between two continuous maps $f, g\colon X \to Y$ is a continuous map $H\colon X \times [0,1] \to Y$ such that $H(\text{–}, 0) = f$ and $H(\text{–}, 1) = g$.

Remark 2.5.2. There are two useful ways to think of a homotopy H: On the one hand H specifies for each point $x \in X$ a path $H(x, \text{–})\colon [0,1] \to Y$ from $f(x)$ to $g(x)$, continuous in the point x. On the other hand, H specifies a continuous map $H_t := H(\text{–}, t)\colon X \to Y$ for $t \in [0,1]$, and this depends continuously on t; thus H is a *continuous deformation* of f into g.

Definition 2.5.3. Two continuous maps $f, g\colon X \to Y$ are *homotopic* if there exists a homotopy from f to g.

Exercise 2.16. Show that being homotopic is an equivalence relation on the set $C(X, Y)$ of continuous maps $X \to Y$.

Definition 2.5.4. The *homotopy category* $h\mathsf{Top}$ has topological spaces as objects, and morphism sets are given by

$$h\mathsf{Top}(X, Y) := \mathsf{Top}(X, Y)/\text{homotopy}.$$

Exercise 2.17. Prove that $h\mathsf{Top}$ is a well-defined category, with composition and identities induced from Top (so that there is a functor $\mathsf{Top} \to h\mathsf{Top}$ that takes each continuous map to its equivalence class). What does Exercise 2.3 then tell you about homotopy equivalences?

Definition 2.5.5. A continuous map $f\colon X \to Y$ is a *homotopy equivalence* if its image in $h\mathsf{Top}$ is an isomorphism. More explicitly, this means there exists a continuous map $g\colon Y \to X$ (a *homotopy inverse*) and homotopies between gf and id_X and between fg and id_Y.

A homotopy inverse, if it exists, is not *unique*, but it is uniquely determined up to homotopy (its homotopy class is unique).

Let's consider a more intuitive special case of homotopy equivalences:

Definition 2.5.6. Let X be a topological space and $A \subseteq X$ a subspace; let $i\colon A \hookrightarrow X$ denote the inclusion.

- A is a *retract* of X if there exists a continuous map $\rho\colon X \to A$ (a *retraction*) such that $\rho i = \mathrm{id}_A$, i.e. $\rho(a) = a$ for all $a \in A$.
- A is a *deformation retract* if in addition there exists a homotopy H between id_X and $i\rho\colon X \to X$.
- A is a *strong deformation retract* if the homotopy H can be chosen so that it fixes A, i.e. $H(a,t) = a$ for $a \in A$, $t \in [0,1]$.

If A is a deformation retract, then $i\colon A \hookrightarrow X$ is a homotopy equivalence with homotopy inverse ρ.

Definition 2.5.7. A topological space X is called *contractible* if it is homotopy-equivalent to a point.

Example 2.5.8. The discs D^n are contractible: Define $H\colon D^n \times [0,1] \to D^n$ by $H(x,t) = t \cdot x$; then H is a homotopy between the constant map with value $(0,\dots,0)$ (at $t = 0$) and id_{D^n} (at $t = 1$). This shows that the point $(0,\dots,0)$ is a (strong) deformation retract of D^n.

Exercise 2.18. Let S be a set. Show that:

(i) if S is equipped with the discrete topology then S is contractible if and only if S has exactly one element,

(ii) if S is equipped with the indiscrete topology then S is contractible if and only if S is non-empty.

[Hint: Prove that with the discrete topology the only continuous paths are the constant ones, while any path is continuous for the indiscrete topology.]

Exercise 2.19 ($**$)**.** (A topological proof that S^1 is not contractible.) View S^1 as $\{z \in \mathbb{C} : |z| = 1\}$ and let $\pi\colon \mathbb{R} \to S^1$ be the continuous map $x \mapsto e^{ix}$. We say that a continuous map $f\colon S^1 \to S^1$ *lifts to* $\mathbb{R}$ if there exists $\bar{f}\colon S^1 \to \mathbb{R}$ such that $f = \pi\bar{f}$.

(i) Show that if $g\colon S^1 \to S^1$ lifts to $\mathbb{R}$ and $f\colon S^1 \to S^1$ is another continuous map such that $f(x)/g(x) \neq -1$ for all $x \in S^1$ then f also lifts to $\mathbb{R}$.

(ii) Let $c_1\colon S^1 \to S^1$ be the constant map with value 1, and suppose f is homotopic to c_1, via a homotopy $H\colon S^1 \times [0,1] \to S^1$. Since $S^1 \times [0,1]$ is compact, we can choose $\delta > 0$ such if $|x - y| < \delta$ then $|H(x) - H(y)| < 2$ for all $x, y \in S^1 \times [0,1]$ (viewed as a subset of $\mathbb{R}^3$). Use this to show that f lifts to $\mathbb{R}$.

(iii) Use (ii) to prove that S^1 is not contractible (i.e. id_{S^1} is not homotopic to a constant map).

2.6 Path-Connectedness and π_0

Let X be a topological space. We define a relation on (the underlying set of) X by defining $x \sim y$ for $x, y \in X$ to mean that there exists a *path* in X from x to y, i.e. a continuous map $p\colon [0,1] \to X$ such that $p(0) = x, p(1) = y$. This is an equivalence relation:

- $x \sim x$ since we always have the constant path with value x,
- $x \sim y$ implies $y \sim x$: If p is a path from x to y we can define a new path p' by doing p in reverse, i.e. $p'(t) = p(1-t)$; then p' is a path from y to x,
- $x \sim y$ and $y \sim z$ implies $x \sim z$: If we have paths p_1 from x to y and p_2 from y to z we can define a new path p by first doing p_1 and then doing p_2,

$$p(t) = \begin{cases} p_1(2t), & t \leq \frac{1}{2}, \\ p_2(2t-1), & t \geq \frac{1}{2}. \end{cases}$$

  Then p is a path from x to z.

Definition 2.6.1. The set $\pi_0 X$ of *path-components* of X is the set of equivalence classes for this equivalence relation,

$$\pi_0 X := X/\sim.$$

The *path-components* of X are the subspaces corresponding to the equivalence classes in $\pi_0 X$. The space X is called *path-connected* if it has a single path-component, i.e. any two points of X are connected by a path.

Thus two points lie in the same path-component if and only if they are connected by a path.

The following exercise shows that $\pi_0 X$ is a homotopy-invariant of the space X:

Exercise 2.20. Let X and Y be topological spaces.

(i) Show that any continuous map $f\colon X \to Y$ induces a function $\pi_0 f\colon \pi_0 X \to \pi_0 Y$, and that this makes π_0 a functor $\mathsf{Top} \to \mathsf{Set}$.

(ii) Show that if $f, g\colon X \to Y$ are homotopic, then $\pi_0 f = \pi_0 g$. [Hence π_0 is a functor $h\mathsf{Top} \to \mathsf{Set}$.]

(iii) Show that if $f\colon X \to Y$ is a homotopy equivalence, then $\pi_0 f$ is an isomorphism.

We would like to think of $\pi_0 X$ as the "set of disjoint pieces" of X in the strong sense that X is a coproduct of its path-components, i.e.

$$X \cong \coprod_{\alpha \in \pi_0 X} X_\alpha, \tag{2.2}$$

with the coproduct topology. Unfortunately, this is false for an arbitrary topological space X for two reasons:

Recall that X is *connected* if it cannot be written as a union $U \cup V$ where $U, V \subseteq X$ are subsets that are both open and closed.

- If $X = A \amalg B$ with the coproduct topology, then the subspaces A and B are both open and closed in X. But there exist spaces that are connected, but not path-connected. Such a space X cannot be decomposed as a coproduct, even though $\pi_0 X$ has more than one element.

An example is the "topologist's sine curve" which is connected, but has two path-components.

- If $X = \coprod_{i \in I} X_i$ (with the coproduct topology) where the X_i are (path-)connected then the X_i are the connected components of X (meaning the maximal connected subsets), and they are both open and closed subsets of X. But the connected components of a space need not be open.

For example, if we view $\mathbb{Q}$ as a subspace of $\mathbb{R}$ then each point is a connected component (and a path-component) but is not open, since any open set around it in $\mathbb{R}$ contains other rational numbers.

However, such issues only arise for rather pathological (or at least non-geometric) examples of topological spaces; for any of the spaces we want to consider in this course, we will have a decomposition as in (2.2). We can ensure this by imposing the following condition on our spaces:

Definition 2.6.2. A topological space X is *locally path-connected* if for any open subset $U \subseteq X$ and any point $x \in U$ there exists a path-connected open subset $V \subseteq U$ with $x \in V$.

Proposition 2.6.3. *Suppose X is a locally path-connected space. Then the path-components of X are open subsets of X.*

Proof. Let x be a point of X and $P \subseteq X$ the path-component containing x. To show that P is open it suffices to show that P contains an open neighbourhood of any point $y \in P$. But since X is locally path-connected there exists some open neighbourhood U of y that is path-connected. It follows that all points in U lie in the same path-component as y, i.e. $U \subseteq P$. □

If we combine this observation with Exercise 2.13 we get the coproduct decomposition (2.2):

Corollary 2.6.4. *If X is locally path-connected, then*

$$X \cong \coprod_{\alpha \in \pi_0 X} X_\alpha,$$

where X_α is the path-component of X corresponding to $\alpha \in \pi_0 X$ and the right-hand side is equipped with the coproduct topology. □

2.7 (⋆) *The Fundamental Group*

Definition 2.7.1. A *pointed space* is a pair (X, x) with X a topological space and $x \in X$ a point. We have a category Top_* of pointed spaces, with $\mathsf{Top}_*((X, x), (Y, y))$ being the set of continous maps $f \colon X \to Y$ such that $f(x) = y$. A *pointed homotopy* of such maps $(X, x) \to (Y, y)$ is a homotopy $H \colon X \times [0, 1] \to Y$ such that $H(x, t) = y$ for all t. Then we define $h\mathsf{Top}_*$ to be the category of pointed topological spaces and pointed continuous maps modulo pointed homotopy.

Definition 2.7.2. The *fundamental group* of a pointed space (X, x) is the set

$$\pi_1(X, x) := h\mathsf{Top}_*((S^1, *), (X, x))$$

of pointed homotopy classes of loops in X that start and end at x.

The fundamental group has a group structure given by concatenation of loops, and any pointed continuous map $f\colon (X, x) \to (Y, y)$ induces a group homomorphism $\pi_1 f\colon \pi_1(X, x) \to \pi_1(Y, y)$. Moreover, any two (pointed) homotopic maps give the same homomorphism on π_1, so we can view the fundamental group as a functor

$$\pi_1\colon h\mathsf{Top}_* \to \mathsf{Grp}.$$

We also have higher *homotopy groups*, which can be defined as

$$\pi_n(X, x) := h\mathsf{Top}_*((S^n, *), (X, x)).$$

For $n \geq 2$ these have canonical abelian group structures, and can be viewed as functors $\pi_n\colon h\mathsf{Top}_* \to \mathsf{Ab}$.

The homotopy groups are fairly simple to define and contain a lot of information about the space X, but they are also notoriously difficult to compute. (For example, it is essentially impossible to compute all the homotopy groups of seemingly simple spaces like S^2.) You can learn more about homotopy groups in the sequel to this course.

3
Simplices and Singular Homology

In this chapter we define the singular homology groups of a topological space, and look at a few simple consequences of the definition. There are three steps in the definition: we first define the sets of *singular simplices* in a topological space in §3.1, and then linearize these to get the abelian groups of *singular chains* in §3.2, before we define boundaries, cycles, and finally homology groups in §3.3. In §3.4 we check that changing signs in homology corresponds to changing the orientation of a path. We then formalize the structure we have obtained in terms of *chain complexes* in §3.5, where we also show that singular homology is a functor. In the two lowest dimensions we can relate homology groups to structures we already know: in §3.6 we show that H_0X is the free abelian group on the set π_0X of path components, while in §3.7 we prove that (for X path-connected) H_1X is the abelianization of the fundamental group π_1X. Finally, in §3.8 we prove that homology takes arbitrary disjoint unions of spaces to the corresponding direct sums of abelian groups.

3.1 Singular Simplices

Simplices are higher-dimensional versions of triangles, tetrahedra, etc. Here is a convenient definition of the n-simplex as a topological space:

Definition 3.1.1. The *n-simplex* is the topological space

$$\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, \quad \sum_i x_i = 1\},$$

with the subspace topology from $\mathbb{R}^{n+1}$.

We see that

- Δ^0 is a point,
- Δ^1 is a closed interval,
- Δ^2 is a (filled) equilateral triangle,
- Δ^3 is a (solid) regular tetrahedron.

Terminology 3.1.2. The ith *vertex* of Δ^n is the point $e_i := (0, \dots, 1, \dots, 0)$ with 1 in the ith coordinate.

Remark 3.1.3. Any point in Δ^n can be written uniquely as $\sum_{i=0}^n a_i e_i$ with a_i non-negative real numbers that satisfy $\sum a_i = 1$. Given any function $f\colon \{0, \ldots, n\} \to \mathbb{R}^k$ we can define a continuous map $F\colon \Delta^n \to \mathbb{R}^k$ by $F(\sum_{i=0}^n a_i e_i) = \sum_{i=0}^n a_i f(i)$.

Definition 3.1.4. The ith *face* of Δ^n is the subset $\partial_i \Delta^n$ of points $(x_0, \ldots, x_n)$ where $x_i = 0$. This is homeomorphic to Δ^{n-1} via the ith *face map* $d^i\colon \Delta^{n-1} \to \Delta^n$, which takes $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$.

Note 3.1.5. The ith face is the face that does *not* contain the ith vertex, or the face *opposite* the ith vertex.

Lemma 3.1.6 (The simplicial identity)**.** *For $0 \leq j < i \leq n+1$ the two maps $d^i d^j$ and $d^j d^{i-1}\colon \Delta^{n-1} \to \Delta^{n+1}$ are the same.*

Proof. The map $d^j d^{i-1}$ first inserts 0 in the $(i-1)$th coordinate and then in the jth coordinate. But as $j < i$ these end up as the ith and jth coordinates in Δ^{n+1} — in other words, $d^j d^{i-1}$ is the inclusion of the subset $\{(x_0, \ldots, x_n) : x_i = x_j = 0\}$. The same holds for $d^i d^j$, where the coordinates don't shift, since $i > j$. □

We are going to study a topological space by looking at all the ways we can map simplices into it:

Definition 3.1.7. A *singular n-simplex* in a topological space X is a continuous map $\Delta^n \to X$. We write $\mathrm{Sing}_n(X)$ for the set of singular n-simplices in X (in the notation we used above, this is $\mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X)$).

Thus $\mathrm{Sing}_0(X)$ is the set of maps $* \to X$, which is just the underlying set of points X, while $\mathrm{Sing}_1(X)$ is the set of continuous paths $[0,1] \cong \Delta^1 \to X$ in X.

Definition 3.1.8. The face map $d^i\colon \Delta^{n-1} \to \Delta^n$ induces a map

$$\partial_i\colon \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X)$$

by composition, i.e.

$$\partial_i \sigma := \sigma \circ d^i\colon \Delta^{n-1} \hookrightarrow \Delta^n \to X$$

is given by restricting σ to the ith face of Δ^n. We call $\partial_i \sigma$ the *ith face* of the simplex σ.

Remarks 3.1.9.

(i) The name *singular* simplices is historical, and refers to the fact that we allow arbitrary continuous maps from simplices, not just ones that are nice embeddings (so they can have singularities).

(ii) We can think of the sets $\mathrm{Sing}_n(X)$ together with the face maps ∂_i as giving a combinatorial description of the space X, recording all the information we can get by "probing" X with simplices. In fact, if X is a reasonably nice space, we can recover X up to homotopy equivalence from this data (but we will not see that in this course).

(iii) The set $\mathrm{Sing}_n(X)$ is typically *huge*, even if the space X is simple — for example $\mathrm{Sing}_n(\Delta^1)$ is uncountable for all $n \geq 0$. (Thus this is not a combinatorial description that is useful for calculations, but one that is good for developing the theory.)

(iv) It is possible to use other families of "test spaces" than simplices to set up homology, without changing the resulting homology groups. For example, it is possible to use n-dimensional cubes.

Cubes are nicer than simplices in some ways (in particular, the product of two cubes is a cube, while the product of two simplices is not a simplex, which will annoy us later), but more complicated in others.

3.2 *Free Abelian Groups and Singular Chains*

The next step will be to *linearize* the sets $\mathrm{Sing}_n(X)$. This will allow us to define the boundary of a singular n-simplex as a linear combination of $(n-1)$-simplices. For example, for $\sigma\colon \Delta^1 \cong [0,1] \to X$, we have $\partial_0\sigma = \sigma(1)$, $\partial_1\sigma = \sigma(0)$, and we want the boundary of σ to be

$$\partial\sigma = \partial_0\sigma - \partial_1\sigma = \sigma(1) - \sigma(0),$$

so that $\partial\sigma = 0$ if σ is a closed loop. For this to make sense we will consider formal linear combinations of simplices, i.e. we take the free abelian group on the set $\mathrm{Sing}_n(X)$. Let's review this notion:

Definition 3.2.1. Let S be a set. Somewhat informally, the *free abelian group* $\mathbb{Z}S$ is the set of formal linear combinations $a_1 s_1 + \cdots + a_n s_n$, with $a_i \in \mathbb{Z}$, $s_i \in S$, with addition defined in the obvious way. More formally we can take $\mathbb{Z}S$ to be the set of functions $f\colon S \to \mathbb{Z}$ such that $f(s) = 0$ for all but finitely many s, with addition defined componentwise (i.e. $(f+g)(s) = f(s) + g(s)$). If we write e_s for the function $S \to \mathbb{Z}$ given by $e_s(s) = 1$, $e_s(s') = 0$ for $s' \neq s$, then any such function f can be written uniquely as a linear combination $f = \sum_{s \in S} f(s) e_s$ (which we can view as a finite linear combination since $f(s)$ is non-zero for only finitely many s).

Remark 3.2.2. The free abelian group $\mathbb{Z}S$ has the following universal property: If A is an abelian group and $\phi\colon S \to A$ is any function, then there exists a *unique* homomorphism $\mathbb{Z}S \to A$ extending ϕ (necessarily given by $\sum a_i s_i \mapsto \sum a_i \phi(s_i)$).

Definition 3.2.3. If A is an abelian group and $T \subseteq A$ is a subset, we say that A is *freely generated* by T if the induced homomorphism $\mathbb{Z}T \to A$ is an isomorphism. (In other words, every a in A can be written as a $\mathbb{Z}$-linear combination of elements of T in a unique way.) We say A has *rank* r if it is freely generated by a set of size r.

Example 3.2.4. The abelian group $\mathbb{Z}^r$ is freely generated by the "standard basis vectors" $e_i = (0, \ldots, 1, \ldots, 0)$ (with 1 in the ith coordinate), so $\mathbb{Z}^r$ is free of rank r.

Example 3.2.5. The trivial group 0 is the free abelian group on $\emptyset$.

Example 3.2.6. The abelian group $\mathbb{Z}/n$ $(n > 1)$ is *not* free: since $(n+1)a = a$ for $a \in \mathbb{Z}/n$, linear combinations are never unique.

Definition 3.2.7. The abelian group $S_n(X)$ of *singular n-chains* on the topological space X is $\mathbb{Z}\operatorname{Sing}_n(X)$. Thus a singular n-chain is a formal $\mathbb{Z}$-linear combination $\sum_{i=0}^n a_i\sigma_i$, $a_i \in \mathbb{Z}$, $\sigma_i \in \operatorname{Sing}_n(X)$.

It is sometimes convenient to extend the definition to $n < 0$, where we take $\operatorname{Sing}_n(X) := \emptyset$ and so $S_n(X) := 0$.

3.3 Boundaries, Cycles, and Singular Homology

Definition 3.3.1. By the universal property of free abelian groups, the face map $\partial_i \colon \operatorname{Sing}_n(X) \to \operatorname{Sing}_{n-1}(X)$ extends uniquely to a homomorphism $S_n(X) \to S_{n-1}(X)$, which we also denote ∂_i, so that the square

$$\begin{array}{ccc} \operatorname{Sing}_n(X) & \xrightarrow{\partial_i} & \operatorname{Sing}_{n-1}(X) \\ \downarrow & & \downarrow \\ S_n(X) & \xrightarrow{\partial_i} & S_{n-1}(X) \end{array}$$

commutes. This is given by $\partial_i(\sum_j a_j\sigma_j) = \sum_j a_j(\partial_i\sigma_j)$.

Definition 3.3.2. The *boundary operator* $\partial \colon S_n(X) \to S_{n-1}(X)$ is the homomorphism given by the alternating sum

$$\partial := \sum_{i=0}^{n} (-1)^i \partial_i.$$

Thus for a 1-simplex $\sigma \colon \Delta^1 \to X$, we have

$$\partial\sigma = \partial_0\sigma - \partial_1\sigma = \sigma(1) - \sigma(0).$$

For a 2-simplex $\sigma \colon \Delta^2 \to X$, we have

$$\partial\sigma = \partial_0\sigma - \partial_1\sigma + \partial_2\sigma.$$

Remark 3.3.3. We can (informally for now) think of the sign as "reversing orientation": if we write $\sigma|_{ij}$ for the path from $\sigma(i)$ to $\sigma(j)$ then the oriented boundary of the 2-simplex decomposes as $\sigma|_{01}$ followed by $\sigma|_{12}$ and then $\sigma|_{20}$, which we can represent as a linear combination $\sigma|_{01} + \sigma|_{12} + \sigma|_{20}$, while the formula for $\partial\sigma$ above gives

$$\partial\sigma = \sigma|_{12} - \sigma|_{02} + \sigma|_{01};$$

these agree if we think of $-\sigma|_{02}$ as corresponding to the reversed path $\sigma|_{20}$.

We give chains with no boundary a special name:

Definition 3.3.4. An *n-cycle* in X is an n-chain $c \in S_nX$ such that $\partial c = 0$. We write

$$Z_n(X) := \ker(\partial \colon S_n(X) \to S_{n-1}(X)) \subseteq S_n(X)$$

for the subgroup of cycles.

Example 3.3.5. If $\sigma \colon \Delta^1 \to X$ is a loop, so that $\sigma(0) = \sigma(1)$, then σ is a cycle. Similarly, if $\sigma_i \colon \Delta^1 \to X$, $i = 1, \ldots, n$ are paths such that $\sigma_i(1) = \sigma_{i+1}(0)$ and $\sigma_1(0) = \sigma_n(1)$, then $\sigma_1 + \cdots + \sigma_n$ is a cycle.

Definition 3.3.6. An *n-boundary* in X is an n-chain c such that there exists an $(n+1)$-chain b with $c = \partial b$. We write

$$B_n(X) := \operatorname{im}(\partial\colon S_{n+1}(X) \to S_n(X)) \subseteq S_n(X)$$

for the subgroup of boundaries.

Proposition 3.3.7. *For every topological space X, the boundary operator ∂ satisfies*

$$\partial^2 = 0,$$

where ∂^2 denotes the composite $S_{n+1}(X) \xrightarrow{\partial} S_n(X) \xrightarrow{\partial} S_{n-1}(X)$.

Proof. Since $S_{n+1}(X)$ is free, it suffices to show that $\partial^2\sigma = 0$ for $\sigma \in \mathrm{Sing}_{n+1}(X)$. We compute

$$\begin{aligned}
\partial^2\sigma &= \partial\left(\sum_{i=0}^{n+1}(-1)^i\partial_i\sigma\right) \\
&= \sum_{i=0}^{n+1}(-1)^i\partial(\partial_i\sigma) \\
&= \sum_{i=0}^{n+1}(-1)^i\left(\sum_{j=0}^{n}(-1)^j\partial_j\partial_i\sigma\right) \\
&= \sum_{i=0}^{n+1}\sum_{j=0}^{n}(-1)^{j+i}\partial_j\partial_i\sigma
\end{aligned}$$

Here $\partial_j\partial_i\sigma = \partial_j(\sigma \circ d^i) = \sigma \circ (d^i d^j)$, so if $j < i$ we can use Lemma 3.1.6 to get

$$\partial_j\partial_i\sigma = \partial_{i-1}\partial_j\sigma.$$

Applying this to the part of the sum where $j < i$ we get

$$\begin{aligned}
\partial^2\sigma &= \sum_{0\le j<i\le n+1}(-1)^{j+i}\partial_j\partial_i\sigma + \sum_{0\le i\le j\le n}(-1)^{j+i}\partial_j\partial_i\sigma \\
&= \sum_{0\le j<i\le n+1}(-1)^{j+i}\partial_{i-1}\partial_j\sigma + \sum_{0\le i\le j\le n}(-1)^{j+i}\partial_j\partial_i\sigma \\
&= \sum_{0\le j<i\le n+1}(-1)^{j+i}\partial_{i-1}\partial_j\sigma + \sum_{0\le j'<i'\le n+1}(-1)^{j'+i'-1}\partial_{i'-1}\partial_{j'}\sigma, \\
&= 0,
\end{aligned}$$

where in the penultimate line we rewrote the second sum using the indices $j' := i, i' := j+1$. □

Corollary 3.3.8. *Every boundary is a cycle, i.e. $B_n(X) \subseteq Z_n(X)$ as subgroups of $S_n(X)$.*

Definition 3.3.9. For X a topological space, the *nth singular homology group $H_n(X)$* is the quotient

$$H_n(X) := Z_n(X)/B_n(X).$$

Typically we can't compute these homology groups directly from the definition of singular chains, but need to use tools we'll develop later in the course. However, we can already compute homology groups in two *very* simple cases:

Example 3.3.10 ($H_*(\emptyset)$)**.** Since there are no maps $\Delta^n \to \emptyset$, we have $\mathrm{Sing}_n(\emptyset) = \emptyset$ and $S_n(\emptyset) = 0$ for all n, so

$$H_n(\emptyset) = 0$$

for all n.

Example 3.3.11 ($H_*(*)$)**.** There is a unique map $c_n \colon \Delta^n \to *$ for any n, so $\mathrm{Sing}_n(*) = \{c_n\}$. Thus $S_n(*) = \mathbb{Z}c_n$ is the free abelian group generated by the single generator c_n. By uniqueness we also know that $\partial_i c_n = c_{n-1}$ for all i, so that

$$\partial c_n = \sum_{i=0}^{n} (-1)^i c_{n-1} = \left(\sum_{i=0}^{n} (-1)^i\right) c_{n-1} = \begin{cases} 0, & n \text{ odd}, \\ c_{n-1}, & n \text{ even}. \end{cases}$$

This implies that we have

$$Z_n(*) = \begin{cases} \mathbb{Z}, & n > 0 \text{ odd}, n = 0, \\ 0, & n > 0 \text{ even}, n < 0, \end{cases} \qquad B_n(*) = \begin{cases} \mathbb{Z}, & n > 0 \text{ odd}, \\ 0, & n \text{ even}, n = 0, n < 0, \end{cases}$$

and so

$$\begin{aligned} H_n(*) &\cong \begin{cases} \mathbb{Z}/\mathbb{Z}, & n > 0 \text{ odd}, \\ 0/0, & n > 0 \text{ even or } n < 0, \\ \mathbb{Z}/0, & n = 0 \end{cases} \\ &\cong \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \neq 0. \end{cases} \end{aligned}$$

3.4 *Signs and Orientations*

Let us make more precise the idea that changing signs in $H_1(X)$ corresponds to reversing the orientation of paths.

Definition 3.4.1. Any function $\phi\colon \{0, \ldots, n\} \to \{0, \ldots, m\}$ gives a continuous map $\Delta(\phi)\colon \Delta^n \to \Delta^m$ given by $(t_0, \ldots, t_n) \mapsto (s_0, \ldots, s_m)$ where

$$s_j = \sum_{i \in \phi^{-1}(j)} t_i.$$

Example 3.4.2. The face map $d^i\colon \Delta^{n-1} \hookrightarrow \Delta^n$ corresponds to

$$\{0, \ldots, n-1\} \cong \{0, \ldots, n\} \setminus \{i\} \hookrightarrow \{0, \ldots, n\}$$

(using the unique order-preserving isomorphism).

Example 3.4.3. Let $\alpha\colon \{0, 1\} \to \{0, 1\}$ be the automorphism that permutes 0 and 1; this induces a homeomorphism $\Delta(\alpha)\colon \Delta^1 \xrightarrow{\sim} \Delta^1$ (given in terms of the embedding in $\mathbb{R}^2$ by flipping the two axes), which reverses the orientation of Δ^1. (In terms of the reparametrization $[0, 1] \cong \Delta^1$, α corresponds to the orientation-reversing automorphism $t \mapsto (1 - t)$.)

Proposition 3.4.4. *For any 1-simplex $\sigma\colon \Delta^1 \to X$, the chain*

$$\sigma \circ \alpha + \sigma$$

is a boundary.

Remark 3.4.5. This proposition says that $\sigma \circ \alpha$ and $-\sigma$ represent the same homology class, so that in $H_1(X)$ we can think of $-[\sigma]$ as the path obtained by reversing the orientation of σ.

Proof of Proposition 3.4.4. Let $\phi\colon \{0,1,2\} \to \{0,1\}$ be the function taking $0,2$ to 0 and 1 to 1. Then $\Delta(\phi)\colon \Delta^2 \to \Delta^1$ is given by

$$(t_0, t_1, t_2) \mapsto (t_0 + t_2, t_1);$$

this "collapses" the 2-simplex to a 1-simplex and sends the entire face $d^1(\Delta^2)$ (where $t_1 = 0$) to the point $(1,0)$. The 2-simplex $\sigma \circ \Delta(\phi)$ satisfies

$$\begin{aligned}
\partial_0(\sigma \circ \Delta(\phi)) &= \sigma \circ \alpha, \\
\partial_1(\sigma \circ \Delta(\phi)) &= \sigma \circ \Delta(\psi), \\
\partial_2(\sigma \circ \Delta(\phi)) &= \sigma,
\end{aligned}$$

where $\psi\colon \{0,1\} \to \{0,1\}$ is the map sending $0,1$ to 0. Then the 1-simplex $\sigma \circ \Delta(\psi)$ is constant at the point $p := \sigma(1,0)$, so if we write $C_n(p)$ for the n-simplex constant at p we have

$$\partial(\sigma \circ \Delta(\phi)) = \sigma \circ \alpha - C_1(p) + \sigma$$

But we also have

$$\partial C_2(p) = C_1(p) - C_1(p) + C_1(p) = C_1(p),$$

so that

$$\sigma \circ \alpha + \sigma = \partial(\sigma \circ \Delta(\phi) - C_2(p)). \qquad \square$$

3.5 Chain Complexes and Functoriality

We now want to prove that the singular homology groups give a sequence of functors from topological spaces to abelian groups. It is convenient to do this in two steps, which requires introducing some terminology for the structure we have obtained on the singular chains:

Definition 3.5.1. A *graded abelian group* $A_\bullet$ is a sequence of abelian groups A_n, $n \in \mathbb{Z}$. A *chain complex* is a graded abelian group $A_\bullet$ together with homomorphisms $\partial\colon A_n \to A_{n-1}$ that satisfy $\partial^2 = 0$ where ∂^2 is the composite $A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1}$.

Example 3.5.2. For a topological space X, we have shown that $(S_\bullet X, \partial)$ is a chain complex, the *singular chain complex* of X.

We can define cycles and boundaries in any chain complex:

Definition 3.5.3. If $(A_\bullet, \partial)$ is a chain complex, we write

$$Z_n(A) := \ker(\partial\colon A_n \to A_{n-1}) \subseteq A_n$$

for the group of *n-cycles* and

$$B_n(A) := \mathrm{im}(\partial\colon A_{n+1} \to A_n) \subseteq A_n$$

for the group of *n-boundaries*. Since $\partial^2 = 0$ by assumption, we again have $B_n(A) \subseteq Z_n(A)$. The nth *homology group* $H_n(A)$ is the quotient

$$H_n(A) := Z_n(A)/B_n(A).$$

Example 3.5.4. The singular homology groups of a topological space X are precisely the homology groups $H_*(S_\bullet(X))$ of the chain complex $S_\bullet(X)$.

Now we can introduce the appropriate notion of morphisms between chain complexes:

Definition 3.5.5. Suppose $A_\bullet$ and $B_\bullet$ are chain complexes. A *chain map* $f_\bullet\colon A_\bullet \to B_\bullet$ is a sequence of homomorphisms $f_n\colon A_n \to B_n$ that are compatible with the boundary maps in the sense that the squares

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & B_n \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} \end{array}$$

commute, i.e. we have

$$\partial f_n = f_{n-1}\partial$$

for all n.

Notation 3.5.6. We define the following categories:

- grAb has graded abelian groups as objects and graded homomorphisms (by which we just mean sequences of homomorphisms) as its morphisms,
- Ch has chain complexes as objects and chain maps as morphisms.

Lemma 3.5.7. *$S_\bullet$ is a functor* $\mathsf{Top} \to \mathsf{Ch}$.

Proof. For a continuous map $f\colon X \to Y$ we first define

$$\mathrm{Sing}_n(f)\colon \mathrm{Sing}_n(X) \to \mathrm{Sing}_n(Y)$$

to be the map of sets taking $\sigma\colon \Delta^n \to X$ to the composite $f \circ \sigma\colon \Delta^n \to Y$. Then

$$\partial_i(\mathrm{Sing}_n(f)(\sigma)) = (f \circ \sigma) \circ d^i = f \circ (\sigma \circ d^i) = \mathrm{Sing}_{n-1}(f)(\partial_i \sigma).$$

Next we define $S_n(f)\colon S_n(X) \to S_n(Y)$ to be the unique homomorphism given by $\mathrm{Sing}_n(f)$ on generators; thus

$$S_n(f)(\sum_i a_i \sigma_i) = \sum_i a_i \,\mathrm{Sing}_n(f)(\sigma_i).$$

Then $S_\bullet(f)$ is a chain map: for $\sigma \in \mathrm{Sing}_n(X)$ we have

$$\partial(S_n f)(\sigma) = \sum_i (-1)^i \partial_i(\mathrm{Sing}_n f)(\sigma) = \sum_i (-1)^i (\mathrm{Sing}_{n-1} f)(\partial_i \sigma) = (S_{n-1}f)(\partial\sigma).$$

We also see immediately from the definition that $S_\bullet$ satisfies $S_\bullet(f \circ g) = S_\bullet(f) \circ S_\bullet(g)$ and $S_\bullet(\mathrm{id}_X) = \mathrm{id}_{S_\bullet(X)}$, so that $S_\bullet$ is indeed a functor. □

Notation 3.5.8. We often write $f_*\colon S_\bullet(X) \to S_\bullet(Y)$ for the chain map $S_\bullet(f)$ induced by a continuous map $f\colon X \to Y$.

Lemma 3.5.9. *H_n is a functor $\mathsf{Ch} \to \mathsf{Ab}$, and H_* is a functor $\mathsf{Ch} \to \mathsf{grAb}$.*

Proof. If $f_\bullet\colon A_\bullet \to B_\bullet$ is a chain map, then we want to define $H_n f\colon H_n(A_\bullet) \to H_n(B_\bullet)$. Since $f_\bullet$ is a chain map we have

$$\partial f_n(z) = f_{n-1}\partial z = 0, \quad z \in Z_n(A),$$

$$f_n(\partial x) = \partial(f_{n+1}x), \quad x \in A_{n+1},$$

and so $f_\bullet$ restricts to homomorphisms $Z_n A \to Z_n B$ and $B_n A \to B_n B$. Thus we get a homomorphism $Z_n A \to Z_n B \to H_n B$ which takes $B_n A$ to 0, and so factors through a canonical homomorphism $H_n A \to H_n B$ (taking a class $[x]$ to $[f_n x]$). It is clear from the definition that this construction is compatible with composition and identities and so is a functor. □

Combining the two lemmas, we see that singular homology is a functor

$$H_*\colon \mathsf{Top} \xrightarrow{S_\bullet} \mathsf{Ch} \xrightarrow{H_*} \mathsf{grAb}.$$

This implies that if two spaces X, X' are homeomorphic, then their homology groups are isomorphic. We will prove later that singular homology actually takes homotopy equivalences to isomorphism, and so gives a functor $h\mathsf{Top} \to \mathsf{grAb}$.

Notation 3.5.10. We often write $f_*\colon H_n(X) \to H_n(Y)$ for the morphisms $H_n(f)$ in homology induced by a continuous map $f\colon X \to Y$.

3.6 H_0 and Path Components

In this section we show that the 0th homology group of a space is closely related to the set of path components. To state this result precisely it is convenient to introduce another concept from category theory:

Definition 3.6.1. Suppose $F, G\colon \mathcal{C} \to \mathcal{D}$ are two functors between the same categories. A *natural transformation* η from F to G (written $\eta\colon F \to G$) consists of morphisms $\eta_x\colon F(x) \to G(x)$ for every $x \in \mathcal{C}$ such that for every morphism $\phi\colon x \to y$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta_x} & G(x) \\ \downarrow{\scriptstyle F(\phi)} & & \downarrow{\scriptstyle G(\phi)} \\ F(y) & \xrightarrow{\eta_y} & G(y) \end{array}$$

commutes, meaning that $G(\phi)\eta_x = \eta_y F(\phi)$. We say that η is a *natural isomorphism* if the component η_x is an isomorphism for every $x \in \mathcal{C}$.

Thus a natural isomorphism is a collection of isomorphisms that are compatible with all morphisms. You can also think of a natural isomorphism as a "homotopy of functors".

Exercise 3.1. If V is a vector space over a field k, consider the linear map $\eta_V : V \to V^{**}$ to the double dual, taking $v \in V$ to the linear functional

$$\eta_V(v) : V^* \to k, \quad \phi \mapsto \phi(v).$$

Prove that these maps are natural, i.e. they determine a natural transformation η of functors $\mathsf{Vect}_k \to \mathsf{Vect}_k$ from the identity to the double dual. Show that if we restrict to finite-dimensional vector spaces η becomes a natural isomorphism.

Proposition 3.6.2. *There is a natural isomorphism*

$$H_0(X) \cong \mathbb{Z}\pi_0 X, \quad X \in \mathsf{Top}$$

of functors $\mathsf{Top} \to \mathsf{Ab}$.

Proof. For any topological space X, we have by definition $Z_0(X) = S_0(X) = \mathbb{Z}X$, the free abelian group on the set of points of X. The map of sets $X \to \pi_0 X$ taking $x \in X$ to its equivalence class $[x] \in \pi_0 X$ is a natural transformation and induces a natural homomorphism $\pi_X : \mathbb{Z}X \to \mathbb{Z}\pi_0 X$, which is clearly surjective; we need to show its kernel is exactly the subgroup $B_0 X$ of boundaries.

Since $\partial\sigma$ for a 1-simplex σ is $\partial_0\sigma - \partial_1\sigma = \sigma(1) - \sigma(0)$, the subgroup $B_0(X)$ of boundaries is freely generated by $x - y$ where x, y are points of X such that there exists a path between x and y, i.e. x and y are in the same path component. In particular, these generators all map to 0 under π_X, so that $B_0(X) \subseteq \ker \pi_X$. Conversely, suppose $\sum_{x \in X}^n a(x)x$ is an element of $\ker \pi_X$ (where $a(x) = 0$ for all but finitely many $x \in X$). We can rewrite this as a sum

$$\sum_{\alpha \in \pi_0 X} \left(\sum_{x \in X_\alpha} a(x)x \right)$$

indexed over the path components X_α of X, which then maps to $\sum_{\alpha \in \pi_0 X} \left(\sum_{x \in X_\alpha} a(x) \right) \alpha$ in $\mathbb{Z}\pi_0 X$. Since $\mathbb{Z}\pi_0 X$ is free, we must have $\sum_{x \in X_\alpha} a(x) = 0$ for each path-component c. Thus the number of positive and negative points in $\sum_{x \in X_\alpha} a(x)x$ must be equal, i.e. we can write this (non-canonically) as a finite sum $\sum_{i=1}^n x_i - y_i$ where x_i, y_i are points in X_α. Since this is a path-component, we can choose paths p_i with $p_i(0) = y_i$, $p_i(1) = x_i$; then we have

$$\sum_{x \in X_\alpha} a(x)x = \sum_{i=1}^n \partial p_i = \partial \left(\sum_{i=1}^n p_i \right).$$

This is true for each path-component, and so we must have that $\sum_{x \in X} a(x)x$ is in $B_0(X)$ as required. ☐

Remark 3.6.3. The *naturality* of the isomorphism in Proposition 3.6.2 is very useful: it lets us explicitly identify the morphism $f_* : H_0(X) \to H_0(Y)$ induced by a continous map $f : X \to Y$ as the homomorphism $\mathbb{Z}\pi_0(X) \to \mathbb{Z}\pi_0(Y)$ that arises by linearizing the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$, which takes the equivalence class $[x]$ containing $x \in X$ to the class $[f(x)]$ containing $f(x)$. In other words, if we write an element a of $H_0(X) \cong \mathbb{Z}\pi_0(X)$ as $a = \sum_{i=1}^n a_i[x_i]$ with $a_i \in \mathbb{Z}, x_i \in X$, then $f_*(a) = \sum_{i=1}^n a_i[f(x_i)]$.

Example 3.6.4. For any topological space X there is a unique continuous map $X \to *$, which gives a homomorphism $H_*X \to H_**$. Since H_** vanishes except in degree 0, the only interesting part is the map $H_0X \to H_0*$. Since the isomorphism of Proposition 3.6.2 is natural, we can identify this with the homomorphism $\mathbb{Z}\pi_0X \to \mathbb{Z}$ induced by the unique morphism of sets $\pi_0X \to *$; this takes each generator in π_0X to $1 \in \mathbb{Z}$.

3.7 H_1 and the Fundamental Group

Suppose X is a path-connected space. In this section we will show that we can identify the first homology group $H_1(X)$ in terms of the *fundamental group* of X.

Recall that if x is a point of X, the fundamental group $\pi_1(X, x)$ consists of pointed homotopy classes of loops, with the group operation given by concatenating loops. More precisely, if we define

$$L(X, x) := \{p\colon I \to X : p(0) = p(1) = x\},$$

then $\pi_1(X, x)$ is the quotient of $L(X, x)$ where we identify paths p, q if there exists a homotopy $H\colon I \times I \to X$ such that $H(s, 0) = p(s), H(s, 1) = q(s)$ and $H(0, t) = H(1, t) = x$ for all $t \in I$.

By definition, $L(X, x)$ is a subset of $\mathrm{Sing}_1(X)$, which gives a function

$$\alpha\colon L(X, x) \to \mathrm{Sing}_1(X) \to S_1(X).$$

Here $\partial\alpha(p) = x - x = 0$, so $\alpha(p)$ is a cycle for every $p \in L(X, x)$; we can therefore view α as a function $L(X, x) \to Z_1(X)$.

Lemma 3.7.1. *The composite*

$$L(X, x) \xrightarrow{\alpha} Z_1(X) \to H_1(X)$$

factors through the quotient $\pi_1(X, x)$.

Proof. We must show that given loops p, q and a pointed homotopy H as above, the chain $q - p$ is a boundary. Dividing the square $I \times I$ into two triangles, we get inclusions $U, L\colon \Delta^2 \hookrightarrow I \times I$ given on vertices by

$$U(e_0) = (0, 0), \quad U(e_1) = (1, 0), \quad U(e_2) = (1, 1),$$

$$L(e_0) = (0, 0), \quad L(e_1) = (0, 1), \quad U(e_2) = (1, 1).$$

Then $H \circ U - H \circ L$ is a chain on X that satisfies

$$\partial(H \circ U - H \circ L) = (q - d + c_x) - (c_x - d + p) = q - p,$$

where c_x is the path constant at x and d is the restriction of H to the diagonal from $(0, 0)$ to $(1, 1)$. □

We thus have a function $\beta\colon \pi_1(X, x) \to H_1(X)$.

Proposition 3.7.2. *Suppose $p, q\colon I \to X$ are paths such that $p(1) = q(0)$. Define a new path $p * q\colon I \to X$ by concatenating p and q, so that*

$$p * q(t) = \begin{cases} p(2t), & 0 \leq t \leq \frac{1}{2}, \\ q(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $p + q - p * q$ is a boundary in $S_1(X)$.

Proof. We abbreviate $r := p * q$. Define a continuous map $\phi\colon \Delta^2 \to I$ by

$$\phi(a_0 e_0 + a_1 e_1 + a_2 e_2) = a_1 \cdot \tfrac{1}{2} + a_2,$$

where the a_i are non-negative real numbers with $a_0 + a_1 + a_2 = 1$. This takes the vertices e_0, e_1, e_2 to $0, \frac{1}{2}, 1$, respectively. If we identify Δ^1 with I using the isomorphism $t \mapsto (1-t)e_0 + te_1$, then the 2-simplex $r \circ \phi\colon \Delta^2 \to X$ satisfies:

$$\partial_0(r \circ \phi)(t) = r(\phi((1-t)e_1 + te_2)) = r(\tfrac{1-t}{2} + t) = r(\tfrac{1+t}{2}) = q(t),$$

$$\partial_1(r \circ \phi)(t) = r(\phi((1-t)e_0 + te_2)) = r(t),$$

$$\partial_2(r \circ \phi)(t) = r(\phi((1-t)e_0 + te_1)) = r(\tfrac{t}{2}) = p(t).$$

Hence $\partial(r \circ \phi) = q - r + p$, as required. □

Corollary 3.7.3. *$\beta\colon \pi_1(X, x) \to H_1(X)$ is a group homomorphism.*

Proof. Given loops $p, q\colon I \to X$ at x, their product in $\pi_1(X, x)$ is represented by the concatenation $p * q$ as in Proposition 3.7.2. Since $p + q - p * q$ is then a boundary, we have $[p] + [q] = [p * q]$ in $H_1(X)$, which implies that β is a homomorphism. □

Proposition 3.7.4. *Suppose $\gamma \in S_1(X)$ is a 1-chain such that $\partial\gamma = x - y$ for two points $x, y \in X$ where either $x \neq y$ or the point $x = y$ occurs as a boundary of a 1-simplex in γ. Then there exists a path p from y to x in X such that $\gamma - p$ is a boundary.*

Proof. Suppose $\gamma = \sum_{i=1}^{n} a_i p_i$ with $p_i \in \mathrm{Sing}_1(X)$, $a_i \in \mathbb{Z}$. Using Proposition 3.4.4 we may assume all the a_i's are positive: if not, replace $a_i p_i$ by $(-a_i)(p_i^{-1})$ where p_i^{-1} is p_i composed with the orientation-reversing automorphism of Δ^1; this only changes γ by a boundary. We may then rewrite γ as $\sum_{j=1}^{N} q_j$ where each term q_j is a path; we induct on the positive integer N (with the case $N = 1$ being trivial). For the inductive step we first note there must exist some path q_i in the sum γ such that $q_i(1) = x$. If $z := q_i(0)$ we have $\gamma = q_i + \gamma'$ where $\partial\gamma' = z - y$. There are two cases to consider:

- $z \neq y$: In this case by hypothesis we can choose a path q' from y to z such that $\gamma' - q'$ is a boundary. Then $\gamma - (\gamma' - q') = q_i + q'$ differs from γ by a boundary. Moreover, if r denotes the concatenation of q' and q_i then $q' * q_i$ is a path from y to x and $q' + q_i - q' * q_i$ is a boundary by Proposition 3.7.2. Then $\gamma - q' * q_i$ is a boundary.

- $z = y$: Then $\partial\gamma' = 0$. The point y need not appear as a term in the boundary, but by hypothesis (as $N > 1$) if we choose a point w that does appears as a boundary point in γ' then we can choose a path q' from w to w such that $\gamma' - q'$ is a boundary. Since X is path-connected we can choose a path r from y to w, and then $q'' := r * (q' * r^{-1})$ differs from q' by a boundary by Proposition 3.7.2 and Proposition 3.4.4. Now we can use Proposition 3.7.2 again to conclude that $\gamma - q'' * q_i$ is a boundary. □

Lemma 3.7.5. *$\beta\colon \pi_1(X, x) \to H_1(X)$ is surjective.*

Proof. Suppose γ is a 1-cycle on X. If $\gamma \neq 0$ we can choose a point y that occurs as a boundary point of one of the 1-simplices that occur in γ. As a special case of Proposition 3.7.4 we can then choose a path $p\colon I \to X$ with $p(0) = p(1) = y$ such that $[\gamma] = [p]$ in $H_1(X)$. It therefore suffices to show that $[p]$ is in the image of β. Since X is by assumption path-connected, we can choose a path q from x to y. Since $q + q \circ \sigma$ is a boundary by Proposition 3.4.4, $[p]$ is also represented by $q + p + q \circ \sigma$; by Proposition 3.7.2 this cycle represents the same class as that of the loop p' obtained by concatenating the paths q, p, and $q \circ \sigma$, which is a loop at x and so in the image of β, as required. □

Definition 3.7.6. Let G be a group. The *commutator subgroup* $[G, G] \subseteq G$ is the (normal) subgroup generated by the *commutators* in G, i.e. the elements of the form $ghg^{-1}h^{-1}$ for $g, h \in G$. The *abelianization* of G is the quotient $G^{\text{ab}} := G/[G, G]$.

Lemma 3.7.7. *If H is an abelian group, then every group homomorphism $\phi\colon G \to H$ factors uniquely through the abelianization G^{ab}, which is an abelian group.*

Proof. Since H is abelian, we have $\phi(g)\phi(h) = \phi(h)\phi(g)$ for all $g, h \in G$, which means $\phi(ghg^{-1}h^{-1}) = 1$ so that ϕ factors through the quotient. If $[g]$ denotes the image of $g \in G$ in G^{ab} then we have $[g][h] = [gh] = [hg] = [h][g]$ since the quotient identifies gh and hg. □

Since $H_1(X)$ is abelian, the homomorphism β factors uniquely through a surjective homomorphism $\gamma\colon \pi_1(X, x)^{\text{ab}} \to H_1(X)$.

Proposition 3.7.8. *The homomorphism γ is injective.*

Proof. Since X is path-connected, we can choose for every point $y \in X$ a path $p_y\colon I \to X$ with $p(0) = x, p(1) = y$; we assume that p_x is the constant path at x. We can then define a function $\Lambda\colon \mathrm{Sing}_1(X) \to L(X, x)$ by

This proof uses the Axiom of Choice, but this is not really necessary.

$$q \mapsto p_{q(0)} * (q * p_{q(1)}^{-1}),$$

where we extend the notation for contatenation to paths that are not loops in the obvious way. Note that if q is a loop at x then $\Lambda(q)$ is homotopic to q (since p_x represents the identity in the fundamental group).

Since $\pi_1(X,x)^{\mathrm{ab}}$ is an abelian group, the composite

$$\mathrm{Sing}_1(X) \xrightarrow{\Lambda} L(X,x) \to \pi_1(X,x) \to \pi_1(X,x)^{\mathrm{ab}}$$

extends uniquely to a homomorphism $\lambda\colon S_1(X) \to \pi_1(X,x)^{\mathrm{ab}}$. Now if q is a loop at x, we see that $\lambda(q)$ is precisely the image in the quotient $\pi_1(X,x)^{\mathrm{ab}}$ of the element of the fundamental group represented by the loop q.

We claim that the restriction λ takes boundaries to 0. To see this, consider a 2-simplex $\sigma\colon \Delta^2 \to X$; we must show that $\lambda(\partial\sigma) = 0$. By definition we have

$$\partial\sigma = \partial_0\sigma - \partial_1\sigma + \partial_2\sigma,$$

so that $\lambda(\partial\sigma)$ is represented by the loop

$$\begin{aligned}\Lambda(\partial_2\sigma) * \Lambda(\partial_0\sigma) * \Lambda(\partial_1\sigma)^{-1} &\simeq p_{\sigma(e_0)} * \partial_2\sigma * p^{-1}_{\sigma(e_1)} * p_{\sigma(e_1)} * \partial_0\sigma * p^{-1}_{\sigma(e_2)} * p_{\sigma(e_2)} * (\partial_1\sigma)^{-1} * p^{-1}_{\sigma(e_0)} \\ &\simeq p_{\sigma(e_0)} * \partial_2\sigma * \partial_0\sigma * (\partial_1\sigma)^{-1} * p^{-1}_{\sigma(e_0)}.\end{aligned}$$

Since the 2-simplex σ gives a nullhomotopy of the loop $\partial_2\sigma * \partial_0\sigma * (\partial_1\sigma)^{-1}$ at $\sigma(e_0)$ that goes around the boundary of σ, we see that $\lambda(\partial\sigma)$ is indeed 0.

It follows that $\lambda|_{Z_1(X)}$ factors through a homomorphism $\lambda'\colon H_1(X) \to \pi_1(X,x)^{\mathrm{ab}}$, which satisfies $\lambda'\gamma = \mathrm{id}$ since by construction for a loop q at x the image of $\gamma(q)$ under λ' is the image of q in $\pi_1(X,x)^{\mathrm{ab}}$. We conclude that γ must be injective, as required. □

Combining our results in this section, we get:

Theorem 3.7.9. *If (X,x) is a path-connected pointed space, there is an isomorphism*

$$\pi_1(X,x)^{\mathrm{ab}} \cong H_1(X).$$

Remark 3.7.10. More precisely, this is a natural isomorphism of functors $\mathsf{Top}_* \to \mathsf{Ab}$.

3.8 *Disjoint Unions*

Definition 3.8.1. Given abelian groups A_i ($i \in I$), their *direct sum* $\bigoplus_{i\in I} A_i$ is the subset of the cartesian product $\prod_{i\in I} A_i$ consisting of lists $(a_i)_{i\in I}$ ($a_i \in A_i$) where $a_i = 0$ for all but finitely many I, with the addition given componentwise.

Exercise 3.2. Let $A_i, i \in I$ be a collection of abelian groups indexed by a set I, and define the inclusion $I_j\colon A_j \to \bigoplus_{i\in I} A_i$ by $I_j(a) = (a_i)_{i\in I}$ where $a_j = a$ and $a_i = 0$ otherwise. Show that the I_j's exhibit the direct sum $\bigoplus_{i\in I} A_i$ as the I-indexed coproduct in Ab, i.e. given homomorphisms $\phi_j\colon A_j \to B$ there exists a unique homomorphism $\phi\colon \bigoplus_{i\in I} A_i \to B$ with $\phi_j = \phi \circ I_j$.

Exercise 3.3. Given sets T_i, $i \in I$, show that there is a natural isomorphism

$$\mathbb{Z}\left(\coprod_i T_i\right) \cong \bigoplus_i \mathbb{Z}T_i.$$

[Hint: We can view the left-hand side as consisting of functions $f\colon \coprod_i T_i \to \mathbb{Z}$ that are 0 except at finitely many elements, while the right-hand side consists of a family of functions $f_i\colon T_i \to \mathbb{Z}$ that are all zero except at finitely many elements, and such that $f_i = 0$ except for finitely many indices i.]

Exercise 3.4 (Direct sums commute with quotients in Ab)**.** Show that given abelian groups A_i with subgroups $B_i \subseteq A_i$ for $i \in I$, there is a canonical isomorphism

$$\bigoplus_{i\in I} A_i/B_i \cong \left(\bigoplus_{i\in I} A_i\right) / \left(\bigoplus_{i\in I} B_i\right).$$

[Hint: Show that homomorphism $\bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} A_i/B_i$ (defined as the sum of the quotient maps) exhibits the target as the quotient by $\bigoplus_{i\in I} B_i$, by checking it satisfies the universal property of the quotient.]

Proposition 3.8.2. *For topological spaces X_i $(i \in I)$ there is a natural isomorphism*

$$H_*\left(\coprod_{i\in I} X_i\right) \cong \bigoplus_{i\in I} H_*(X_i).$$

Proof. Since Δ^n is connected, any n-simplex $\sigma\colon \Delta^n \to \coprod_i X_i$ must lie in one of the X_i's. Thus we have $\mathrm{Sing}_n(\coprod_i X_i) \cong \coprod_i \mathrm{Sing}_n(X_i)$.

We can write $\Delta^n = \coprod_i \sigma^{-1}(X_i)$ where the preimages $\sigma^{-1}(X_i)$ are disjoint open subsets, hence all but one must be empty.

By Exercise 3.3 this implies that we have a natural isomorphism

$$S_n\left(\coprod_i X_i\right) \cong \bigoplus_i S_n(X_i).$$

Moreover, the boundary map ∂ for $\coprod_i X_i$ decomposes as the sum of the boundary maps for each X_i, so that

$$Z_n\left(\coprod_i X_i\right) \cong \bigoplus_i Z_n(X_i), \qquad B_n\left(\coprod_i X_i\right) \cong \bigoplus_i B_n(X_i),$$

and so

$$H_n\left(\coprod_i X_i\right) \cong \bigoplus_i Z_n(X_i)/\bigoplus_i B_n(X_i) \cong \bigoplus_i Z_n(X_i)/B_n(X_i) \cong \bigoplus_i H_n(X_i),$$

where the second isomorphism is that of Exercise 3.4. □

Example 3.8.3. Let S be a set, equipped with the discrete topology. We can identify this topological space with $\coprod_{s\in S} *$. Since we already computed $H_*(*)$ we can conclude

$$H_*(S) \cong \bigoplus_{s\in S} H_*(*) \cong \begin{cases} 0, & * \neq 0, \\ \mathbb{Z}S, & * = 0. \end{cases}$$

4
Relative Homology and Long Exact Sequences

In this chapter we first introduce *relative* homology for a pair (X, A) (where A is a subspace of X) in §4.1 and *exact sequences* in §4.2, where we also prove that there is a long exact sequence that relates the relative homology of (X, A) to the homologies of the spaces X and A. We generalize this construction in §4.3 where we see that any short exact sequence of chain complexes give a long exact sequence in homology.

The long exact sequence for relative homology is one of the key properties of homology, known as the *Eilenberg–Steenrod axioms*, which we state in §4.4 (we will prove them later). Perhaps the most important of these axioms is the *excision* axiom, which we reformulate in §4.5 as an isomorphism between the relative homology of (X, A) and the reduced homology of the quotient X/A, under some technical assumptions on the pair (X, A).

Using the Eilenberg–Steenrod axioms we can compute the homology of the n-sphere S^n in §4.6. In §4.7 we then look at some topological applications of the homology groups of spheres.

In §4.8 we construct another long exact sequence, the *Mayer–Vietoris sequence*, which allows us to compute homology by decomposing a space into two parts whose homology we know. Finally, in §4.9 we use our computation of the homology of S^n to extract from every continuous map $f: S^n \to S^n$ an integer, the *degree* of f, and discuss how to compute this.

4.1 Relative Homology

Definition 4.1.1. Suppose (X, A) is a *subspace pair*, i.e. a pair consisting of a topological space X and a subspace $A \subseteq X$. Then $\mathrm{Sing}_n(A)$ is a subset of $\mathrm{Sing}_n(X)$, which makes $S_n(A)$ a subgroup of $S_n(X)$. We then define the group $S_n(X, A)$ of *singular n-chains on X relative to A* as the quotient

$$S_n(X, A) := S_n(X)/S_n(A).$$

Since the boundary map $\partial: S_n(X) \to S_{n-1}(X)$ takes the subgroup $S_n(A)$ into $S_{n-1}(A)$, it induces a homomorphism on quotients

$$\bar{\partial}: S_n(X, A) \to S_{n-1}(X, A),$$

which again satisfies $\overline{\partial}^2 = 0$. This means that $(S_\bullet(X,A), \overline{\partial})$ is a chain complex, so that we have homology groups

$$H_*(X,A) := H_*(S_\bullet(X,A)),$$

the (singular) homology of X *relative to* A, or *relative homology* of (X,A).

Let us unpack the definitions and make it a bit more explicit what is going on here:

- First of all, an element of $S_n(X,A)$ is an equivalence class $[x]$ of chains $x \in S_n(X)$, where $[x] = [x']$ if $x - x'$ lies in the subgroup $S_n(A)$, i.e. is a linear combination of simplices whose images lie in the subspace $A \subseteq X$. (Informally, $[x] = [x']$ means that the chains x and x' agree outside of A.)
- Such a class $[x]$ is a *cycle* if $\overline{\partial}[x] = [\partial x] = 0$ — this means that the boundary of x only contains $(n-1)$-simplices in A, i.e. the boundary of x lies in the subspace A.
- On the other hand, $[x]$ is a *boundary* if $[x] = \overline{\partial}[y] = [\partial y]$, which means that $x - \partial y$ lies in $S_n(A)$. In other words, we can write x as a boundary plus a chain in A.
- A relative homology class in $H_n(X,A)$ is then represented by a relative cycle $[x]$, which in turn is represented by a chain x whose boundary lies in A.

Remark 4.1.2. The relative homology $H_*(X,A)$ is in some sense the part of the homology of X where we ignore A. We'll make this more precise later, but the basic idea is that we can (sometimes) compute the homology of X by computing $H_*(A)$ and $H_*(X,A)$ separately, and then putting them back together.

Construction 4.1.3. Any n-cycle on X is in particular a relative cycle, and any boundary is a relative boundary, so we have a canonical homomorphism $H_n(X) \to H_n(X,A)$. We can also make a homomorphism $\partial\colon H_n(X,A) \to H_{n-1}(A)$, called the boundary map, as follows:

- Suppose the homology class $\gamma H_n(X,A)$ is represented by a relative cycle $[x] \in Z_n(X,A)$. Then ∂x is a chain in $S_{n-1}(A)$. Since $\partial^2 = 0$ we see that ∂x is a cycle, but it is not necessarily a boundary in $S_\bullet(A)$ (it *is* a boundary in $S_\bullet(X)$, but it need not be the boundary of any chain on A). Thus ∂x represents a homology class $[\partial x] \in H_{n-1}(A)$.
- We claim that this homology class is independent of the choice of representative $[x]$ and x. First of all, if $[x] = [x']$ then $x - x'$ is a chain in A; then we have $\partial x' = \partial x + \partial(x - x')$ so that $[\partial x'] = [\partial x]$ in $H_{n-1}(A)$ since $\partial(x - x')$ is a boundary *in* A.
- Second, if γ is also represented by $[y]$, then $[x] = [y] + \overline{\partial}[z] = [y + \partial z]$. By the previous point, this means $[\partial x] = [\partial y + \partial^2 z] = [\partial y]$, as required.

This shows that we have defined a function $\partial\colon H_n(X, A) \to H_{n-1}(A)$; to check that this is a homomorphism it is enough to observe that we can represent a sum in $H_n(X, A)$ by a sum of representatives.

The three maps $H_n(A) \to H_n(X)$ (coming from the inclusion $A \hookrightarrow X$), $H_n(X) \to H_n(X, A)$, and $\partial\colon H_n(X, A) \to H_{n-1}(A)$ fit together in a precise way: they form a *long exact sequence*.

4.2 Exact Sequences

Definition 4.2.1. A (bounded or unbounded) sequence of abelian groups and homomorphisms

$$\cdots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \cdots$$

is *exact* at A_n if $\ker f_n = \operatorname{im} f_{n+1}$. The sequence is *exact* if it is exact at A_n for all n (excluding the end points, if there are any, where the definition doesn't make sense). A *long exact sequence* is an exact sequence that's unbounded in both directions.

Remark 4.2.2. If

$$\cdots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \cdots$$

is a long exact sequence, then in particular $\ker f_n = \operatorname{im} f_{n+1}$ so that $f_n \circ f_{n+1} = 0$. Thus $(A_\bullet, f_\bullet)$ is a chain complex. Conversely, a chain complex $A_\bullet$ is an exact sequence if and only if $H_n(A) = 0$ for all n.

Examples 4.2.3. We can express several common algebraic notions in terms of exactness:

(i) $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective,

(ii) $A \xrightarrow{f} B \to 0$ is exact if and only if f is surjective,

(iii) $0 \to A \xrightarrow{f} B \to 0$ is exact if and only if f is both injective and surjective, i.e. an isomorphism.

Definition 4.2.4. A *short exact sequence* is an exact sequence of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0.$$

The exactness conditions amount to saying

- i is injective,
- q is surjective,
- $\operatorname{im} i = kerq$.

Thus the short exact sequence exhibits C as the quotient B/A.

Here is a key property of exact sequences.

Lemma 4.2.5 (The 5-Lemma)**.** *Suppose we have a commutative diagram of abelian groups and homomorphisms*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

where the rows are exact and $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are all isomorphisms. Then α_3 is also an isomorphism.

As with many proofs in homological algebra, this is far more instructive to work out by oneself:

Exercise 4.1. Prove the 5-Lemma.

Exercise 4.2. Suppose we have an exact sequence

$$(\cdots)A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E(\cdots).$$

Show that there is a short exact sequence

$$0 \to \operatorname{coker} f \to C \to \ker i \to 0,$$

where the *cokernel* $\operatorname{coker} f$ is the quotient $B/\operatorname{im} f$. (Thus we can in a sense "decompose" a long exact sequence into a series of short exact sequences.)

Exercise 4.3. Let $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$ be a short exact sequence (SES). A *splitting* of the SES is a section $s\colon C \to B$, so that $qs = \mathrm{id}_C$. (The SES is *splittable* if a splitting exists, while a *split* SES is a SES together with a choice of splitting.)

(i) Show that a splitting s induces an isomorphism $A \oplus C \xrightarrow{\sim} B$. [Note that different splittings can give different isomorphisms.]

(ii) Show that if C is a free abelian group then the SES above is splittable. [Hint: Use the universal property of free abelian groups.]

(iii) Give an example of a SES that is not splittable.

Let's return to the case of relative homology:

Proposition 4.2.6. *Let (X, A) be a subspace pair. Then the sequence of homomorphisms*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

is a long exact sequence. Here i_* is induced by the inclusion $i\colon A \hookrightarrow X$, q_* arises from the quotient map $S_\bullet(X) \to S_\bullet(X, A)$, and ∂ is the boundary map defined above.

Proof. We must show that the composite of any pair of adjacent maps is zero, which we can interpret as im $\subseteq$ ker at every point in the sequence, and that the sequence is exact, which is equivalent to the additional condition ker $\subseteq$ im at every point. Since there are three kinds of groups in the sequence, this means we have 6 things to check:

- $\operatorname{im} i_* \subseteq \ker q_*$: Given a homology class $[\alpha] \in H_n(A)$ represented by $\alpha \in S_n(A)$, the image $i_*[\alpha]$ is represented by $i_*\alpha \in S_n(X)$; this chain lies entirely in A, so it goes to 0 in $S_n(X, A)$. In particular, $q_*i_*[\alpha] = 0$.

- $\operatorname{im} q_* \subseteq \ker \partial$: Suppose $[\gamma] \in H_n(X)$ is represented by a cycle $\gamma \in Z_n(X)$. Then $q_*[\gamma]$ is represented by the image of γ in the quotient $S_n(X, A)$, and $\partial q_*[\gamma]$ is represented by $\partial\gamma$ — but γ is a cycle, so this is indeed 0.

- $\operatorname{im} \partial \subseteq \ker i_*$: Suppose $[[\beta]] \in H_n(X, A)$ is represented by $\beta \in S_n(X)$ with $\partial\beta \in S_{n-1}(A)$. Then $\partial[[\beta]]$ is represented by $\partial\beta$. The image of this in $S_{n-1}(X)$ is a boundary, and so represents 0 in $H_{n-1}(X)$.

- $\ker q_* \subseteq \operatorname{im} i_*$: Suppose $q_*[\gamma] = 0$, where $[\gamma] \in H_n(X)$ is represented by a cycle $\gamma \in S_n(X)$. Then the image of γ in $S_n(X, A)$ is a boundary, meaning there exists $\beta \in S_{n+1}(X)$ such that $\alpha := \gamma - \partial\beta$ lies in $S_n(A)$. But then $[\gamma] = i_*[\alpha]$ in $H_n(X)$, as required.

- $\ker \partial \subseteq imq_*$: Suppose $[[\beta]] \in H_n(X, A)$ is represented by $\beta \in S_n(X)$. If $\partial[[\beta]] = 0$ in $H_{n-1}(A)$, then $\partial\beta$ is a boundary in $S_{n-1}(A)$, so there exists $\alpha \in S_n(A)$ such that $\partial\beta = \partial\alpha$. Then $\beta - \alpha$ is a cycle in $S_n(X)$, and $[[\beta]] = q_*[\beta - \alpha]$.

- $\ker i_* \subseteq \operatorname{im} \partial$: Suppose $[\alpha] \in H_n(A)$ is represented by $\alpha \in S_n(A)$ and $i_*[\alpha] = 0$. That means α is a boundary in $S_n(X)$, i.e. there exists $\gamma \in S_{n+1}(X)$ with $\partial\gamma = \alpha$. In particular, the boundary of γ lies in A, so γ represents a class $[[\gamma]] \in H_{n+1}(X, A)$. By definition we then have $[\alpha] = \partial[[\gamma]]$, as required. □

Remark 4.2.7. We will soon see that there is an alternative description of relative homology, which will make it possible to carry out computations using this long exact sequence.

Example 4.2.8 (Reduced homology)**.** Let's look at a very simple example of relative homology: Let (X, x) be a pointed space (i.e. x is a point of X). If we don't want to explicitly denote the point x, we often write

$$\tilde{H}_*(X) := H_*(X, x),$$

and call this the *reduced homology* of X. We have a long exact sequence

$$\cdots \to H_n(*) \to H_n(X) \to \tilde{H}_n(X) \to H_{n-1}(*) \to \cdots .$$

If $n > 1$ then both $H_n(*)$ and $H_{n-1}(*)$ are 0, so $H_n(X) \xrightarrow{\sim} \tilde{H}_n(X)$. That leaves us with the end of the long exact sequence,

$$0 \to H_1(X) \to \tilde{H}_1(X) \to H_0(*) \to H_0(X) \to \tilde{H}_0(X) \to 0.$$

We computed H_0, so we know $H_0(*) \cong \mathbb{Z}$, $H_0(X) \cong \mathbb{Z}\pi_0 X$, and the map $H_0(*) \to H_0(X)$ takes $1 \in \mathbb{Z}$ to the generator $[x] \in \pi_0 X$. In particular this map is injective, so $\tilde{H}_1(X) \to H_0(*)$ must be the zero map, and so $H_1(X) \to \tilde{H}_1(X)$ is surjective; thus $H_1(X) \cong \tilde{H}_1(X)$. Finally, we see that $\tilde{H}_0(X)$ is the quotient

$$H_0(X)/\mathbb{Z}[x] \cong \mathbb{Z}(\pi_0(X) \setminus \{[x]\}).$$

In particular, if X is path-connected then

$$\tilde{H}_*(X) \cong \begin{cases} H_*(X), & * > 0, \\ 0, & * = 0. \end{cases}$$

Example 4.2.9 ($H_*(I, \partial I)$)**.** Let's look at the subspace pair $(I, \partial I)$ where $I = [0, 1]$ is the closed unit interval and $\partial I = \{0, 1\}$ is its boundary, i.e. the two end points. The space I is contractible, i.e. homotopy equivalent to a point. We will see later that homology takes homotopy equivalences to isomorphisms; assuming this for now, we have an isomorphism

$$H_n(I) \cong H_n(*) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

On the other hand, ∂I is the disjoint union of two points (with the discrete topology), so we have

$$H_n(\partial I) \cong H_n(\{0\}) \oplus H_n(\{1\}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

We can then use the long exact sequence of the pair $(I, \partial I)$ to find the relative homology groups $H_n(I, \partial I)$ — we will later see that these are isomorphic to $\tilde{H}_n(I/\partial I)$ so this computes the reduced homology of the circle $S^1 \cong I/\partial I$.

For $n > 1$ the part of the long exact sequence surrounding $H_n(I, \partial I)$ looks like

$$\cdots \to H_n(I) \to H_n(I, \partial I) \to H_{n-1}(\partial I) \to \cdots,$$

where $H_n(I) = H_{n-1}(\partial I) = 0$. This means we also have $H_n(I, \partial I) = 0$: by exactness the image of $H_n(I)$ is the kernel of the map to $H_{n-1}(\partial I)$ — but $H_n(I) = 0$ so its image is 0, and since $H_{n-1}(\partial I) = 0$ the kernel is all of $H_n(I, \partial I)$, which means

$$H_n(I, \partial I) = 0, \quad n > 1.$$

To understand the remaining cases $n = 0, 1$ we look at the "tail" of the long exact sequence:

$$\cdots \to H_1(I) \to H_1(I, \partial I) \to H_0(\partial I) \xrightarrow{\phi} H_0(I) \to H_0(I, \partial I) \to 0,$$

which becomes

$$0 \to H_1(I, \partial I) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to H_0(I, \partial I) \to 0$$

after plugging in the homology groups we already know. By exactness we see that $H_1(I, \partial I)$ is the kernel of the homomorphism α and $H_0(I, \partial I)$ is the cokernel of α. But here α is the homomorphism $i_* \colon H_0(\partial I) \to H_0(I)$ induced by the inclusion $i \colon \partial I \hookrightarrow I$, which we have identified with the homomorphism $\mathbb{Z}\pi_0(\partial I) \to \mathbb{Z}\pi_0(I)$ induced by i on path components. Both components of ∂I are sent to the unique component of I, so identifying $\mathbb{Z}\pi_0(\partial I)$ with $\mathbb{Z} \oplus \mathbb{Z}$ by identifying the generators with $(1, 0)$ and $(0, 1)$, we see that α is given by $\alpha(a, b) = a + b$. Hence $\ker \alpha$ is the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $(1, -1)$, which is isomorphic to $\mathbb{Z}$, while $\operatorname{coker} \alpha = 0$ since α is surjective. In summary, we have shown

$$H_n(I, \partial I) \cong \begin{cases} \mathbb{Z}, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

4.3 Short Exact Sequences of Chain Complexes

We can generalize the construction of the long exact sequence for relative homology to obtain a long exact sequence from any *short* exact sequence of chain complexes. This is an important tool in homological algebra that we will use several variations of later on in the course, so we take the time to introduce this here.

Definition 4.3.1. Suppose $i_\bullet \colon A_\bullet \to B_\bullet$ is a chain map such that each homomorphism i_n is injective. Then we can define a *quotient* chain complex $C_\bullet := B_\bullet/A_\bullet$ with $C_n := B_n/A_n$: since we have commutative squares

$$\begin{array}{ccc} A_n & \xrightarrow{i_n} & B_n \\ \big\downarrow{\scriptstyle\partial} & & \big\downarrow{\scriptstyle\partial} \\ A_{n-1} & \xrightarrow{i_{n-1}} & B_{n-1} \end{array}$$

the boundary map $B_n \to B_{n-1}$ induces a homomorphism $\overline{\partial} \colon C_n \to C_{n-1}$ which again satisfies $\overline{\partial}^2 = 0$. Note that the quotient maps $q_n \colon B_n \to B_n/A_n$ give a chain map $q_\bullet \colon B_\bullet \to C_\bullet$.

Example 4.3.2. If (X, A) is a subspace pair, then the relative singular chain complex $S_\bullet(X, A)$ is precisely the quotient chain complex $S_\bullet(X)/S_\bullet(A)$.

Definition 4.3.3. A *short exact sequence* of chain complexes consists of chain maps

$$0 \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{q_\bullet} C_\bullet \to 0$$

(where we write 0 for the chain complex with 0 in each degree) such that

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{q_n} C_n \to 0$$

is a short exact sequence for every n.

Construction 4.3.4. Continuing the notation of Definition 4.3.1, we have for every n homomorphisms of homology groups

$$H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(q)} H_n(C).$$

Just as in Construction 4.1.3 we can use the boundary map for B to define another homomorphism $H_n(C) \xrightarrow{\partial} H_{n-1}(A)$:

- Suppose $[c]$ is an element of $H_n(C)$; this is represented by an element $c \in C_n = B_n/A_n$ such that $\overline{\partial} c = 0$ in C_{n-1}.
- Since q_n is surjective, we can choose $b \in B_n$ such that $c = q_n(b)$. Then $\overline{\partial} c = q_{n-1}(\partial b)$ and $\overline{\partial} c = 0$ means that ∂b lies in $\ker q_{n-1} = \operatorname{im} i_{n-1}$ and so $\partial b = i_{n-1}(a)$ for a unique element $a \in A_{n-1}$.
- Then $i_{n-2}(\partial a) = \partial i_{n-1}(a) = \partial^2 b = 0$, so as i_{n-2} is injective we have $\partial a = 0$. Thus a is a cycle and represents a homology class $[a] \in H_{n-1}(A)$. (Note that a is not necessarily a boundary, even though its image in B_{n-1} is a boundary.)

- We claim the class $[a]$ is independent of the choices we made. First suppose b' is another element such that $q_n(b') = c$, with $\partial b' = i_{n-1}(a')$; then $b' - b$ is in $\ker q_n = \operatorname{im} i_n$ so there exists a (unique) element $x \in A_n$ with $b' - b = i_n(x)$. Thus
$$i_{n-1}(a' - a) = \partial b' - \partial b = \partial i_n(x) = i_{n-1}(\partial x),$$
and so $a' - a = \partial x$, i.e. a' and a differ by a boundary, hence $[a'] = [a]$.

- Next suppose c' is another element of C_n that represents $[c]$, so $c' = c + \partial y$ with $y \in C_{n+1}$. If y is the image of $z \in B_{n+1}$ then one lift of c' is $b + \partial z$, with $\partial(b + \partial z) = \partial b$, giving the same element of A_{n-1}.

We thus have a well-defined map of sets $\partial\colon H_n(C) \to H_{n-1}(A)$, and it is now easy to see that this is a homomorphism: for $[c_1], [c_2]$ in $H_n(C)$, with preimages b_1, b_2 in B_n, the homology class $\partial([c_1] + [c_2]) = \partial([c_1 + c_2])$ is the class represented by the preimage of $\partial(b_1 + b_2) = \partial b_1 + \partial b_2$. In addition, we have

- $\partial \circ H_n(q_n) = 0$, since for $[c] = H_n(q_n)[b]$ we can choose the preimage to be b with $\partial b = 0$,

- $i_{n-1} \circ \partial = 0$ since $\partial[c]$ is represented by the preimage of a boundary in B_{n-1}.

Exercise 4.4. Given a commutative diagram of chain complexes and chain maps
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \longrightarrow & B_\bullet & \longrightarrow & C_\bullet & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & A'_\bullet & \longrightarrow & B'_\bullet & \longrightarrow & C'_\bullet & \longrightarrow & 0, \end{array}$$
where the rows are exact, check that the boundary map on homology gives commutative squares
$$\begin{array}{ccc} H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) \\ \downarrow & & \downarrow \\ H_n(C') & \xrightarrow{\partial} & H_{n-1}(A'). \end{array}$$

Exercise 4.5. Suppose $A \subseteq B$ are subspaces of a topological space X, and the inclusion $i\colon A \hookrightarrow B$ induces isomorphisms $i_*\colon H_n(A) \xrightarrow{\sim} H_n(B)$ for all n. Prove that the natural homomorphism $H_n(X, A) \to H_n(X, B)$ is an isomorphism for all n. [Hint: Use the 5-Lemma and Exercise 4.4.]

Proposition 4.3.5. *From a short exact sequence of chain complexes as above we get a sequence of homology groups and homomorphisms*
$$\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(q)} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots.$$
This is a long exact sequence.

In other words, a short exact sequence of chain complexes gives a long exact sequence in homology. We leave the proof as an exercise, since it is really the same as that we did above in Proposition 4.2.6 for relative homology.

Exercise 4.6. Let $0 \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{q_\bullet} C_\bullet \to 0$ be a short exact sequence of chain complexes. Show that the induced sequence of homology groups

$$\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(q)} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

is a long exact sequence.

4.4 *The Eilenberg–Steenrod Axioms*

We can now articulate the key properties of homology, in the form of the axioms given by Eilenberg and Steenrod. We will eventually prove that these are satisfied for our definition of homology, but for the next part of the course we will instead assume they hold and use them to compute some examples of homology groups.

Definition 4.4.1. Let Pair be the category whose objects are pairs (X, A) where X is a topological space and $A \subseteq X$ is a subspace, with a morphism $f\colon (X, A) \to (Y, B)$ given by a continuous map $f\colon X \to Y$ such that $f(A) \subseteq B$. If $f, g\colon (X, A) \to (Y, B)$ are morphisms in Pair, a *homotopy* from f to g is a map $H\colon (X \times [0,1], A \times [0,1]) \to (Y, B)$ in Pair such that $H(–, 0) = f$, $H(–, 1) = g$. (In other words, H is a homotopy from f to g such that $H(a, t) \in B$ for all $a \in A, t \in [0,1]$.) We say two maps of pairs $f, g\colon (X, A) \to (Y, B)$ are *homotopic* if there exists a homotopy between them in this sense.

Notation 4.4.2. If $A \subseteq X$ is a subset of a topological space, we write A° for the *interior* of A, the largest open subset of X contained in A, and $\overline{A}$ for the *closure* of A, the smallest closed subset of X that contains A.

Definition 4.4.3 (Eilenberg–Steenrod)**.** An (ordinary) *homology theory* consists of

- functors $h_n\colon \mathsf{Pair} \to \mathsf{Ab}$, $n \in \mathbb{Z}$ (we abbreviate $h_n(X) := h_n(X, \emptyset)$),
- natural boundary maps $\partial\colon h_n(X, A) \to h_{n-1}(A)$, so that the squares

$$\begin{array}{ccc} h_n(X,A) & \xrightarrow{\partial} & h_{n-1}(A) \\ \downarrow{\scriptstyle h_n(f)} & & \downarrow{\scriptstyle h_n(f|_A)} \\ h_n(Y,B) & \xrightarrow{\partial} & h_{n-1}(B) \end{array}$$

  commute for every map $f\colon (X, A) \to (Y, B)$,

with the following properties:

(1) (Long exact sequences) For every pair (X, A), the sequence of maps

$$\cdots \to h_n(A) \to h_n(X) \to h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \to \cdots$$

induced by the maps of pairs $(A, \emptyset) \to (X, \emptyset) \to (X, A)$, is a long exact sequence.

(2) (Homotopy axiom) If $f, g\colon (X, A) \to (Y, B)$ are homotopic, then $h_n(f) = h_n(g)$ for all $n \in \mathbb{Z}$.

(3) (Excision axiom) For $(X, A) \in \mathsf{Pair}$, if $U \subseteq A$ is a subset such that $\overline{U} \subseteq A^\circ$, then the homomorphisms

$$h_n(X \setminus U, A \setminus U) \to h_n(X, A)$$

induced by the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$, are isomorphisms for all $n \in \mathbb{Z}$.

(4) (Additivity axiom) If $X = \coprod_{i \in I} X_i$ is a disjoint union, then the inclusions $X_i \hookrightarrow X$ induce an isomorphism

$$\bigoplus_{i \in I} h_n(X_i) \xrightarrow{\sim} h_n(X)$$

for all n.

(5) (Dimension axiom) $h_n(*) = 0$ if $n \neq 0$.

If we drop the dimension axiom, there are many interesting examples of homology theories, such as K-theory and cobordism theory, which are no longer determined by their value at the point. These are sometimes called *extraordinary* homology theories, and are the starting point for the subject of *stable homotopy theory*.

Remark 4.4.4. Eilenberg and Steenrod showed in 1945 that (if we restrict to a class of "reasonable" spaces) there is for every abelian group A a *unique* ordinary homology theory in this sense such that $h_0(*) \cong A$. We will prove later that singular homology satisfies the axioms and so is indeed a homology theory in this sense. (We have already proved all but two of the axioms, namely the homotopy and excision axioms, but these are also the less formal ones!) Before we do so, we will explore some consequences of the axioms, and in particular the excision axiom.

We will also define a variant of singular homology with coefficients in an abelian group A, which gives the ordinary homology theory with $h_0(*) \cong A$.

Remark 4.4.5. Note that the homotopy axiom implies that a homotopy equivalence induces isomorphisms in homology: If $f\colon X \to Y$ is a continous map with homotopy inverse $g\colon Y \to X$, so that there are homotopies between gf and id_X and between fg and id_Y, then for the induced homomorphisms

$$f_*\colon H_*(X) \to H_*(Y), \quad g_*\colon H_*(Y) \to H_*(X)$$

the homotopy axiom together with functoriality implies

$$g_*f_* = (gf)_* = (\mathrm{id}_X)_* = \mathrm{id}_{H_*X}, \quad f_*g_* = (fg)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{H_*Y},$$

so that f_* is an isomorphism with inverse g_*.

Exercise 4.7. Suppose h_* is an ordinary homology theory satisfying the Eilenberg–Steenrod axioms.

(i) Suppose $X = \coprod_{i \in I} X_i$ is a coproduct and $A_i \subseteq X_i$ is a collection of subspaces. If $A := \coprod_{i \in I} A_i$, show that the inclusions $(X_i, A_i) \hookrightarrow (X, A)$ induce an isomorphism

$$\bigoplus_{i \in I} h_*(X_i, A_i) \cong h_*(X, A).$$

[Hint: Use the long exact sequence.]

(ii) If (X_i, x_i) is a collection of pointed spaces, their *wedge* is the quotient space

$$\bigvee_{i \in I} X_i := \left(\coprod_{i \in I} X_i \right) / \{x_i : i \in I\}$$

where we identify all the base points to a single point x. Show that if $(X_i, \{x_i\})$ is a good pair for every i then there is a canonical isomorphism

$$\bigoplus_{i\in I} \tilde{h}_*(X_i) \cong \tilde{h}_*(\bigvee_{i\in I} X_i),$$

where for a pointed space (X, x) we write $\tilde{h}_*(X) := h_*(X, x)$.

4.5 From Excision to Quotients

The excision axiom is in a sense the crucial axiom that allows us to do some computations. Before we do so, it is convenient to give a reformulation of it in terms of quotients of spaces:

Definition 4.5.1. Suppose X is a topological space and $A \subseteq X$ is a subspace. The *quotient* X/A is the quotient space $X/\sim$ where $\sim$ is the relation generated by $x \sim x'$ if either $x = x'$ or x and x' are both in A. (Equivalently, this is the pushout

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/A \end{array}$$

in Top.)

Definition 4.5.2. A pair $(X, A) \in$ Pair is called a *good pair* if there exists a subspace $B \subseteq X$ such that $\overline{A} \subseteq B^\circ$ and $A \hookrightarrow B$ is a deformation retract, i.e. there exists a retraction $\rho: B \to A$ of the inclusion $i: A \hookrightarrow B$ and a homotopy $B \times I \to B$ from id to $i\rho$, which we furthermore require to satisfy $H(a, t) \in A$ for all $a \in A, t \in I$.

This is really a very mild condition, so arguably it would be more correct to say that pairs that *don't* satisfy this condition are bad or pathological.

Proposition 4.5.3. *Suppose (X, A) is a good pair. Then the map*

$$H_n(X, A) \xrightarrow{\sim} H_n(X/A, *)$$

induced by $(X, A) \to (X/A, *)$ is an isomorphism for all n.

Proof. Choose B as in Definition 4.5.2. We then have a commutative diagram of pairs

$$\begin{array}{ccccc} (X,A) & \xrightarrow{i} & (X,B) & \xleftarrow{j} & (X\setminus A, B\setminus A) \\ \downarrow & & \downarrow & & \downarrow k \\ (X/A,*) & \xrightarrow{\bar{\imath}} & (X/A, B/A) & \xleftarrow{\bar{\jmath}} & ((X/A)\setminus *, (B/A)\setminus *). \end{array}$$

We want to show that the left vertical map gives isomorphisms in homology. To see this we will check that all the other maps in the diagram give isomorphisms:

- k is a homeomorphism of pairs.
- j gives an isomorphism by the excision axiom (which we can apply to A by our assumption).
- The same holds for $\bar{\jmath}$ since by definition of the quotient topology we have $\overline{\{*\}} \subseteq (B/A)^\circ$.

- Looking at the right square we see that the middle vertical map gives isomorphisms in homology, since this is true for the three other maps in the square.
- The inclusion $A \hookrightarrow B$ gives an isomorphism $H_n(A) \to H_n(B)$ for all n, by the homotopy axiom, and so by Exercise 4.5 the map $H_n(X, A) \to H_n(X, B)$ induced by i is also an isomorphism.
- The same argument applies to $\bar{\imath}$: By assumption we have a retraction $B \xrightarrow{\rho} A \hookrightarrow B$ fixing A, which induces $B/A \xrightarrow{\bar{\rho}} * \to B/A$, and a homotopy $B \times I \to B$ between ρ and the identity that takes A to itself; this induces a homotopy $B/A \times I \to B/A$ between $\bar{\rho}$ and the identity, so $* \subseteq B/A$ is also a deformation retract.

We can then conclude the left vertical morphism gives isomorphisms in homology, since we know this for the three other maps in the left square. □

4.6 *Homology of Spheres*

Now we are finally in a position to compute the homology of the n-dimensional sphere S^n.

Proposition 4.6.1. *For* $n > 0$, $H_*(S^n) \cong \begin{cases} \mathbb{Z}, & * = 0, n, \\ 0, & \text{otherwise.} \end{cases}$

Proof. We consider the long exact sequence for the pair $(D^n, \partial D^n)$, which looks like

$$\cdots \to H_i(\partial D^n) \to H_i(D^n) \to H_i(D^n, \partial D^n) \to H_{i-1}(\partial D^n) \to \cdots .$$

Here $\partial D^n \cong S^{n-1}$, while D^n is contractible, so the homotopy axiom implies

$$H_i(D^n) \cong H_i(*) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, by Proposition 4.5.3 we have an isomorphism

$$H_i(D^n, \partial D^n) \cong H_i(D^n/\partial D^n, *) \cong \tilde{H}_i(S^n).$$

Thus we can express the long exact sequence as

$$\cdots \to H_i(*) \to \tilde{H}_i(S^n) \to H_{i-1}(S^{n-1}) \to H_{i-1}(*) \to \cdots .$$

If $i > 1$ then we get $\tilde{H}_i(S^n) \cong H_{i-1}(S^{n-1})$, while at the end of the exact sequence we have

$$0 \to \tilde{H}_1(S^n) \to H_0(S^{n-1}) \to H_0(*) \to \tilde{H}_0(S^n) \to 0.$$

We already considered the case $n = 1$ in Example 4.2.9, so we may assume $n > 1$. Then S^{n-1} is path-connected and $H_0(S^{n-1}) \to H_0(*)$ is an isomorphism, so we also have $\tilde{H}_1(S^n) = \tilde{H}_0(S^n) = 0$. From this the result follows by induction (using Example 4.2.8 to pass between reduced and unreduced homology). □

The computation of $H_*(S^n)$ will play an important role later on in the course, so it is worth looking at it a little more carefully, and explicitly identify a generator of the free abelian group $H_n(S^n)$:

Proposition 4.6.2. *The homology class $[q]$ represented by the quotient map $q\colon \Delta^n \to \Delta^n/\partial\Delta^n \cong S^n$ is a generator of $H_n(S^n) \cong \mathbb{Z}$.*

For $n = 1$, this also follows from the isomorphism $\pi_1 S^1 \cong H_1 S^1$, since q corresponds to the identity of S^1, a generator of $\pi_1 S^1$.

We have to go back to our computation of $H_*(S^n)$ and work out the generator inductively. Instead of the long exact sequence for $(D^n, \partial D^n)$ we could just as well have looked at the homeomorphic pair $(\Delta^n, \partial\Delta^n)$. Then we identify $\tilde{H}_n(S^n)$ through the isomorphism

$$q_*\colon H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\sim} \tilde{H}_n(S^n),$$

induced by the map of pairs $q\colon (\Delta^n, \partial\Delta^n) \to (\Delta^n/\partial\Delta^n, *)$. We first see what the class $[q]$ corresponds to under this isomorphism:

The identity map of Δ^n is a singular n-simplex of Δ^n; it is not a cycle in $S_\bullet(\Delta^n)$, but its boundary lies (tautologously) in $\partial\Delta^n$ and so id_{Δ^n} determines a class $I_n = [\mathrm{id}_{\Delta^n}] \in H_n(\Delta^n, \partial\Delta^n)$. Going back to the definition of the homomorphism q_* we see that $q_*[\sigma] = [q \circ \sigma]$ and so

$$q_* I_n = [q \circ \mathrm{id}_{\Delta^n}] = [q].$$

We can therefore reformulate the proposition as:

Proposition 4.6.3. *The homology class $I_n \in H_n(\Delta^n, \partial\Delta^n)$ is a generator.*

We will prove this by induction, using the following observation:

Lemma 4.6.4. *For $n > 1$ there is an isomorphism*

$$H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}),$$

under which I_n corresponds to I_{n-1}.

Proof. In the long exact sequence for the pair $(\Delta^n, \partial\Delta^n)$, we saw that the boundary map

$$\partial\colon H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\sim} H_{n-1}(\partial\Delta^n)$$

is an isomorphism.

From the definition of the boundary map we have that ∂I_n is represented by $\partial \mathrm{id}_{\Delta^n} = \sum_i (-1)^i d^i$ in $S_{n-1}(\partial\Delta^n)$ (where we think of the face map d^i as an inclusion $\Delta^{n-1} \hookrightarrow \partial\Delta^n$).

To complete the proof we want an identification of $H_{n-1}(\partial\Delta^n)$ with $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$ under which the class represented by $\sum_i (-1)^i d^i$ corresponds to I_{n-1}.

Let Λ^n_i denote the subspace of $\partial\Delta^n$ where we remove the interior of the ith face (so this is the union of all but one face in Δ^n). (This is known as the ith *horn* of Δ^n.) We have continuous maps of pairs

$$(\Delta^{n-1}, \partial\Delta^{n-1}) \xrightarrow{d^0} (\partial\Delta^n, \Lambda^n_0) \leftarrow (\partial\Delta^n, *),$$

where the first is the inclusion of the 0th face and the latter picks out some point in Λ^n_0. Both give isomorphisms in homology: the first by excision and the second by the homotopy axiom and Exercise 4.5 (since the point is a deformation retraction of Λ^n_0). We thus have isomorphisms

$$H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \xrightarrow{d^0_*} H_{n-1}(\partial\Delta^n, \Lambda^n_0) \xleftarrow{\phi} \tilde{H}_{n-1}(\partial\Delta^n).$$

Here $d^0_* I_{n-1} = [d^0 \circ \mathrm{id}_{\Delta^{n-1}}] = [d^0]$ (which is a relative cycle since its boundary lies in Λ^n_0). On the other hand, the alternating sum $\sum_i (-1)^i d^i$ in $S_{n-1}(\partial\Delta^n, *)$ is also taken to $[d_0]$ in $H_{n-1}(\partial\Delta^n, \Lambda^n_0)$ since the other terms lie in $S_{n-1}(\Lambda^n_0)$.

Thus under the composite isomorphism $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \cong \tilde{H}_{n-1}(\partial\Delta^n)$ the class I_{n-1} is identified with ∂I_n, as required. □

Proof of Proposition 4.6.3. The lemma implies that, for $n > 1$, the class I_n is a generator in $H_n(\Delta^n, \partial\Delta^n)$ if and only if I_{n-1} is generator in $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. By induction, this means it's enough to show that I_1 is a generator in $H_1(\Delta^1, \partial\Delta^1)$. Here we saw that the boundary map $H_1(\Delta^1, \partial\Delta^1) \to H_0(\partial\Delta^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ corresponded to the injective map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ taking 1 to $(1, -1)$. But $(1, -1) = (1, 0) - (0, 1)$ is (up to ordering the two points in $\partial\Delta^1$, and so up to a sign) precisely ∂I_1, which is represented by $d^0 - d^1$. Hence I_1 is a generator as required. □

4.7 *Topological Applications*

Let's discuss some topological applications of this computation. First of all, we have:

Corollary 4.7.1.

- *The sphere S^n is not contractible for any n.*
- *The spheres S^n and S^m are not homotopy equivalent if $n \neq m$.*

Proof. This is just because we know $H_*(S^n)$ is not isomorphic to the homology of a point, or to $H_*(S^m)$ when $m \neq n$. □

Corollary 4.7.2 (Invariance of dimension)**.** *The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$.*

Proof. Suppose $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism. Then it restricts to a homeomorphism $\mathbb{R}^n \setminus \{0\} \xrightarrow{\sim} \mathbb{R}^m \setminus \{\phi(0)\}$. But the inclusion $i\colon S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is a deformation retract (and so in particular a homotopy equivalence): we can define a retraction $\rho\colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ by $x \mapsto \frac{x}{|x|}$ with homotopy $H\colon \mathbb{R}^n \setminus \{0\} \times I \to \mathbb{R}^n \setminus \{0\}$ given by $H(x, t) = (1 - t)\rho(x) + tx$. Since homotopy equivalence is an equivalence relation we see that S^{n-1} is homotopy equivalent to S^{m-1}, contradicting the previous corollary. □

Corollary 4.7.3. *The inclusion $i\colon S^{n-1} \cong \partial D^n \hookrightarrow D^n$ does not admit a retraction.*

Proof. Suppose we had a retraction $\rho\colon D^n \to S^{n-1}$, so that $\rho i = \mathrm{id}_{S^{n-1}}$. Then since homology is functorial we would have $H_*(\rho) \circ H_*(i) = \mathrm{id}_{H_*(S^{n-1})}$. But then in degree $n-1$ we'd have composites

$$\mathbb{Z} \xrightarrow{H_{n-1}(\rho)} 0 \xrightarrow{H_{n-1}(i)} \mathbb{Z}, \quad (n > 1)$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{H_0(\rho)} \mathbb{Z} \xrightarrow{H_0(i)} \mathbb{Z} \oplus \mathbb{Z}, \quad (n = 1),$$

both of which clearly can't be the identity. □

Remark 4.7.4. Note that here we used not just the homology *groups* but also the homomorphisms arising from continuous maps.

Corollary 4.7.5 (Brouwer's fixed point theorem)**.** *Any continuous map $D^n \xrightarrow{f} D^n$ must have a fixed point, i.e. there is some point $x \in D^n$ such that $f(x) = x$.*

Proof. Suppose f does not have a fixed point. Then we can define $\rho\colon D^n \to D^n$ to take x to the point where the ray from $f(x)$ to x intersects ∂D^n. This is continuous and if $x \in \partial D^n$ then by construction $\rho(x) = x$. Thus ρ is a retract of the inclusion $\partial D^n \hookrightarrow D^n$, contradicting our previous corollary. □

4.8 *The Mayer–Vietoris Sequence*

Now we'll derive a variant of the long exact sequence of a pair that is often more convenient for computations. We start with a purely algebraic version:

Proposition 4.8.1 (Algebraic Mayer–Vietoris sequence)**.** *Consider a commutative diagram of abelian groups*

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A''_{n+1} & \xrightarrow{\delta_{n+1}} & A'_n & \xrightarrow{i_n} & A_n & \xrightarrow{q_n} & A''_n & \xrightarrow{\delta_n} & A'_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \phi''_{n+1}} & & \downarrow{\scriptstyle \phi'_n} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi''_n} & & \downarrow{\scriptstyle \phi'_{n-1}} & & \\
\cdots & \longrightarrow & B''_{n+1} & \xrightarrow{\epsilon_{n+1}} & B'_n & \xrightarrow{j_n} & B_n & \xrightarrow{r_n} & B''_n & \xrightarrow{\epsilon_n} & B'_{n-1} & \longrightarrow & \cdots
\end{array}$$

such that the rows are long exact sequences and the homomorphisms ϕ''_n are isomorphisms for all n. Then there is another long exact sequence

$$\cdots \to A'_n \xrightarrow{(i_n, \phi'_n)} A_n \oplus B'_n \xrightarrow{\phi_n - j_n} B_n \xrightarrow{\Delta_n} A'_{n-1} \to \cdots,$$

where $\Delta_n = \delta_n \circ (\phi''_n)^{-1} \circ r_n$.

Proof. We first check composites of successive maps are 0 (or in other words that at each step the image is contained in the kernel):

- $(\phi_n - j_n) \circ (i_n, \phi'_n) = \phi_n i_n - j_n \phi'_n = 0$, since the original diagram commutes,
- $\Delta_n \circ (\phi_n - j_n) = \delta_n (\phi''_n)^{-1} r_n \phi_n - \delta_n (\phi''_n)^{-1} r_n j_n = \delta_n q_n - 0 = 0$, since $\delta_n q_n = 0$ and $r_n j_n = 0$,

- $(i_{n-1}, \phi'_{n-1}) \circ \Delta_n = (i_{n-1}\delta_n(\phi''_n)^{-1}r_n, \phi'_{n-1}\delta_n(\phi''_n)^{-1}r_n) = (0, \epsilon_n r_n) = 0$, since $i_{n-1}\delta_n = 0$ and $\epsilon_n r_n = 0$.

Now to check exactness we must show that at each step the kernel is contained in the image:

- Take $x \in B_n$ with $\Delta_n x = 0$. Then $(\phi''_n)^{-1}r_n x$ is in the kernel of δ_n so there exists $y \in A_n$ with $q_n(y) = (\phi''_n)^{-1}r_n x$, which we can rewrite as $r_n\phi_n(y) = r_n x$. Hence $x - \phi_n(y)$ is in the kernel of r_n, and so there exists $z \in B'_n$ with $j_n z = x - \phi_n(y)$. Thus $x = \phi_n(y) + j_n(z) = (\phi_n - j_n)(y, -z)$.
- Take $(a, b) \in A_n \oplus B'_n$ with $\phi_n(a) - j_n(b) = 0$. Then $\phi''_n q_n a = r_n\phi_n a = r_n j_n b = 0$, so $q_n a = 0$ since ϕ''_n is an isomorphism. This means there exists $x \in A'_n$ with $a = i_n x$. Then $j_n b = j_n\phi'_n x$ and so there exists $y \in B''_{n+1}$ with $\epsilon_{n+1}(y) = b - \phi'_n x$. Then $x' := x + \delta_{n+1}(\phi''_{n+1})^{-1}(y)$ satisfies $i_n x' = i_n x + 0 = a$ and $\phi'_n x' = \phi'_n x + \epsilon_{n+1} y = b$, as required.
- Take $a \in A'_{n-1}$ with $i_{n-1}a = 0$ and $\phi'_{n-1}a = 0$. Then there exists $x \in A''_n$ with $a = \delta_n x$. Moreover, $\epsilon_n\phi''_n x = \phi'_{n-1}a = 0$ so there exists $y \in B_n$ with $r_n y = \phi''_n x$. But then $\Delta_n y = \delta_n(\phi''_n)^{-1}r_n y = \delta_n x = a$, as required. □

Note that the choice of signs in this exact sequence is somewhat arbitrary, we could have made other compatible choices of signs.

Now we want to apply this algebraic construction to singular homology. Consider a topological space X with subspaces $A, B \subseteq X$ such that $A^\circ \cup B^\circ = X$. We have inclusions

$$\begin{array}{ccc} A \cap B & \overset{j}{\hookrightarrow} & A \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle i} \\ B & \overset{i'}{\hookrightarrow} & X, \end{array}$$

and so a morphism of pairs $\iota = (i, j') \colon (A, A \cap B) \to (X, B)$. Then ι induces isomorphisms in homology $H_*(A, A \cap B) \xrightarrow{\sim} H_*(X, B)$: we can apply excision to the subset $U = X \setminus A$ since our assumptions imply $\overline{U} = X \setminus A^\circ \subseteq B^\circ$. The long exact sequences for these two pairs then give a commutative diagram

For a continuous map $f \colon X \to Y$ we'll write f_* instead of $H_n(f)$ for the induced map $H_n(X) \to H_n(Y)$ when n is obvious from context, and similarly for relative homology.

$$\begin{array}{ccccccccccc} \cdots \to & H_{n+1}(A, A \cap B) & \xrightarrow{\partial} & H_n(A \cap B) & \xrightarrow{j_*} & H_n(A) & \to & H_n(A, A \cap B) & \xrightarrow{\partial} & H_{n-1}(A \cap B) & \to \cdots \\ & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle j'_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle j'_*} & \\ \cdots \to & H_{n+1}(X, B) & \xrightarrow{\partial} & H_n(B) & \xrightarrow{i'_*} & H_n(X) & \to & H_n(X, B) & \xrightarrow{\partial} & H_{n-1}(B) & \to \cdots, \end{array}$$

where the rows are exact and the maps ι_* are isomorphisms. We can then apply Proposition 4.8.1 to obtain a new long exact sequence:

Corollary 4.8.2 (Mayer–Vietoris sequence)**.** *Let X be a topological space with subspaces $A, B \subseteq X$ such that $A^\circ \cup B^\circ = X$. Then there is a long exact sequence (called the* Mayer–Vietoris sequence*)*

$$\cdots \to H_n(A \cap B) \xrightarrow{(j_*, j'_*)} H_n(A) \oplus H_n(B) \xrightarrow{i_* - i'_*} H_n(X) \xrightarrow{\Delta} H_{n-1}(A \cap B) \to \cdots,$$

where Δ is the composite

$$H_n(X) \to H_n(X, B) \xrightarrow{(\iota_*)^{-1}} H_n(A, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B).$$

Example 4.8.3. Let's use the Mayer–Vietoris sequence to compute the homology of the torus $T = S^1 \times S^1$. If we cut the torus in half we get two (bent) cylinders, and we can take the subsets A and B to be two slight thickenings of these, so that they overlap. Then $A \cap B$ consists of two disjoint cylinders, so we have

We'll use the notation $X \simeq Y$ for "X is homotopy equivalent to Y".

$$A \simeq S^1, \quad B \simeq S^1, \quad A \cap B \simeq S^1 \amalg S^1.$$

The non-zero part of the Mayer–Vietoris sequence is

$$0 \to H_2T \to H_1(S^1 \amalg S^1) \to H_1S^1 \oplus H_1S^1 \to H_1T \to H_0(S^1 \amalg S^1) \to H_0(S^1) \oplus H_0(S^1) \to H_0T \to 0,$$

which using our computation of H_*S^1 becomes

$$0 \to H_2T \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to H_1T \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z} \to H_0T \to 0.$$

We thus need to understand the two maps $\alpha, \beta \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, for which we need to be a little careful about orientations. We can think of the homotopy equivalences $A, B \simeq S^1$ as shrinking each cylinder to its central cross-section, and the inclusions of the two components of $A \cap B$ as the inclusions of the two boundary circles of the cylinder. We conclude that both maps $A \cap B \to A, B$ are homotopic to the map $S^1 \amalg S^1 \to S^1$ given by the identity on each copy of S^1. These induce a homomorphism $H_*(S^1) \oplus H_*(S^1) \to H_*(S^1)$ that restricts to the identity on each copy of $H_*(S^1)$, which must be given by $(x, y) \mapsto x + y$. Thus α and β are both given by $(x, y) \mapsto (x + y, x + y)$, with

$$\ker \alpha = \mathbb{Z}(1, -1), \operatorname{coker} \alpha = (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z}(1, 1) \cong \mathbb{Z}.$$

We have $H_2T \cong \ker \alpha \cong \mathbb{Z}$, $H_0T = \operatorname{coker} \beta \cong \mathbb{Z}$, and a short exact sequence

$$0 \to \operatorname{coker} \alpha \xrightarrow{i} H_1T \xrightarrow{q} \ker \beta \to 0.$$

Since $\ker \beta \cong \mathbb{Z}$ is free, we can choose a splitting by Exercise 4.3, i.e. a section of q, which gives an isomorphism

$$H_1T \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Example 4.8.4. Now let's consider the Klein bottle K, which can also be built by gluing two ends of a cylinder together, but with a twist. We can take A to be an untwisted segment of the cylinder, and B a neighbourhood of the twist that overlaps A on both sides (then B is still a cylinder, it's just been "folded over" itself). The situation is very similar to what we had for the torus: $A \cap B$ is again two disjoint cylinders, and we have

$$A \cap B \simeq S^1 \amalg S^1, \quad A \simeq S^1, \quad B \simeq S^1.$$

The inclusion $A \cap B \hookrightarrow A$ also still correponds to the map $S^1 \amalg S^1 \to S^1$ that's the identity on each copy of S^1. The difference is that $A \cap B \hookrightarrow B$ correponds to the map $S^1 \amalg S^1 \to S^1$ that's the identity on one component and an orientation-reversing diffeomorphism on the other; we'll see in the next section that this gives multiplication by -1 on $H_1 S^1$. The Mayer–Vietoris sequence again looks like

$$0 \to H_2 K \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to H_1 K \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z} \to H_0 K \to 0,$$

but now α is given by $(x, y) \mapsto (x + y, x - y)$ while β is as before (since that only depends on what happens on connected components). Thus we now get $H_2 K = \ker \alpha = 0$ while $\operatorname{coker} \alpha \cong (\mathbb{Z} \oplus \mathbb{Z})/((1, 1), (1, -1)) \cong (\mathbb{Z} \oplus \mathbb{Z})/((1, 1), (2, 0)) \cong \mathbb{Z}/2$, so now the short exact sequence for $H_1 K$ looks like

$$0 \to \mathbb{Z}/2 \to H_1 K \to \mathbb{Z} \to 0.$$

We can again choose a splitting since $\mathbb{Z}$ is free, giving

$$H_1 K \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

Exercise 4.8. Use the Mayer–Vietoris sequence to compute the homology of the orientable surface Σ_g of genus g. [Hint: Find a way to induct on g.]

Exercise 4.9. Think of $\mathbb{RP}^2$ as the quotient of D^2 where we identify x with $-x$ for $x \in \partial D^2$. Compute the homology of $\mathbb{RP}^2$ using the Mayer–Vietoris sequence with $A =$ a neighbourhood of the image of ∂D^2 and $B =$ the image of a smaller disc inside D^2. [Assume that the map $S^1 \to S^1$ that loops around twice is given on $H_1(S^1)$ by multiplication by 2.]

Exercise 4.10. The *cone* on a topological space X is the quotient $(X \times [0, 1])/(X \times \{0\})$, and the *suspension* ΣX of X is the quotient of $(X \times [0, 1])$ where we collapse $X \times \{0\}$ to a point and $X \times \{1\}$ to a different point.

(i) Show that CX is contractible for any X, and that ΣX is the union of two copies of CX with intersection X.

(ii) Use the Mayer–Vietoris sequence to show that $H_n(\Sigma X) \cong H_{n-1}(X)$ for $n > 1$.

(iii) By looking at what happens at the bottom of the Mayer–Vietoris sequence, show that $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$ for all n.

(iv) If $X = S^n$, convince yourself that ΣS^n is homeomorphic to S^{n+1}. [Think of the two cones as two "hemispheres" glued along the "equator".] Use (iii) to compute $\tilde{H}_*(S^n)$ again.

4.9 *Degrees*

Under the isomorphism $\pi_1 S^1 \cong \mathbb{Z}$, the map $\ell_k \colon S^1 \to S^1$ that loops around the circle k times (with positive orientation) corresponds to $k \in \mathbb{Z}$; we say that such a loop has *winding number k*. In this section we will see that the homomorphism $\ell_{k,*} \colon H_1 S^1 \to H_1 S^1$ is also given by multiplication by k, and this gives a generalization of the winding number to continuous maps $f \colon S^n \to S^n$ for any $n > 0$, the *degree* of f:

Definition 4.9.1. We know that $H_n S^n \cong \mathbb{Z}$. For a continuous map $f \colon S^n \to S^n$ $(n > 0)$, the induced homomorphism $f_* \colon H_n(S^n) \to H_n(S^n)$ must therefore be given by multiplication with an integer $\deg(f)$, since this is true for all homomorphisms $\mathbb{Z} \to \mathbb{Z}$. This integer is called the *degree* of f.

Here are some easy properties of degrees:

- $\deg(\mathrm{id}_{S^n}) = 1$ (since $\mathrm{id}_{S^n,*} = \mathrm{id}_{H_*S^n}$)
- $\deg(S^n \to * \to S^n) = 0$ (since this factors through $H_n(*) = 0$),
- $\deg(g \circ f) = \deg(g) \cdot \deg(f)$ (since $(g \circ f)_* = g_* \circ f_*$).

Our first goal is to compute the degree of the restriction to S^n of an orthogonal linear transformation. This boils down to computing the degree of a reflection, which we do first for S^1:

Lemma 4.9.2. *Let $r_1 \colon S^1 \to S^1$ be an orientation-reversing automorphism of S^1 (which is the restriction of a reflection in a line through the origin if we view S^1 as a subspace of $\mathbb{R}^2$). Then*

$$\deg(r_1) = -1.$$

Proof. We know from Proposition 4.6.2 that a generator of $H_1(S^1)$ is the class represented by the quotient map

$$q \colon \Delta^1 \to \Delta^1/\partial\Delta^1 \cong S^1.$$

Then $r_{1,*}[q] = [r_1 \circ q]$ corresponds to reversing the orientation of the 1-simplex q, and we saw in Proposition 3.4.4 that this means $r_{1,*}[q] = -[q]$, i.e. $\deg r_1 = -1$. □

We can generalize this to higher dimensions:

Proposition 4.9.3. *Let $r_n \colon S^n \to S^n$ ($n \geq 1$) be the restriction of the reflection $(x_0, \ldots, x_n) \mapsto (-x_0, x_1 \ldots, x_n)$ in $\mathbb{R}^{n+1}$. Then $\deg(r_n) = -1$.*

Proof. View S^n as the unit sphere in $\mathbb{R}^{n+1}$, and let D^n_+ and D^n_- be the two hemispheres in S^n consisting of points $(x_0, \ldots, x_n)$ with $x_n \geq 0$ and ≤ 0, respectively. Then r_n takes each hemisphere to itself, and restricts to r_{n-1} on $S^{n-1} \cong D^n_+ \cap D^n_-$. (Strictly speaking we are taking little open neighbourhoods of these discs.) Then r_n gives a map of Mayer–Vietoris sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_n(D^n_+) \oplus H_n(D^n_-) & \longrightarrow & H_n(S^n) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow & H_{n-1}(D^n_+) \oplus H_{n-1}(D^n_-) & \longrightarrow \cdots \\
& \downarrow & & \downarrow{\scriptstyle r_{n,*}} & & \downarrow{\scriptstyle r_{n-1,*}} & & \downarrow & \\
\cdots \longrightarrow & H_n(D^n_+) \oplus H_n(D^n_-) & \longrightarrow & H_n(S^n) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow & H_{n-1}(D^n_+) \oplus H_{n-1}(D^n_-) & \longrightarrow \cdots
\end{array}$$

If $n > 1$ then $H_n(D^n) = H_{n-1}(D^n) = 0$ so we have a commutative square

$$\begin{array}{ccc}
H_n(S^n) & \xrightarrow{\sim} & H_{n-1}(S^{n-1}) \\
\downarrow{\scriptstyle r_{n,*}} & & \downarrow{\scriptstyle r_{n-1,*}} \\
H_n(S^n) & \xrightarrow{\sim} & H_{n-1}(S^{n-1}).
\end{array}$$

Hence $\deg r_n = \deg r_{n-1}$ for $n > 1$, and hence $\deg r_n = \deg r_1 = -1$. □

Corollary 4.9.4. *For any orthogonal matrix $A \in O(n+1)$ ($n > 0$) the map $\alpha \colon S^n \to S^n$ obtained by restricting multiplication by A to the unit sphere satisfies*

$$\deg \alpha = \det A.$$

Proof. Write A as a product of reflections and use that degrees are compatible with compositions. $\square$

Our next goal is to give way to *add* degrees. For this we need to think a bit about wedges:

Definition 4.9.5. If (X, x) and (Y, y) are pointed spaces, their *wedge* is the pointed space obtained by gluing X and Y together along their base points. More formally, we have

$$X \vee Y := (X \amalg Y)/\{x, y\},$$

i.e. we take the disjoint union and then identify the two base points.

Remark 4.9.6. The (pointed) inclusions

$$i_X \colon X \hookrightarrow X \vee Y, \quad i_Y \colon Y \hookrightarrow X \vee Y$$

give a homomorphism

$$i_{X,*} + i_{Y,*} \colon \tilde{H}_* X \oplus \tilde{H}_* Y \to \tilde{H}_*(X \vee Y),$$

and we saw in Exercise 4.7 that this is an isomorphism. But we also have *projections*

$$p_X \colon X \vee Y \to X, \quad p_Y \colon X \vee Y \to Y,$$

where p_X sends all of Y to the base point $x \in X$ and restricts to the identity on X, and similarly for p_Y.

Proposition 4.9.7. *The homorphism*

$$(p_{X,*}, p_{Y,*}) \colon \tilde{H}_*(X \vee Y) \to \tilde{H}_* X \oplus \tilde{H}_* Y$$

is an isomorphism, inverse to $i_{X,*} + i_{Y,*}$.

Proof. It suffices to show that the composite $(p_{X,*}, p_{Y,*}) \circ (i_{X,*} + i_{Y,*})$ is the identity, since we already know that $(i_{X,*} + i_{Y,*})$ is an isomorphism. For $\xi \in \tilde{H}_n X, \eta \in \tilde{H}_n Y$, we have

$$(p_{X,*}, p_{Y,*})(i_{X,*} + i_{Y,*})(\xi, \eta) = (p_{X,*}, p_{Y,*})(i_{X,*}\xi + i_{Y,*}\eta) = ((p_X i_X)_*\xi + (p_X i_Y)_*\eta, (p_Y i_X)_*\xi + (p_Y i_Y)_*\eta).$$

Now we use that $p_X i_X = \mathrm{id}_X, p_Y i_Y = \mathrm{id}_Y$, while $p_X i_Y$ and $p_Y i_X$ are constant at the base points of X and Y, respectively. Thus $p_X i_Y$ factors as

$$X \to \{x\} \to X$$

and so gives zero on reduced homology since $\tilde{H}_*(*) = 0$, and similarly for $p_Y i_X$. We then have

$$(p_{X,*}, p_{Y,*})(i_{X,*} + i_{Y,*})(\xi, \eta) = (\xi, \eta),$$

as required. $\square$

Corollary 4.9.8. *Suppose we have a continuous pointed map* $\phi \colon Z \to X \vee Y$*. Then under the isomorphism* $\tilde{H}_*(X \vee Y) \cong \tilde{H}_* X \oplus \tilde{H}_* Y$*, the map* $\phi_* \colon \tilde{H}_* Z \to \tilde{H}_*(X \vee Y)$ *corresponds to*

$$((p_X \phi)_*, (p_Y \phi)_*) \colon \tilde{H}_* Z \to \tilde{H}_* X \oplus \tilde{H}_* Y.$$

Proof. Immediate from functoriality since we know this isomorphism is given by $(p_{X,*}, p_{Y,*})$. □

Proposition 4.9.9. *Let $\mu\colon S^n \to S^n \vee S^n$ be the map that pinches the equator to a point. Given pointed maps $f, g\colon S^n \to S^n$ we have a map $(f,g)\colon S^n \vee S^n \to S^n$ that does f on one sphere and g on the other. Then*

$$\deg((f,g)\circ\mu) = \deg f + \deg g.$$

Proof. Let $p_1, p_2\colon S^n \vee S^n \to S^n$ be the two projection maps that are given by the identity on one sphere and takes the other to the base point. As we saw in Corollary 4.9.8, we then have that under the isomorphism $H_*(S^n) \oplus H_*(S^n) \xrightarrow{\sim} H_*(S^n \vee S^n)$, the map $\mu_*\colon H_*(S^n) \to H_*(S^n \vee S^n)$ corresponds to

$$((p_1\mu)_*, (p_2\mu)_*) = (\mathrm{id}, \mathrm{id})\colon H_*(S^n) \to H_*(S^n) \oplus H_*(S^n),$$

since both $p_1\mu$ and $p_2\mu$ are homotopic to id_{S^n}. Moreover, the map $(f,g)_*\colon H_*(S^n \vee S^n) \to H_*(S^n)$ corresponds under the same isomorphism to the map $(x,y) \mapsto f_*x + g_*y$, since we know it restricts to f_* and g_* on the two summands. It follows that the map $((f,g)\mu)_*\colon H_*(S^n) \to H_*(S^n)$ is given by

$$x \mapsto f_*x + g_*x,$$

and so a generator $\gamma \in H_n(S^n)$ is mapped to

$$f_*\gamma + g_*\gamma = \deg(f)\gamma + \deg(g)\gamma = (\deg f + \deg g)\gamma,$$

as required. □

Corollary 4.9.10. *For any $n > 0$ and $d \in \mathbb{Z}$, there exists a map $f\colon S^n \to S^n$ of degree d.* □

Remark 4.9.11. Since homology is homotopy-invariant, if two maps $S^n \to S^n$ are homotopic then they must have the same degree. In fact, the converse is also true: two maps $S^n \to S^n$ are homotopic if and only if they have the same degree.

5
Cellular Homology

In this chapter we introduce *cell complexes* (or *CW-complexes*), which are spaces built by iteratively gluing on discs along their boundaries. We will see that this decomposition into discs can be used to define a small chain complex whose homology, the *cellular homology*, agrees with singular homology. We first briefly look at another categorical notion, that of *pushouts* in §5.1 before we use this to define cell complexes in §5.2. We then look at a special case of cell complexes, namely Δ-*complexes*, which are spaces built by gluing simplices together along their faces. Such a space can be described by certain combinatorial data, called a Δ-*set*; in §5.4 we will see that a Δ-set also determines a chain complex, and so a notion of homology, called *simplicial homology*. We will see that the simplicial homology of a Δ-complex is isomorphic to its singular homology, but before we can prove this we must first show, in §5.5 that the homology of a cell complex is determined by its finite-dimensional subspaces, and then analyze these homology groups a bit further in §5.6; we are then ready to prove the comparison result in §5.7. In §5.8 we will see that simplicial homology can be generalized to the *cellular* chain complex associated to any cell complex, and in §5.9 we show that the differential in this chain complex can be described in terms of the degrees of certain maps of spheres. Finally, in §5.10 we apply the cellular chains to compute the homology of the real projective spaces $\mathbb{RP}^n$.

5.1 Pushouts

To define cell complexes it is convenient to first define another categorical notion:

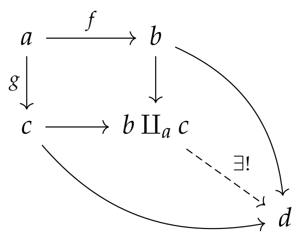
Definition 5.1.1. Let $\mathcal{C}$ be a category. Given morphisms $f\colon a \to b$, $g\colon a \to c$ in $\mathcal{C}$, their *pushout* (if it exists) is the universal commutative square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow \\ c & \longrightarrow & b \amalg_a c. \end{array}$$

Thus given any other commutative square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow \\ c & \longrightarrow & d. \end{array}$$

there exists a unique morphism $b \amalg_a c \to d$ such that the diagram



commutes.

Example 5.1.2. In the category Set of sets, the pushout of two morphisms $f\colon S \to T$, $g\colon S \to U$ is the quotient

$$(T \amalg U)/(f(s) \sim g(s) : s \in S).$$

Example 5.1.3. In the category Top of topological spaces, the pushout of two continuous maps $f\colon X \to Y$, $g\colon X \to Z$, has as its underlying set the pushout $Y \amalg_X Z$ in Set, with the topology where a subset U is open if and only if its preimages in Y and Z are both open.

Exercise 5.1. Show that in the category Ab of abelian groups, the pushout of two homomorphisms $f\colon A \to B$, $g\colon A \to C$ is the cokernel of the homomorphism $(f, -g)\colon A \to B \oplus C$.

Examples 5.1.4.

(i) If X is $\emptyset$ then $Y \amalg_\emptyset Z$ is just the disjoint union $Y \amalg Z$.

(ii) If $A \hookrightarrow X$ is a subspace then the pushout $* \amalg_A X$ along the unique map $A \to *$ is the quotient X/A.

(iii) The commutative square

$$\begin{array}{ccc} S^n & \hookrightarrow & D^{n+1} \\ \downarrow & & \downarrow \\ D^{n+1} & \hookrightarrow & S^{n+1}, \end{array}$$

given by the inclusions of D^{n+1} as the upper and lower "hemispheres" in S^{n+1} and by S^n as the boundary of D^{n+1} (so that the composite $S^n \to S^{n+1}$ is the inclusion of the "equator"), is a pushout: the pushout of these two inclusions $S^n \hookrightarrow D^{n+1}$ is built by gluing two $(n+1)$-discs together along their boundary.

Remark 5.1.5. In the special case where we have $X \hookrightarrow Y$ a subspace inclusion and a continuous map $f\colon X \to Z$, then we say that the pushout $Y \amalg_X Z$ is built by *attaching* Y to Z along f.

Here are some useful properties of pushouts:

Exercise 5.2. Consider a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & A' & \longrightarrow & A'' \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & B' & \longrightarrow & B'', \end{array}$$

in a category $\mathcal{C}$. If the left square is a pushout, then the right square is a pushout if and only if the outer (composite) square is a pushout.

Exercise 5.3 (Pushouts commute with coproducts)**.** Suppose we have pushout squares

$$\begin{array}{ccc} A_i & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ C_i & \longrightarrow & D_i \end{array}$$

for $i \in I$ in some category $\mathcal{C}$. If I-indexed coproducts exist in $\mathcal{C}$, then the canonical square

$$\begin{array}{ccc} \coprod_{i\in I} A_i & \longrightarrow & \coprod_{i\in I} B_i \\ \downarrow & & \downarrow \\ \coprod_{i\in I} C_i & \longrightarrow & \coprod_{i\in I} D_i \end{array}$$

is also a pushout. [Hint: Use the universal properties.]

5.2 *Cell Complexes*

Cell complexes are topological spaces built by attaching discs (which in this context are called *cells*) along their boundary — they are spaces that can be built by a sequence of pushouts of the form

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \amalg_{S^{n-1}} D^n \end{array} \quad \text{or} \quad \begin{array}{ccc} \coprod_{i\in I} S^{n-1} & \hookrightarrow & \coprod_{i\in I} D^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \amalg_{\coprod S^{n-1}} \coprod D^n. \end{array}$$

More precisely, we make the following definition:

Definition 5.2.1. A *cell complex* (or *CW-complex*) is a topological space X equipped with a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq \bigcup_n X_n = X,$$

a set of continuous maps $f_\alpha : S^n \to X_n$, $\alpha \in \Gamma_n$, and pushout squares

$$\begin{array}{ccc} \coprod_{\alpha\in\Gamma_n} S^{n-1} & \hookrightarrow & \coprod_{\alpha\in\Gamma_n} D^n \\ \downarrow{\scriptstyle (f_\alpha)_{\alpha\in\Gamma_n}} & & \downarrow{\scriptstyle (e_\alpha)_{\alpha\in\Gamma_n}} \\ X_{n-1} & \hookrightarrow & X_n. \end{array}$$

The map f_α is called the *attaching map* of the cell α, and the map e_α the *characteristic map* of α. In addition, X must have the topology where a subset $U \subseteq X$ is open if and only if $U \cap X_n$ is open in X_n for all n.

Thus X_n is built from X_{n-1} by attaching an n-disc along each of the maps f_α.

Remark 5.2.2. Here we make the convention that $S^{-1} = \partial\Delta^0 = \emptyset$ (since $D^0 \subseteq \mathbb{R}^0$ is a single point). Thus X_0 is built from $X_{-1} = \emptyset$ by adding a set of points, since we have a pushout

$$\begin{array}{ccc} \coprod_{\alpha\in\Gamma_0} \emptyset & \hookrightarrow & \coprod_{\alpha\in\Gamma_0} * \\ \downarrow & & \downarrow \\ X_{-1} & \hookrightarrow & X_0. \end{array}$$

which exhibits X_0 as the discrete set of points Γ_0. Next we build X_1 by picking out a set of pairs of points in X_0, i.e. maps $S^0 \to X_0$, and connecting them with intervals, and so forth.

Remark 5.2.3. If X is a cell complex, then $U \subseteq X$ is open (closed) if and only if for every cell $D^n \xrightarrow{e_\alpha} X_n \hookrightarrow X$, the preimage of U is open in D^n. This follows by combining the definition of the topology on X with that of the topology on pushouts and disjoint unions.

Fact 5.2.4. *Every cell complex is a Hausdorff space. (See Hatcher, Proposition A.3 for a proof.)*

Fact 5.2.5. *Every cell complex is locally contractible (and so in particular locally path-connected). (See Hatcher, Proposition A.4.)*

Definition 5.2.6. A cell complex X is *finite-dimensional* if $X = X_n$ for some n, and of *finite type* if each set Γ_n of cells is finite (so X has finitely many cells in each dimension). We say X is *finite* if it is both finite-dimensional and of finite type, i.e. if X is built from finitely many cells.

Remark 5.2.7. It is easy to see that a finite cell complex is compact, since it is built from finitely many compact spaces, hence a finite cell complex is a compact Hausdorff space.

Fact 5.2.8. *Every compact smooth manifold can be given the structure of a cell complex.*

Remark 5.2.9. A harder result is that every compact *topological* manifold can be given the structure of a cell complex; this is known in all dimensions except 4 (which is still open).

Now let's look at some examples of cell complexes:

Examples 5.2.10.

(1) The n-sphere S^n can be described as the quotient D^n/S^{n-1}, so we have a pushout square

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^n. \end{array}$$

This means we can give S^n a cell structure with a single 0-cell and a single n-cell, so that

$$(S^n)_i = \begin{cases} *, & 0 \leq i < n, \\ S^n, & i \geq n. \end{cases}$$

(2) S^n is also that pushout $D^n \amalg_{S^{n-1}} D^n$, where two n-discs are glued together along their boundary. We can rewrite this as a pushout square

$$\begin{array}{ccc} S^{n-1} \amalg S^{n-1} & \hookrightarrow & D^n \amalg D^n \\ {\scriptstyle (\mathrm{id},\mathrm{id})}\downarrow & & \downarrow \\ S^{n-1} & \hookrightarrow & S^n. \end{array}$$

Combining these pushouts in dimensions up to n, we can give S^n a cell structure with two i-cells in each dimension $\leq n$ and $(S^n)_i = S^i$ for $i \leq n$.

(3) n-dimensional real projective space $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Since every line intersects the unit sphere in two antipodal points, we can describe this as the quotient space $S^n/(x \sim -x)$. Since every point in the upper hemisphere is identified with a single point in the lower hemisphere, we can equivalently describe $\mathbb{RP}^n$ as the quotient

$$D^n/(x \sim -x : x \in \partial D^n).$$

The resulting map $D^n \to \mathbb{RP}^n$ restricts on the boundary to the quotient map $S^{n-1} \to \mathbb{RP}^{n-1}$ and is a homeomorphism away from the boundary, so that we have a pushout

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \downarrow & & \downarrow \\ \mathbb{RP}^{n-1} & \hookrightarrow & \mathbb{RP}^n. \end{array}$$

This allows us to describe $\mathbb{RP}^n$ as a cell complex with a single cell in each dimension $\leq n$ and with $(\mathbb{RP}^n)_i = \mathbb{RP}^i$ for $i \leq n$ (where $\mathbb{RP}^0$, the space of lines in $\mathbb{R}$, is a single point).

(4) We can also continue adding cells in this way forever, getting infinite-dimensional real projective space $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n$. This has a single cell in each dimension, and $(\mathbb{RP}^\infty)_i = \mathbb{RP}^i$ for all i.

(5) n-dimensional complex projective space $\mathbb{CP}^n$ is the space of complex lines through the origin (or 1-dimensional sub-vector spaces) in $\mathbb{C}^{n+1}$. Such a line intersects the unit sphere in $\mathbb{C}^{n+1}$ in a copy of the unit complex numbers, which is homeomorpic to S^1. Thus

$$\mathbb{CP}^n \cong S^{2n+1}/(x \sim \lambda x : \lambda \in \mathbb{C}, |\lambda| = 1)$$

We can define a map $D^{2n} \to \mathbb{CP}^n$ by taking $(z_0, \ldots, z_{n-1}) \in D^{2n}$ ($z_i \in \mathbb{C}$) to $(z_0 : z_1 : \cdots : z_{n-1} : 1 - \left(\sum_i |z_i|^2\right)^{1/2})$ in $\mathbb{CP}^n$. This is a homeomorphism away from the boundary, and takes ∂D^{2n} to the subspace of $\mathbb{CP}^n$ where the last projective coordinate is 0. This subspace is a copy of $\mathbb{CP}^{n-1}$, and the restricted map $\partial D^{2n} \to \mathbb{CP}^{n-1}$ is exactly the quotient map, so that we have a pushout

$$\begin{array}{ccc} S^{2n-1} & \hookrightarrow & D^{2n} \\ \downarrow & & \downarrow \\ \mathbb{CP}^{n-1} & \hookrightarrow & \mathbb{CP}^n. \end{array}$$

Thus $\mathbb{CP}^n$ has a cell structure with a single cell in each even dimensional $\leq 2n$, and

$$(\mathbb{CP}^n)_i = \begin{cases} \mathbb{CP}^{i/2}, & i \text{ even,} \\ \mathbb{CP}^{(i-1)/2} & i \text{ odd.} \end{cases}$$

We can also keep going and define infinite-dimensional complex projective space $\mathbb{CP}^\infty = \bigcup_n \mathbb{CP}^n$. This has a cell structure with a single cell in every even dimension.

Definition 5.2.11. A *relative cell complex* (X, A) is a subspace pair (i.e. $A \subseteq X$) with a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq \bigcup_n X_n = X,$$

a set of continuous maps $f_\alpha \colon S^n \to X_n$, $\alpha \in \Gamma_n$, and pushout squares

$$\begin{array}{ccc} \coprod_{\alpha \in \Gamma_n} S^n & \hookrightarrow & \coprod_{\alpha \in \Gamma_n} D^n \\ \downarrow (f_\alpha)_{\alpha \in \Gamma_n} & & \downarrow (e_\alpha)_{\alpha \in \Gamma_n} \\ X_n & \hookrightarrow & X_{n+1}. \end{array}$$

X must have the topology where a subset $U \subseteq X$ is open if and only if $U \cap X_n$ is open in X_n for all $n \geq -1$. (So it's the same as a cell complex except that we *start* with A, rather than with $\emptyset$, i.e. X is built from A by attaching cells.)

Fact 5.2.12. *A relative cell complex is a good pair. (See Hatcher, Proposition A.5 for a proof.)*

Example 5.2.13. If X is a cell complex, then (X, X_n) and (X_n, X_{n-k}) are relative cell complexes, and so in particular good pairs.

5.3 Δ-Sets and Δ-Complexes

In this section we will look at a special case of cell complexes, called Δ-*complexes*, which are built by gluing together simplices along their faces. These cell structures can be described by the following entirely combinatorial data:

Definition 5.3.1. A Δ-*set* S is a collection of sets S_n $(n = 0, 1, \ldots)$ together with face maps $\partial_i \colon S_n \to S_{n-1}$ $(i = 0, \ldots, n)$ satisfying the simplicial identity,

$$\partial_j \partial_i = \partial_{i-1} \partial_j$$

for $0 \leq j < i \leq n+1$.

A more "grown-up" name for Δ-sets is *semisimplicial sets*, because they have part of the structure of certain objects called *simplicial sets*.

Example 5.3.2. If X is a topological space, the singular simplices $\mathrm{Sing}_n(X)$ together with the face maps between them form a Δ-set $\mathrm{Sing}(X)$.

We think of a Δ-set S as a recipe for building a topological space out of simplices: we take an n-simplex Δ^n for every element s of S_n and glue its ith face to the $(n-1)$-simplex corresponding to $\partial_i s$. More precisely, we have the following definition:

Definition 5.3.3. Let S be a Δ-set. The *geometric realization* $|S|$ is defined inductively as follows: Starting with $|S|_{-1} = \emptyset$ we will define topological spaces $|S|_n$ with $|S|_{n-1} \subseteq |S|_n$ and maps $e_\sigma \colon \Delta^n \to |S|_n$ for every $\sigma \in S_n$, such that for every σ and $0 \leq i \leq n$ we have a commutative square

$$\begin{array}{ccc} \Delta^{n-1} & \xrightarrow{d^i} & \Delta^n \\ \downarrow{\scriptstyle e_{\partial_i \sigma}} & & \downarrow{\scriptstyle e_\sigma} \\ |S|_{n-1} & \hookrightarrow & |S|_n. \end{array}$$

Informally, we define $|S|_{n+1}$ by gluing on an $(n+1)$-simplex for every $\sigma \in S_{n+1}$ such that the boundary matches with the boundary of σ in S_n.

Given this, for every $\sigma \in S_{n+1}$ we can define a map $\partial e_\sigma \colon \partial\Delta^{n+1} \to |S|_n$ by defining ∂e_σ to be given by $e_{\partial_i \sigma}$ on the ith face $\partial_i \Delta^n$; this makes sense because these maps agree on the $n-1$-simplices where the faces overlap since we have

$$e_{\partial_i \sigma} \circ d_j = e_{\partial_j \partial_i \sigma} = e_{\partial_{i-1} \partial_j \sigma} = e_{\partial_j \sigma} \circ d_{i-1}$$

when $j < i$. Then we define $|S|_{n+1}$ and e_σ for $\sigma \in S_{n+1}$ by the pushout

$$\begin{array}{ccc} \coprod_{\sigma \in S_{n+1}} \partial\Delta^{n+1} & \hookrightarrow & \coprod_{\sigma \in S_{n+1}} \Delta^{n+1} \\ \downarrow{\scriptstyle \coprod_\sigma \partial e_\sigma} & & \downarrow{\scriptstyle \coprod_\sigma e_\sigma} \\ |S|_n & \hookrightarrow & |S|_{n+1}. \end{array}$$

We then take $|S| = \bigcup_n |S|_n$, with the topology where a subset is open if and only if its intersection with $|S|_n$ is open for all n.

Remark 5.3.4. Since $(D^n, \partial D^n) \cong (\Delta^n, \partial\Delta^n)$, the construction of $|S|$ tautologically makes this space a cell complex with an n-cell for every element of S_n.

Remark 5.3.5. The geometric realization $|S|$ can also be defined "all at once" as the topological space

$$\left(\coprod_{n=0}^{\infty} S_n \times \Delta^n \right) / \sim,$$

where $\sim$ is the relation generated by $(s, d^i p) \sim (\partial_i s, p)$ for $s \in S_n, p \in \Delta^{n-1}$. This is a more convenient description for some purposes, but we will not prove here that it is equivalent to ours as this is best done using a bit more category theory than we have covered.

Definition 5.3.6. A Δ-*complex* is a topological space X together with a Δ-set S and a homeomorphism $X \cong |S|$.

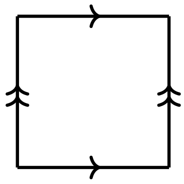
Thus giving a topological space X the structure of a Δ-complex amounts to specifying a way to build X by gluing together simplices; this can also be viewed as a special kind of cell structure.

Examples 5.3.7.

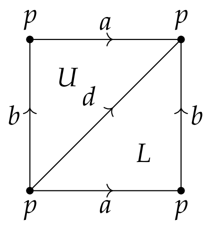
(i) We can build the circle S^1 by taking a point and a 1-simplex and gluing both ends of the 1-simplex to the point. More precisely we consider the Δ-set S with $S_0 = \{p\}, S_1 = \{x\}, S_n = \emptyset$ for $n > 1$, with $\partial_i x = p$ $(i = 0, 1)$.

(ii) We can build the sphere S^2 by taking two 2-simplices and gluing them together along the boundary. This corresponds to the Δ-set S with $S_0 = \{[0], [1], [2]\}, S_1 = \{[01], [12], [02]\}, S_2 = \{U, L\}$ where $\partial_0[ij] = [j]$, $\partial_1[ij] = [i]$, and $\partial_i U = \partial_i L = [jk]$ where $i \neq j, k$.

(iii) We can build the torus by taking a square and gluing opposite edges together:



If we add the diagonal, we get a description of the torus as a Δ-complex with two 2-simplices, three 1-simplices and one 0-simplex:



Here

$$\partial_0 U = a, \quad \partial_1 U = d, \quad \partial_2 U = b,$$

$$\partial_0 L = b, \quad \partial_1 L = d, \quad \partial_2 L = a.$$

Definition 5.3.8. If S, T are Δ-sets, then a *morphism of* Δ*-sets* $f \colon S \to T$ consists of functions $f_n \colon S_n \to T_n$ such that the squares

$$\begin{array}{ccc} S_n & \xrightarrow{f_n} & T_n \\ \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} \\ S_{n-1} & \xrightarrow{f_{n-1}} & T_{n-1} \end{array}$$

commute for all n, i. We write Set_Δ for the category whose objects are Δ-sets and whose morphisms are morphisms of Δ-sets.

Exercise 5.4. Show that a morphism of Δ-sets $f \colon S \to T$ induces a canonical continuous map $|f| \colon |S| \to |T|$ between geometric realizations such that for every $\sigma \in S_n$ the triangle

$$\begin{array}{ccc} & \Delta^n & \\ {\scriptstyle e_\sigma} \swarrow & & \searrow {\scriptstyle e_{f(\sigma)}} \\ |S| & \xrightarrow{|f|} & |T| \end{array}$$

commutes. Check that this makes $|{-}|$ a functor $\mathrm{Set}_\Delta \to \mathrm{Top}$.

5.4 Simplicial Homology

Definition 5.4.1. Let S be a Δ-set. Then we can define a chain complex $C_\bullet(S)$ as follows: The group $C_n(S)$ is the free abelian group $\mathbb{Z}S_n$, and the boundary map $C_n(S) \to C_{n-1}(S)$ is given by

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i.$$

This satisfies $\partial^2 = 0$ by the same proof as for the singular chain complex. We write $H_n(S) := H_n(C_\bullet(S))$ for the corresponding homology groups.

We can think of the homology groups $H_n(S)$ as "a" simplicial homology of the topological space $|S|$.

Example 5.4.2. Let X be a topological space. Then the singular chain complex $S_\bullet X$ is the same as $C_\bullet(\mathrm{Sing}(X))$ using the singular Δ-set we defined above.

Examples 5.4.3. Let's compute the homology of the three Δ-complexes from Examples 5.3.7.

(i) Let S be the Δ-set describing S^1, then

$$C_0 S = \mathbb{Z}p, \quad C_1 S = \mathbb{Z}x, \quad C_n S = 0, n \neq 0, 1.$$

We have $\partial x = p - p = 0$, so this chain complex looks like

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \cdots,$$

and we get

$$H_*(S) = \begin{cases} \mathbb{Z}, & * = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Let S be the Δ-set describing S^2, then

$$C_0 S = \mathbb{Z}\{[0], [1], [2]\}, \quad C_1 S = \mathbb{Z}\{[01], [02], [12]\}, \quad C_2 S = \mathbb{Z}\{U, L\}, \quad C_n S = 0, n > 2$$

with $\partial[ij] = [j] - [i]$ and

$$\partial U = \partial L = [12] - [02] + [01].$$

Thus $B_0 S$ is generated by $[1] - [0], [2] - [0], [2] - [1]$ and so $H_0 S$ is $\mathbb{Z}$, generated by any of the three 0-simplices, which are identified in the quotient. We also see that $Z_2 S$ is generated by $U - L$ so that $H_2 S \cong \mathbb{Z}$, generated by this chain. (Note that geometrically $U - L$ represents the entire sphere, with the signs giving the correct orientation.) Finally we need to compute $Z_1 S$: suppose $a[01] + b[02] + c[12]$ is in Z_1, then

$$\partial(a[01] + b[02] + c[12]) = (a - c)[1] + (b + c)[2] - (a + b)[0] = 0$$

which forces $a = c, b = -c$, so that $Z_1 S$ is generated by the single cycle $[01] - [02] + [12]$ which is $\partial U = \partial L$, so that $Z_1 S = B_1 S$ and $H_1 S = 0$. Thus

$$H_* S = \begin{cases} \mathbb{Z}, & n = 0, 2, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Let S be the Δ-set describing the torus, then

$$C_0S = \mathbb{Z}p, \quad C_1S = \mathbb{Z}\{a,b,d\}, \quad C_2S = \mathbb{Z}\{U,L\}, \quad C_nS = 0, n > 2.$$

We have $\partial a = \partial b = \partial d = p - p = 0$, and

$$\partial U = d - b + a, \quad \partial L = a - b + d.$$

Thus $B_0S = 0$ and $H_0S \cong \mathbb{Z}$, while $H_2S \cong Z_2S \cong \mathbb{Z}$, generated by $U - L$. Moreover $Z_1S = C_1S$ while B_1S is generated by the single chain $a - b + d$. Thus

$$H_1S \cong \mathbb{Z}\{a,b,d\}/(a - b + d) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

In summary,

$$H_*S = \begin{cases} \mathbb{Z}, & n = 0,2, \\ \mathbb{Z} \oplus \mathbb{Z}, n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 5.5. The *combinatorial n-simplex* is the Δ-set Δ^n_{comb} with $(\Delta^n_{\text{comb}})_k$ being the set of subsets of $\{0,\ldots,n\}$ of size $k+1$; it is convenient to label these as $[i_0 \cdots i_k]$ with $0 \leq i_0 \leq i_1 \leq \cdots \leq i_k \leq n$. Then the face map $\partial_j \colon (\Delta^n_{\text{comb}})_k \to (\Delta^n_{\text{comb}})_{k-1}$ is given by

$$[i_0 \cdots i_k] \mapsto [i_0 \cdots i_{j-1} i_{j+1} \cdots i_k].$$

The *boundary* of Δ^n_{comb} is the Δ-set $\partial\Delta^n_{\text{comb}}$ obtained by removing the single n-simplex $[01 \cdots n]$, so that

$$(\partial\Delta^n_{\text{comb}})_k = \begin{cases} (\Delta^n_{\text{comb}})_k, & 0 \leq k \leq n-1, \\ \emptyset, & k \geq n, \end{cases}$$

with the same face maps in degrees $< n$.

(i) Convince yourself that $|\Delta^n_{\text{comb}}|$ is homeomorphic to Δ^n and $|\partial\Delta^n_{\text{comb}}|$ to the boundary of Δ^n.

(ii) Compute the simplicial homology of the Δ-sets Δ^3_{comb} and $\partial\Delta^3_{\text{comb}}$. [The space $|\partial\Delta^3_{\text{comb}}|$ is a tetrahedron, which is topologically a sphere, so the result should agree with the usual homology of the sphere.]

Remark 5.4.4. $C_\bullet$ is a functor $\mathsf{Set}_\Delta \to \mathsf{Ch}$, by the same argument as for $S_\bullet$ above. As part of that proof we also essentially showed that Sing is a functor $\mathsf{Top} \to \mathsf{Set}_\Delta$. It follows that simplicial homology is a functor

$$H_* \colon \mathsf{Set}_\Delta \xrightarrow{C_\bullet} \mathsf{Ch} \xrightarrow{H_*} \mathsf{grAb},$$

while singular homology decomposes as the composite of functors

$$\mathsf{Top} \xrightarrow{\text{Sing}} \mathsf{Set}_\Delta \xrightarrow{C_\bullet} \mathsf{Ch} \xrightarrow{H_*} \mathsf{grAb}.$$

Definition 5.4.5. Let S be a Δ-set. Then there is a canonical morphism of Δ-sets $S \to \text{Sing}\,|S|$, which takes $s \in S_n$ to the inclusion $e_s \colon \Delta^n \to |S|$ of the simplex labelled by s. This is a morphism of Δ-sets since from the definition of e_s we see that $\partial_i s$ maps to

$$e_{\partial_i s} = e_s \circ d^i = \partial_i e_s.$$

This gives a canonical chain map $C_\bullet S \to S_\bullet|S|$ and so a homomorphism $H_*S \to H_*|S|$.

We will prove that this homomorphism of homology groups is always an *isomorphism*, i.e.

$$H_n(S) \cong H_n(|S|).$$

Thus it was not a coincidence that the simplicial homology groups we computed in Examples 5.4.3 were the same as the singular homology groups of S^1, S^2, and the torus. This gives us one recipe for computing the homology of a topological space X:

(1) Find a combinatorial description of X by decomposing it into simplices, i.e. find a Δ-set S such that $|S| \cong X$.

(2) Try to compute the simplicial homology H_*S as Z_nS/B_nS; if we're lucky, we can choose S so that $C_\bullet S$ is very small, and this is actually feasible.

Remark 5.4.6. If X is a topological space, there is also a canonical continuous map $|\operatorname{Sing}(X)| \to X$, which is determined by commutative diagrams

$$\begin{array}{ccc} \Delta^n & \xrightarrow{e_\sigma} & |\operatorname{Sing} X| \\ & \searrow^{\sigma} & \downarrow \\ & & X, \end{array}$$

i.e. on the simplex labelled by σ we use the map σ to get to X. If X is a reasonable space, then this map actually turns out to be a homotopy equivalence; this means that in a sense the Δ-set $\operatorname{Sing} X$ knows all the homotopy-invariant information about X.

5.5 *Homology of Sequential Colimits*

We are going to prove that simplicial and singular homology agree. In order to do this, we first need to understand the relation between the homology of a Δ-complex $X \cong |S|$ and the spaces $|S|_n$ in its filtration, which we may as well do for a general cell complex. For this it is convenient to introduce another categorical construction:

Definition 5.5.1. Let $\mathcal{C}$ be a category and suppose we have a sequence

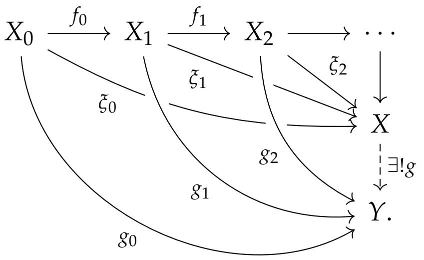
$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{\cdots}$$

of morphisms $f_i \colon X_i \to X_{i+1}$ in $\mathcal{C}$. The *colimit* of the sequence (if it exists) is an object X together with maps $\xi_i \colon X_i \to X$ such that the triangles

$$\begin{array}{ccc} X_i & \xrightarrow{\quad f_i \quad} & X_{i+1} \\ & \searrow^{\xi_i} \quad \swarrow_{\xi_{i+1}} & \\ & X & \end{array}$$

commute (i.e. we have $\xi_i = \xi_{i+1} \circ f_i$ for all i), and with the universal property that given any other family of maps $g_i \colon X_i \to Y$ such that

$g_i = g_{i+1} f_i$ there exists a unique map $g\colon X \to Y$ such that $g \circ \xi_i = g_i$:



Examples 5.5.2.

(i) In Set, the colimit $\operatorname{colim}_n X_n$ as above is given by the quotient

$$\left(\coprod_n X_n\right) / \sim,$$

where $\sim$ is the equivalence relation generated by $x \sim f_i x$ for $x \in X_i$, with $\xi_i\colon X_i \to \operatorname{colim}_n X_n$ given as the composite

$$X_i \hookrightarrow \coprod_n X_n \to \left(\coprod_n X_n\right) / \sim$$

of the inclusion of the factor X_i in the coproduct and the quotient map.

(ii) In Top, the colimit $\operatorname{colim}_n X_n$ is the colimit in Set with the quotient topology, which is here the topology where $U \subseteq \operatorname{colim}_n X_n$ is open if and only if $\xi_i^{-1}(U)$ is open in X_i for all i.

(iii) In Ab, the colimit $\operatorname{colim}_n X_n$ is the quotient

$$\left(\bigoplus_n X_n\right) / A$$

where A is the subgroup generated by $\iota_k x - \iota_{k+1} f_k x$ for $x \in X_k$, where ι_k is the inclusion $X_k \hookrightarrow \bigoplus_n X_n$.

(iv) In Ch, the colimit $\operatorname{colim}_n X_n$ is given by the degreewise colimit in Ab, i.e. $(\operatorname{colim}_n X_n)_k \cong \operatorname{colim}_n X_{n,k}$, with differential $(\operatorname{colim}_n X_n)_k \to (\operatorname{colim}_n X_n)_{k-1}$ the unique homomorphism such that the squares

$$\begin{array}{ccc} X_{i,k} & \longrightarrow & \operatorname{colim}_n X_{n,k} \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ X_{i,k-1} & \longrightarrow & \operatorname{colim}_n X_{n,k-1} \end{array}$$

commute (which exists since the maps $X_i \to X_{i+1}$ are chain maps).

Exercise 5.6.

(i) Suppose we have subsets

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots.$$

Show that the union $\bigcup_{n=0}^{\infty} S_n$ is isomorphic to the sequential colimit of the inclusions $S_n \hookrightarrow S_{n+1}$.

(ii) Suppose we have subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots.$$

Show that the union $\bigcup_{n=0}^{\infty} X_n$ is homeomorphic to the sequential colimit of the continuous inclusions $X_n \hookrightarrow X_{n+1}$ if we give the union the topology where a subspace $U \subseteq \bigcup_{n=0}^{\infty} X_n$ is open if and only if $U \cap X_n$ is open in X_n for all n.

(iii) Also check the analogous statement for abelian groups.

Exercise 5.7. Suppose we have a sequence

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{\cdots}$$

of morphisms $f_i \colon X_i \to X_{i+1}$ in a category $\mathcal{C}$ such that f_i is an isomorphism for all $i \geq N$. Show that then X_N is a colimit of the sequence. [Hint: We have compatible isomorphisms $X_N \xrightarrow{\sim} X_i$ for $i > N$, inverting these we get compatible morphisms $X_i \to X_N$ for all N. Now check the universal property.]

Exercise 5.8. Show that the free abelian group functor $\mathbb{Z}(-)\colon \mathsf{Set} \to \mathsf{Ab}$ preserves sequential colimits.

Given Exercise 5.6 we can think of sequential colimits as a generalization of unions. This is convenient because there are functors that preserve sequential colimits but do not preserve unions (because they don't preserve injections). In particular, this is true for homology as a functor $H_* \colon \mathsf{Ch} \to \mathsf{Ab}$, which we'll see in the next pair of exercises:

Exercise 5.9. Given a diagram of abelian groups

$$A_0 \xrightarrow{f_0} A_1 \to \cdots,$$

and subgroups $B_i \hookrightarrow A_i$ such that $f_i(B_i) \subseteq B_{i+1}$, show that there is a canonical isomorphism

$$\operatorname{colim}_i A_i/B_i \cong (\operatorname{colim}_i A_i) \,/\, (\operatorname{colim}_i B_i).$$

Exercise 5.10. Suppose we have $C_\bullet \cong \operatorname{colim}_n C_{n,\bullet}$. Show that

$$Z_k(C) \cong \operatorname{colim}_n Z_k(C_n), \quad B_k(C) \cong \operatorname{colim}_n B_k(C_n),$$

and conclude using the previous exercise that

$$H_k(C) \cong \operatorname{colim}_n H_k(C_n).$$

We now want to apply this to singular homology. However, it is *not* true in general that the functor $S_\bullet$ preserves sequential colimits. Nevertheless, this is true with some hypotheses on the diagram in question, which do hold for the cellular filtration of a cell complex. The key technical point is the following:

Proposition 5.5.3. *Suppose K is a compact Hausdorff space and there exists a sequence of closed inclusions*

$$K_0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K = \bigcup_{i=0}^{\infty} K_i$$

such that a subset $U \subseteq K$ is open if and only if $U \cap K_n$ is open in K_n for all n. Then we must have $K = K_n$ for some finite n.

This proof uses a lot of point-set topology we don't otherwise need in the course.

Proof. Since K is compact Hausdorff, points are closed, and the subspaces K_n are also compact Hausdorff since they are closed in K. If $K_n \subsetneq K$ for all n, then we can choose points $x_n \in K_n \setminus K_{n-1}$. Set $S = \{x_1, x_2, \ldots\}$. Then $S \cap K_n$ is finite for all n, hence closed in K_n, which implies S is closed in K. The same argument shows any subset of S is closed, so S has the discrete topology. But then S is not compact, contradicting it being a closed subset of the compact space K. □

Corollary 5.5.4. *Suppose X is a Hausdorff space and we are given an increasing sequence of subspaces*

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X = \bigcup_{i=1}^{\infty} X_i$$

such that

- $X_i \hookrightarrow X_{i+1}$ *is a closed inclusion for all i,*
- *a subset U of X is open if and only if $U \cap X_i$ is open in X_i for all i.*

If C is a compact space, then any continuous map $f\colon C \to X$ factors through X_n for some n, so that we have

$$\mathrm{Hom}_{\mathsf{Top}}(C, X) \cong \bigcup_{n=0}^{\infty} \mathrm{Hom}_{\mathsf{Top}}(C, X_n).$$

Proof. For $f\colon C \to X$, the subspace $f(C)$ is a compact subset of X, and hence a compact Hausdorff space. We can then apply Proposition 5.5.3 to the filtration $f(C) \cap X_i$ and conclude that $f(C) = f(C) \cap X_n$ for some n, i.e. $f(C) \subseteq X_n$. Since $\mathrm{Hom}_{\mathsf{Top}}(C, X_n) \to \mathrm{Hom}_{\mathsf{Top}}(C, X)$ is an injection for all n (with image those continuous maps whose image lies in X_n), the induced map

$$\mathrm{colim}_n \mathrm{Hom}_{\mathsf{Top}}(C, X_n) \to \mathrm{Hom}_{\mathsf{Top}}(C, X)$$

is injective (with image those maps whose image lies in X_n for some n). We have just shown that this map is also surjective, and hence an isomorphism. □

Since the simplices Δ^n are compact for all n, as a special case we have:

Corollary 5.5.5. *For X as in Corollary 5.5.4, we have*

$$\mathrm{Sing}_n(X) = \bigcup_{i=0}^{\infty} \mathrm{Sing}_n(X_i)$$

for all $n = 0, 1, \ldots$. □

Corollary 5.5.6. *Suppose X is a Hausdorff space and we are given an increasing sequence of subspaces*

$$X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X = \bigcup_{i=1}^{\infty} X_i$$

such that

- *$X_i \hookrightarrow X_{i+1}$ is a closed inclusion for all i,*
- *a subset U of X is open if and only if $U \cap X_i$ is open in X_i for all i.*

Then we have

$$H_*(X) \cong \operatorname{colim}_i H_*(X_i).$$

Proof. From Exercise 5.8 and Corollary 5.5.5 we have

$$S_\bullet(X) \cong \operatorname{colim}_i S_\bullet(X_i).$$

Now Exercise 5.10 gives the corresponding isomorphism in homology. □

As a special case, we get:

Corollary 5.5.7. *Let X be a cell complex. Then we have*

$$H_*(X) \cong \operatorname{colim}_n H_*(X_n)$$

for all $* = 0, 1, \ldots$. □

5.6 *Homology of Cell Complexes*

If X is a cell complex, Corollary 5.5.7 tells us that we can extract the homology of X from the homology of the subspaces X_n. In this section we will compute the relative homology of the pairs (X_n, X_{n-1}) and use this together with the long exact sequence for this pair to extract more information about the homology of X.

Proposition 5.6.1. *Let X be a cell complex. Then*

$$X_n/X_{n-1} \cong \bigvee_{\alpha \in \Gamma_n} S^n.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \coprod_{\alpha\in\Gamma_n} S^{n-1} & \longrightarrow & \coprod_{\alpha\in\Gamma_n} D^n \\ \downarrow & & \downarrow \\ X_{n-1} & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ * & \longrightarrow & X_n/X_{n-1}. \end{array}$$

Here both squares are pushouts, so by Exercise 5.2 the composite square

$$\begin{array}{ccc} \coprod_{\alpha\in\Gamma_n} S^{n-1} & \longrightarrow & \coprod_{\alpha\in\Gamma_n} D^n \\ \downarrow & & \downarrow \\ * & \longrightarrow & X_n/X_{n-1} \end{array}$$

is also a pushout. We can expand to a diagram

$$\begin{array}{ccc} \coprod_{\alpha\in\Gamma_n} S^{n-1} & \longrightarrow & \coprod_{\alpha\in\Gamma_n} D^n \\ \downarrow & & \downarrow \\ \coprod_{\alpha\in\Gamma_n} * & \longrightarrow & \coprod_{\alpha\in\Gamma_n} D^n/S^{n-1} \\ \downarrow & & \downarrow \\ * & \longrightarrow & X_n/X_{n-1}, \end{array}$$

where the top square is a pushout by Exercise 5.3. Applying Exercise 5.2 again we conclude that the bottom square is a pushout. But this means X_n/X_{n-1} is homeomorphic to

$$\bigvee_{\alpha\in\Gamma_n} D^n/S^{n-1} := \left(\coprod_{\alpha\in\Gamma_n} D^n/S^{n-1}\right) / \left(\coprod_{\alpha\in\Gamma_n} *\right),$$

as required. □

Remark 5.6.2. Let X be a cell complex. Since (X_n, X_{n-1}) is a good pair by Fact 5.2.12, it follows that we have isomorphisms

$$H_*(X_n, X_{n-1}) \cong \tilde{H}_*(\bigvee_{\alpha\in\Gamma_n} S^n) \cong \begin{cases} \mathbb{Z}\Gamma_n, & * = n, \\ 0, & * \neq n. \end{cases}$$

These groups show up in the long exact sequence for the pair (X_n, X_{n-1}):

$$\cdots \to H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1}) \to \cdots$$

We have segments of the form

$$0 \to H_k(X_{n-1}) \to H_k(X_n) \to 0, \qquad (k \neq n, n-1)$$

$$0 \to H_n(X_{n-1}) \to H_n(X_n) \to \mathbb{Z}\Gamma_n$$

$$\mathbb{Z}\Gamma_n \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_n) \to 0.$$

In particular, we see that $H_k(X_{n-1}) \cong H_k(X_n)$ if $k \neq n, n-1$.

Let's consider what this means for the groups $H_k(X_n)$ with k fixed as we vary n:

- For $n = -1$ we have $H_k(X_{-1}) = H_k(\emptyset) = 0$.
- For $n < k$ we have

$$H_k(X_n) \cong H_k(X_{n-1}) \cong \cdots \cong H_k(X_{-1}) = 0.$$

- For $n = k$ we have the exact sequence (as we know $H_k(X_{k-1}) = 0$)

$$0 \to H_k(X_k) \to \mathbb{Z}\Gamma_k.$$

  Thus $H_k(X_k)$ is a subgroup of the free group $\mathbb{Z}\Gamma_k$ and hence is itself a free abelian group.

- For $n = k+1$ we have the exact sequence

$$\mathbb{Z}\Gamma_{k+1} \to H_k(X_k) \to H_k(X_{k+1}) \to 0,$$

  which exhibits $H_k(X_{k+1})$ as a quotient of $H_k(X_k)$.

- For $n > k+1$ we have

$$H_k(X_n) \cong H_k(X_{n-1}) \cong \cdots \cong H_k(X_{k+1}).$$

Moreover, we know from Corollary 5.5.7 that $H_k(X)$ is the sequential colimit of the groups $H_k(X_n)$ as n goes to ∞; since these groups stabilize at $n = k+1$, applying Exercise 5.7 this means that we also have

$$H_k(X) \cong H_k(X_{k+1}).$$

To summarize, we have shown:

Proposition 5.6.3. *For X a cell complex, we have*

$$\begin{aligned} H_k(X_n) &= 0, \quad n < k, \\ H_k(X_n) &= H_k(X), \quad n > k. \end{aligned}$$

As a special case, we see:

Corollary 5.6.4. *If X is a d-dimensional cell complex, so that $X = X_d$, then $H_k(X) = H_k(X_d) = 0$ if $k > d$.* □

5.7 *Singular and Simplicial Homology*

Now we can return to the comparison of singular and simplicial homology. If S is a Δ-set, recall that we defined a canonical morphism of Δ-sets $S \to \mathrm{Sing}\,|S|$ and so a chain map $C_\bullet S \to S_\bullet|S|$, which we want to prove gives an isomorphism $H_n S \xrightarrow{\sim} H_n|S|$ in homology.

Definition 5.7.1. Let S be a Δ-set. The *n-skeleton* $\mathrm{sk}_n\,S$ is the Δ-set given by

$$(\mathrm{sk}_n\,S)_k = \begin{cases} S_k, & k \leq n, \\ \emptyset, & k > n. \end{cases}$$

This gives a filtration of S by sub-Δ-sets

$$\mathrm{sk}_0\,S \subseteq \mathrm{sk}_1\,S \subseteq \mathrm{sk}_2\,S \subseteq S = \bigcup_{n=0}^{\infty} \mathrm{sk}_n\,S.$$

Since $\mathrm{sk}_n\,S$ has no k-simplices for $k > n$, it is clear from the definitions that

$$|\,\mathrm{sk}_n\,S| \cong |S|_n,$$

so that the cellular filtration on $|S|$ is the image of the skeletal filtration of S under geometric realization.

By Exercise 5.8 and the description of sequential colimits in Ch we see that

$$C_\bullet(S) \cong \mathrm{colim}_n\, C_\bullet(\mathrm{sk}_n\,S).$$

Hence from Exercise 5.10 we have an isomorphism

$$H_*(S) \cong \mathrm{colim}_n\, H_*(\mathrm{sk}_n\,S),$$

compatible with the isomorphism

$$H_*(|S|) \cong \mathrm{colim}_n\, H_*(|\,\mathrm{sk}_n\,S|),$$

from Corollary 5.5.7. It is therefore enough to show that the canonical map $H_*(\mathrm{sk}_n\,S) \to H_*(|\,\mathrm{sk}_n\,S|)$ is an isomorphism for each n. We want to prove this by induction on n (we already know the case $n = 0$ since $|\operatorname{sk}_0 S|$ is the discrete set S_0). Set

$$C_\bullet(\operatorname{sk}_n S, \operatorname{sk}_{n-1} S) := C_\bullet(\operatorname{sk}_n S)/C_\bullet(\operatorname{sk}_{n-1} S),$$

then we have a commutative diagram of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_\bullet(\operatorname{sk}_{n-1} S) & \longrightarrow & C_\bullet(\operatorname{sk}_n S) & \longrightarrow & C_\bullet(\operatorname{sk}_n S, \operatorname{sk}_{n-1} S) & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & S_\bullet(|\operatorname{sk}_{n-1} S|) & \longrightarrow & S_\bullet(|\operatorname{sk}_n S|) & \longrightarrow & S_\bullet(|\operatorname{sk}_n S|, |\operatorname{sk}_{n-1} S|) & \longrightarrow & 0 \end{array}$$

where the rows are short exact sequences. From this we get a commutative diagram of homology long exact sequences from Exercise 4.4:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_{i+1}(\operatorname{sk}_n S, \operatorname{sk}_{n-1} S) & \xrightarrow{\partial} & H_i(\operatorname{sk}_{n-1} S) & \longrightarrow & H_i(\operatorname{sk}_n S) & \longrightarrow & H_i(\operatorname{sk}_n S, \operatorname{sk}_{n-1} S) & \xrightarrow{\partial} & H_{i-1}(\operatorname{sk}_{n-1} S) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_{i+1}(|\operatorname{sk}_n S|, |\operatorname{sk}_{n-1} S|) & \xrightarrow{\partial} & H_i(|\operatorname{sk}_{n-1} S|) & \longrightarrow & H_i(|\operatorname{sk}_n S|) & \longrightarrow & H_i(|\operatorname{sk}_n S|, |\operatorname{sk}_{n-1} S|) & \xrightarrow{\partial} & H_{i-1}(|\operatorname{sk}_{n-1} S|) & \longrightarrow & \cdots \end{array}$$

Suppose we already know that $H_*(\operatorname{sk}_{n-1} S) \to H_*(|\operatorname{sk}_{n-1} S|)$ is an isomorphism. The chain complex $C_\bullet(\operatorname{sk}_n S, \operatorname{sk}_{n-1} S)$ is very simple: it has $\mathbb{Z}S_n$ in degree n and 0 everywhere else, so

$$H_*(\operatorname{sk}_n S, \operatorname{sk}_{n-1} S) = \begin{cases} \mathbb{Z}S_n, & * = n, \\ 0, & \text{otherwise.} \end{cases}$$

By Remark 5.6.2 this is the same as the relative singular homology $H_*(|\operatorname{sk}_n S|, |\operatorname{sk}_{n-1} S|)$. If we can show that the map between the two is an isomorphism, we could then apply 5-Lemma to conclude that $H_*(\operatorname{sk}_n S) \to H_*(|\operatorname{sk}_n S|)$ must be an isomorphism too.

Unwinding the definitions, the map

$$H_n(\operatorname{sk}_n S, \operatorname{sk}_{n-1} S) \to H_n(|\operatorname{sk}_n S|, |\operatorname{sk}_{n-1} S|)$$

takes $s \in S_n$ to the homology class represented by the composite

$$\Delta^n \xrightarrow{e_s} |\operatorname{sk}_n S| \to |\operatorname{sk}_n S|/|\operatorname{sk}_{n-1} S| \cong \bigvee_{s \in S_n} S^n,$$

where e_s is the inclusion of the n-simplex labelled by s. This factors through

$$\Delta^n \to \Delta^n/\partial\Delta^n \cong S^n \to \bigvee_{s \in S_n} S^n$$

where the last map is the inclusion of the sphere labelled by s. But we know from Proposition 4.6.2 that the quotient map $\Delta^n \to \Delta^n/\partial\Delta^n$ is a generator of $H_n(S^n)$. Hence the map we're looking at takes generators to generators and so is an isomorphism. In conclusion, we have proved:

Theorem 5.7.2. *Let S be a Δ-set. Then the canonical map $S \to \operatorname{Sing}|S|$ induces an isomorphism*

$$H_*(S) \xrightarrow{\sim} H_*(|S|)$$

between the simplicial and singular homology of $|S|$. □

5.8 Cellular Chains

We can exploit the long exact sequences we have been looking at further to obtain a new method for computing homology of cell complexes in general, not just for Δ-complexes.

Definition 5.8.1. Let X be a cell complex with Γ_n its set of n-cells. The group of *cellular n-chains* is the free abelian group

$$C_n^{\text{cell}}(X) := H_n(X_n, X_{n-1}) \cong \mathbb{Z}\Gamma_n.$$

The differential $d\colon C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X)$ is the composite

$$H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X_{n-1}, X_{n-2}),$$

where ∂ is the boundary map for the pair (X_n, X_{n-1}) and the second comes from the map of pairs $(X_{n-1}, \varnothing) \to (X_{n-1}, X_{n-2})$.

Remark 5.8.2. Note that the two maps we are composing appear in two *different* long exact sequences: ∂ occurs in the long exact sequence for (X_n, X_{n-1}) while j_{n-1} occurs in that for (X_{n-1}, X_{n-2}).

Remark 5.8.3. The map $j_n\colon H_n(X_n) \to H_n(X_n, X_{n-1})$ that appears in the definition of d is injective: the preceding term in the long exact sequence for (X_n, X_{n-1}) is $H_n(X_{n-1}) = 0$.

Lemma 5.8.4. *$d^2 = 0$, so that $(C_\bullet^{\text{cell}}(X), d)$ is a chain complex.*

Proof. The map $d^2\colon C_n^{\text{cell}}(X) \to C_{n-2}^{\text{cell}}(X)$ is the composite

$$H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X_{n-1}, X_{n-2}) \xrightarrow{\partial} H_{n-2}(X_{n-2}) \xrightarrow{j_{n-2}} H_{n-2}(X_{n-2}, X_{n-3}).$$

Here the two middle maps are adjacent in the long exact sequence for (X_{n-1}, X_{n-2}) and so their composite is 0. □

Proposition 5.8.5. *The homology of the cellular chain complex of X is the homology of X, i.e. there are isomorphisms*

$$H_*(C_\bullet^{\text{cell}}(X), d) \cong H_*(X).$$

Proof. We compute the cycles and boundaries in the cellular chain complex:

$$\begin{aligned} Z_n^{\text{cell}}(X) &= \ker(d\colon C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X)) \\ &\cong \ker(\partial\colon H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1})) && (j_{n-1} \text{ injective}) \\ &\cong \operatorname{im}(j_n\colon H_n(X_n) \to H_n(X_n, X_{n-1})) && (\text{by exactness}) \\ &\cong H_n(X_n) && (j_n \text{ injective}) \end{aligned}$$

$$\begin{aligned} B_n^{\text{cell}}(X) &= \operatorname{im}(d\colon C_{n+1}^{\text{cell}}(X) \to C_n^{\text{cell}}(X)) \\ &\cong \operatorname{im}(\partial\colon H_{n+1}(X_{n+1}, X_n) \to H_n(X_n)) && (j_n \text{ injective}) \end{aligned}$$

Thus the homology group $H_n(C^{\text{cell}}(X), d) = Z_n^{\text{cell}}(X)/B_n^{\text{cell}}(X)$ is isomorphic (via the injective map j_n) to $H_n(X_n)/(\operatorname{im}\partial)$. But in the long exact sequence for (X_{n+1}, X_n) we have

$$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial} H_n(X_n) \to H_n(X_{n+1}) \to 0$$

(since $H_n(X_{n+1}, X_n) = 0$), so that this quotient is isomorphic to $H_n(X_{n+1}) \cong H_n(X)$, as required. □

Corollary 5.8.6. *If X has a cell structure with only even-dimensional cells, then*

$$H_*(X) \cong C_*^{\text{cell}}(X) = \begin{cases} \mathbb{Z}\Gamma_*, & * \text{ even,} \\ 0, & * \text{ odd.} \end{cases}$$

Proof. In this case every other group in $C_*^{\text{cell}}(X)$ vanishes, so that the differentials must all be zero. Thus $Z_*^{\text{cell}}(X) = C_*^{\text{cell}}(X)$ and $B_*^{\text{cell}}(X) = 0$. □

Example 5.8.7. We saw that complex projective space $\mathbb{CP}^n$ has a cell structure with a single cell in every even dimension $\leq 2n$. Hence we get

$$H_*(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z}, & 0 \leq * \leq 2n \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we have

$$H_*(\mathbb{CP}^\infty) \cong \begin{cases} \mathbb{Z}, & 0 \leq * \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

5.9 *The Cellular Differential*

To do more computations we need a better understanding of the differential in the cellular chain complex. This was defined as the composite of two maps, the first being the boundary map $\partial \colon H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1})$. Here the cell structure provides a pushout square

$$\begin{array}{ccc} \coprod_{\alpha \in \Gamma_n} S^{n-1} & \hookrightarrow & \coprod_{\alpha \in \Gamma_n} D^n \\ \downarrow{\scriptstyle (f_\alpha)_{\alpha \in \Gamma_n}} & & \downarrow{\scriptstyle (e_\alpha)_{\alpha \in \Gamma_n}} \\ X_{n-1} & \hookrightarrow & X_n. \end{array}$$

Since the boundary map is natural, the map of pairs $(\coprod D^n, \coprod S^{n-1}) \to (X_n, X_{n-1})$ gives a commutative square

$$\begin{array}{ccc} H_n(\coprod D^n, \coprod S^{n-1}) & \xrightarrow{\partial} & H_{n-1}(\coprod S^{n-1}) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle H_{n-1}((f_\alpha)_{\alpha \in \Gamma_n})} \\ H_n(X_n, X_{n-1}) & \xrightarrow{\partial} & H_{n-1}(X_{n-1}). \end{array}$$

For $n > 1$, the boundary map $H_n(\coprod D^n, \coprod S^{n-1}) \to H_{n-1}(\coprod S^{n-1})$ is an isomorphism (since the two neighbouring terms in the long exact sequence are homology groups of D^n in dimensions $\neq 0$). Thus to understand the boundary map to $H_{n-1}(X_{n-1})$ it's equivalent to understand the map $H_{n-1}((f_\alpha)_{\alpha \in \Gamma_n})$ induced by the attaching maps $f_\alpha \colon S^{n-1} \to X_{n-1}$.

The second step in the boundary map is the map $j_{n-1}\colon H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$, which we can also (since $n-1>0$) think of as the map in homology induced by the quotient map

$$\pi\colon X_{n-1} \to X_{n-1}/X_{n-2} \cong \bigvee_{\Gamma_{n-1}} S^{n-1}.$$

We conclude that $d\colon C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X)$ corresponds under appropriate isomorphisms to the map on H_{n-1} induced by the composite

$$\coprod_{\Gamma_n} S^{n-1} \xrightarrow{(f_\alpha)_{\alpha\in\Gamma_n}} X_{n-1} \xrightarrow{\pi} \bigvee_{\Gamma_{n-1}} S^{n-1},$$

i.e. d is given by

$$\mathbb{Z}\Gamma_n \cong \bigoplus_{\Gamma_n} H_{n-1}(S^{n-1}) \cong H_{n-1}\left(\coprod_{\Gamma_n} S^{n-1}\right) \to H_{n-1}(X_{n-1}) \to \tilde{H}_{n-1}\left(\bigvee_{\Gamma_{n-1}} S^{n-1}\right) \cong \bigoplus \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}\Gamma_{n-1}.$$

We can isolate the effect on a single generator in Γ_n by restricting to the corresponding sphere in the coproduct, giving the composite map

$$S^{n-1} \xrightarrow{f_\alpha} X_{n-1} \xrightarrow{\pi} \bigvee_{\beta\in\Gamma_{n-1}} S^{n-1}.$$

In homology, the generalization of Corollary 4.9.8 to an arbitrary wedge implies that the resulting map $H_{n-1}(S^{n-1}) \to \bigoplus_{\beta\in\Gamma_{n-1}} H_{n-1}(S^{n-1})$ takes $x \in H_{n-1}(S^{n-1})$ to $\sum_\beta (p_\beta \pi f_\alpha)_* x$, where $p_\beta\colon \bigvee S^{n-1} \to S^{n-1}$ is the projection that takes the copy of S^{n-1} corresponding to β to S^{n-1} by the identity and collapses all the other copies to the base point.

If we write $q_\beta := p_\beta\pi\colon X_{n-1} \to S^{n-1}$ (which we can also think of as the quotient $X_{n-1}/(X_{n-1} \setminus e_\beta(D^{n,\circ}))$ where we collapse everything outside the interior of the cell β to a single point), then this means d is a sum of the maps $H_{n-1}(q_\beta f_\alpha)\colon H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$.

Then we have proved the following:

Proposition 5.9.1. *The cellular differential $d\colon \mathbb{Z}\Gamma_n \to \mathbb{Z}\Gamma_{n-1}$ takes the generator $\alpha \in \mathbb{Z}\Gamma_n$ to*

$$d(\alpha) = \sum_{\beta\in\Gamma_{n-1}} \deg(q_\beta f_\alpha)\beta$$

if $n > 1$.

Remark 5.9.2. For $n = 1$, the cellular differential is just the boundary map

$$H_1(X_1, X_0) \to H_0(X_0).$$

Here the map of pairs $(\coprod_{\Gamma_1} D^1, \coprod_{\Gamma_1} S^0) \to (X_1, X_0)$ gives a commutative square

$$\begin{array}{ccc} \bigoplus_{\Gamma_1} H_1(D^1, S^0) & \longrightarrow & \bigoplus_{\Gamma_1} H_0(S^0) \\ \downarrow\wr & & \downarrow \\ H_1(X_1, X_0) & \longrightarrow & H_0(X_0). \end{array}$$

We know that the boundary map $H_1(D^1, S^0) \to H_0(S^0)$ takes the generator represented by the relative cycle id_{D^1} to the difference

$[1] - [0]$ if these are the generators of $H_0 S^0$ corresponding to the two points of S^0. The generator in $H_1(X_1, X_0) \cong \mathbb{Z}\Gamma_1$ corresponding to $\alpha \in \Gamma_1$ is then sent to $f_\alpha(1) - f_\alpha(0)$ in $H_0(X_0) \cong \mathbb{Z}\Gamma_0$ for $f_\alpha \colon S^0 \to X_0$ the corresponding attaching map, so that

$$d\alpha = f_\alpha(1) - f_\alpha(0).$$

Remark 5.9.3. Since S^{n-1} is compact, $f_\alpha(S^{n-1})$ can only intersect the interiors of finitely many $(n-1)$-cells, hence only finitely many of these degrees can be non-zero.

Exercise 5.11. Compute the cellular homology of S^n using the cell structure with two cells in each dimension $\leq n$. [You need to keep track of the orientations of the generators. There are two ways to define this cell structure: either attach both cells using the identity, or attach one cell using the identity and one using an orientation-reversing map; it may be instructive to look at both.]

Exercise 5.12. Find a cell structure on the torus and compute the cellular homology.

Exercise 5.13. By the classification of finitely generated abelian groups, we can write any finitely generated abelian group A as a direct sum $\mathbb{Z}^r \oplus T$ where T is a torsion group (i.e. all its elements have finite order). The integer r is called the *rank* $\operatorname{rk} A$ of A. If $C_\bullet$ is a chain complex such that C_n is a finitely generated abelian group for all n, and vanishes except for finitely many n, then the *Euler characteristic* of $C_\bullet$ is

$$\chi(C_\bullet) = \sum_i (-1)^i \operatorname{rk} C_i.$$

(i) Show that
$$\chi(C_\bullet) = \sum_i (-1)^i \operatorname{rk} H_i(C).$$
[Assume that rank is additive in short exact sequences: if $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then $\operatorname{rk} B = \operatorname{rk} A + \operatorname{rk} C$.]

(ii) If X is a finite cell complex with Γ_n its set of of n-cells, the *Euler characteristic* of X is the alternating sum
$$\chi(X) = \sum_i (-1)^i |\Gamma_n|.$$
Prove that $\chi(X)$ is independent of the cell structure of X, and only depends on X up to homotopy equivalence.

5.10 The Homology of $\mathbb{RP}^n$

Using our earlier results on degrees, we can now compute the homology of real projective space $\mathbb{RP}^n$. Recall that this has a cell structure with a single cell e_n in each dimension $\leq n$, with attaching map the quotient map

$$f_i \colon S^{i-1} \to S^{i-1}/(x \sim -x) \cong \mathbb{RP}^{i-1}.$$

But we have to be careful here: when we form the pushout square

$$\begin{array}{ccc} S^{i-1} & \hookrightarrow & D^i \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle e_i} \\ \mathbb{RP}^{i-1} & \hookrightarrow & \mathbb{RP}^i \end{array}$$

we are thinking of $\mathbb{RP}^i$ as the quotient of D^i where we identify x with $-x$ for $x \in \partial D^i$. To form the next step in the filtration we need to identify this quotient with the quotient of S^i where we identify all x with $-x$ — and there are two natural ways to do this: we can map S^i to the quotient of D^i by identifying the interior of one hemisphere with the interior D^i and by first composing with $x \mapsto -x$ on the other hemisphere — but we have to choose on which hemisphere we do what. This gives two possible quotient maps $q_1, q_2 \colon S^i \to \mathbb{RP}^i$, related by $q_1 = q_2 \circ \mathrm{inv}_i$ where $\mathrm{inv}_i \colon S^i \to S^i$ is the inversion map $x \mapsto -x$. To precisely define the cell structure we must (arbitrarily) choose which map to use as f_{i+1}, though these choices will not affect the computation.

The degree of inv_i on S^i is $\det(-I_{i+1}) = (-1)^{i+1}$ by Corollary 4.9.4 (where I_{i+1} is the identity matrix in dimension $i+1$), so if i is even the map inv_i is definitely not homotopic to id.

The cellular chain complex looks like

$$\cdots \to 0 \to \mathbb{Z}e_n \to \mathbb{Z}e_{n-1} \to \cdots \to \mathbb{Z}e_0.$$

We need to compute $d(e_i)$, which for $i > 1$ is given by the degree of the composite map

$$g \colon S^{i-1} \xrightarrow{f_i} \mathbb{RP}^{i-1} \to \mathbb{RP}^{i-1}/\mathbb{RP}^{i-2} \xrightarrow{\sim} S^{i-1}.$$

The quotient map f_i takes the equator in S^{i-1} to $\mathbb{RP}^{i-2}$ so we can factor g as

$$S^{i-1} \xrightarrow{\mu} S^{i-1} \vee S^{i-1} \xrightarrow{g'} S^{i-1},$$

where μ is the map that collapses the equator to a point. From our choice of f_i we know that on one hemisphere it restricts to the quotient map $D^{i-1}/\partial D^{i-1} \to S^{i-1}$, and on the other it is this quotient composed with the antipodal map $\mathrm{inv}_{i-1} \colon S^{i-1} \to S^{i-1}$.

By Proposition 4.9.9 this means that $\deg g = \deg(\mathrm{id}) + \deg(\mathrm{inv}_{i-1})$. Moreover, by Corollary 4.9.4 we also know that $\deg(\mathrm{inv}_{i-1}) = (-1)^i$, so that

$$\deg g = 1 + (-1)^i = \begin{cases} 0, & i \text{ odd}, \\ 2, & i \text{ even}. \end{cases}$$

Thus $C_*^{\mathrm{cell}}(\mathbb{RP}^n)$ looks like

$$0 \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Here the lowest differential is just the boundary map $H_1(\mathbb{RP}^1, \mathbb{RP}^0) \to H_0(\mathbb{RP}^0)$ where $\mathbb{RP}^1 \cong S^1$ and $\mathbb{RP}^0 = *$, which is zero, for example since it takes a generator of $\tilde{H}_1(S^1)$ to its boundary, which is 0.

Computing the homology of this chain complex, we get:

Proposition 5.10.1. $H_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2, & i \text{ odd}, i < n, \\ 0, & i \text{ even} > 0 \text{ or } i > n, \\ \mathbb{Z}, & i = 0 \text{ or } i = n \text{ odd}. \end{cases}$

6
Homotopy Invariance and Excision

Having convinced ourselves that the Eilenberg–Steenrod axioms are useful for computing homology, we now proceed to prove that they are indeed satisfied for singular homology. Recall that the two properties we need to establish are homotopy invariance and excision. We start by considering homotopy invariance; the proof of this has two ingredients, which we first introduce separately: in §6.1 we consider the (somewhat) geometric construction of *exterior products* of singular chains, and in §6.2 we look at the algebraic notion of *chain homotopies*. We then put these together in §6.3 to prove that singular homology is homotopy-invariant. Next we turn to excision: In §6.4 we reduce the proof of excision to a *locality* property of singular chains, and in §6.5 we establish this using the *barycentric subdivision* of singular chains.

6.1 Exterior Product of Singular Chains

One ingredient in the proof of homotopy-invariance is a definition of "multiplication" of singular chains, which we will introduce first. These will be maps

$$\mu_{n,m}\colon S_n(X) \times S_m(Y) \to S_{n+m}(X \times Y).$$

The idea for defining $\mu_{n,m}$ is that given singular simplices $\sigma\colon \Delta^n \to X$ and $\tau\colon \Delta^m \to Y$, we have $\sigma \times \tau\colon \Delta^n \times \Delta^m \to X \times X'$. This is of course not an $(n+m)$-simplex (if $n, m \neq 0$). However, we can certainly *decompose* the product $\Delta^n \times \Delta^m$ into $(n+m)$-simplices, and if we pick some reasonable way of doing this (using maps $\alpha_i\colon \Delta^{n+m} \to \Delta^n \times \Delta^m$) then we can take $\mu_{n,m}(\sigma, \tau)$ to be the chain on $X \times Y$ given by the sum $\sum_i \pm(\sigma \times \tau) \circ \alpha_i$ of the restrictions of $\sigma \times \tau$ to these simplices, perhaps with some signs to get the correct orientations. The precise result we want is the following:

Theorem 6.1.1. *For topological spaces X, Y there exist bilinear maps*

$$S_n(X) \times S_m(Y) \xrightarrow{\mu_{n,m}} S_{n+m}(X \times Y),$$

such that

(i) $\mu_{n,m}$ is natural, i.e. for maps $f\colon X \to X'$, $g\colon Y \to Y'$, we have

$$\mu_{n,m}(f_*\sigma, g_*\tau) = (f \times g)_*\mu_{n,m}(\sigma, \tau).$$

(ii) ∂ *is a derivation for* μ*, i.e. we have*

$$\partial\mu_{n,m}(\sigma,\tau) = \mu_{n-1,m}(\partial\sigma,\tau) + (-1)^n\mu_{n,m-1}(\sigma,\partial\tau)$$

(where we interpret $\mu_{-1,m}$ *and* $\mu_{m,-1}$ *as* 0 *if* $n = 0$ *or* $m = 0$*).*

(iii) *For* $m = 0$*,* $\sigma\colon \Delta^n \to X$ *and* $x\colon \Delta^0 \cong * \to Y$*,* $\mu_{n,0}(\sigma,x)$ *is the* n*-simplex*

$$\Delta^n \cong \Delta^n \times \Delta^0 \xrightarrow{\sigma\times x} Y \times Y',$$

and similarly for $n = 0$*.*

Remark 6.1.2. If $\mu_{n,m}(\sigma,\tau)$ represents the product of σ and τ as a chain, then $\partial\mu_{n,m}(\sigma,\tau)$ should represent the oriented boundary of $\sigma\times\tau$. This boundary is $\partial\sigma\times\tau\cup\sigma\times\partial\tau$, which explains the two terms in the formula for $\partial\mu_{n,m}$; the sign is needed to get the correct orientations. For example, the boundary of $\Delta^1\times\Delta^1$ we can represent as the chain

$$\begin{aligned}\partial(\Delta^1\times\Delta^1) &= [00\to 10] + [10\to 11] - [01\to 11] - [00\to 01]\\ &= \Delta^1\times[0] + [1]\times\Delta^1 - \Delta^1\times[1] - [0]\times\Delta^1\\ &= ([1]-[0])\times\Delta^1 - \Delta^1\times([1]-[0])\\ &= \partial\Delta^1\times\Delta^1 + (-1)^1\Delta^1\times\partial\Delta^1.\end{aligned}$$

Exercise 6.1. Show that the exterior product $\mu_{n,m}$ induces bilinear maps in homology $H_n(X)\times H_m(Y)\to H_{n+m}(X\times Y)$.

Let $\iota_n\in S_n(\Delta^n)$ denote the singular n-simplex corresponding to the identity of Δ^n. Then for $\sigma\colon\Delta^n\to X$ any singular simplex, we have the tautologous identity $\sigma = \sigma_*\iota_n$. But then if we set $\iota_{n,m} := \mu_{n,m}(\iota_n,\iota_m)$, the naturality property of $\mu_{n,m}$ means that if $\mu_{n,m}$ exists it must satisfy

$$\mu_{n,m}(\sigma,\tau) = \mu_{n,m}(\sigma_*\iota_n,\tau_*\iota_m) = (\sigma\times\tau)_*\iota_{n,m}$$

for any $\sigma\in\mathrm{Sing}_n(X),\tau\in\mathrm{Sing}_m(Y)$; if σ and τ are chains we can similarly represent $\mu_{n,m}(\sigma,\tau)$ as a linear combination of terms of this form. Thus $\mu_{n,m}$ is completely determined by $\iota_{n,m}$. Thus to define $\mu_{n,m}$ is suffices to construct chains $\iota_{n,m}$ with properties that imply those in Theorem 6.1.1:

Proposition 6.1.3. *For all* n,m *there exists a chain* $\iota_{n,m}\in S_{n+m}(\Delta^n\times\Delta^m)$ *such that*

(i) $\partial\iota_{n,m} = \sum_{i=0}^n(-1)^i(d^i\times\mathrm{id})_*\iota_{n-1,m} + (-1)^n\sum_{j=0}^m(-1)^j(\mathrm{id}\times d^j)_*\iota_{n,m-1}$ *(where we interpret* $\iota_{-1,m}$ *and* $\iota_{n,-1}$ *as* 0 *if* $n=0$ *or* $m=0$*)*

(ii) $\iota_{0,m} = \iota_m$ *and* $\iota_{n,0} = \iota_n$*.*

Remark 6.1.4. Here the formula for $\partial\iota_{n,m}$ comes from the requirement

$$\partial\mu_{n,m}(\iota_n,\iota_m) = \mu_{n-1,m}(\partial\iota_n,\iota_m) + (-1)^n\mu_{n,m-1}(\iota_n,\partial\iota_m),$$

since we must have

$$\begin{aligned}\mu_{n-1,m}(\partial\iota_n,\iota_m) &= \sum_{i=0}^n(-1)^i\mu_{n-1,m}(\partial_i\iota_n,\iota_m)\\ &= \sum_{i=0}^n(-1)^i\mu_{n-1,m}(d^i_*\iota_{n-1},\iota_m)\\ &= \sum_{i=0}^n(-1)^i(d^i\times\mathrm{id})_*\iota_{n-1,m},\end{aligned}$$

and similarly for the other term. We can also think of this as a geometric decomposition of the oriented boundary of $\Delta^n \times \Delta^m$ into (faces of Δ^n)$\times\Delta^m$ and $\Delta^n\times$(faces of Δ^m).

Proof of Theorem 6.1.1. Given $\iota_{n,m}$ as in Proposition 6.1.3, we define

$$\mu_{n,m} \colon \operatorname{Sing}_n(X) \times \operatorname{Sing}_m(Y) \to S_{n+m}(X \times Y)$$

as $(\sigma, \tau) \mapsto (\sigma \times \tau)_* \iota_{n,m}$, and extend linearly to a map on $S_n(X) \times S_m(Y)$. By linearity it suffices to check the required properties for $\sigma \colon \Delta^n \to X$ and $\tau \colon \Delta^m \to Y$. For $f \colon X \to X'$, $g \colon Y \to Y'$ we have

$$\mu_{n,m}(f_*\sigma, g_*\tau) = \mu_{n,m}(f \circ \sigma, g \circ \tau) = ((f \circ \sigma) \times (g \circ \tau))_* \iota_{n,m} = (f \times g)_*(\sigma \times \tau)_* \iota_{n,m} = (f \times g)_* \mu_{n,m}(\sigma, \tau).$$

For the boundary, we get

$$\begin{aligned}
\partial \mu_{n,m}(\sigma, \tau) &= \partial(\sigma \times \tau)_* \iota_{n,m} \\
&= (\sigma \times \tau)_* \partial \iota_{n,m} \\
&= (\sigma \times \tau)_* \left(\sum_{i=0}^{n} (-1)^i (d^i \times \mathrm{id})_* \iota_{n-1,m} + (-1)^n \sum_{j=0}^{m} (-1)^j (\mathrm{id} \times d^j)_* \iota_{n,m-1} \right) \\
&= \sum_{i=0}^{n} (-1)^i (\sigma d^i \times \tau)_* \iota_{n-1,m} + (-1)^n \sum_{j=0}^{m} (-1)^j (\sigma \times \tau d^j)_* \iota_{n,m-1} \\
&= \sum_{i=0}^{n} (-1)^i \mu_{n-1,m}(\partial_i \sigma, \tau) + (-1)^n \sum_{j=0}^{m} (-1)^j \mu_{n,m-1}(\sigma, \partial_j \tau) \\
&= \mu_{n-1,m}(\partial\sigma, \tau) + (-1)^n \mu_{n,m-1}(\sigma, \partial\tau).
\end{aligned}$$

Finally, for $m = 0$ we have

$$\mu_{n,0}(\sigma, x) = (\sigma \times x)_* \iota_{n,0} = (\sigma \times x)_* \iota_n,$$

as required, and similarly for $n = 0$. □

We'll first give a concrete proof of Proposition 6.1.3 in the case $n = 1$ (which is all we need for homotopy invariance). This will use an explicit decomposition of the cylinder $\Delta^1 \times \Delta^m$; a proof for general n can be obtained by similarly decomposing $\Delta^n \times \Delta^m$ but this unsurprisingly gets more complicated. Alternatively, there is an abstract proof that shows that appropriate classes $\iota_{n,m}$ exist without giving any explicit choice thereof.

Concrete proof of Proposition 6.1.3, $n = 1$. We label the vertices of $\Delta^1 \times \Delta^n$ as $[i] = ([0], [i])$ and $[\bar{i}] = ([1], [i])$ and denote the face with vertices $v_1, \ldots, v_n$ as $[v_1 \cdots v_n]$.

The *face* means the polygon with these vertices, i.e. their convex hull in $\Delta^1 \times \Delta^n$ (which is embedded in $\mathbb{R}^{n+3}$).

Define $R_{i,n} \colon \Delta^{n+1} \to \Delta^1 \times \Delta^n$ as the inclusion of the face $[0 \cdots i\bar{i} \cdots \bar{n}]$ (so we *repeat* the vertex i in both the lower and upper copy of Δ^n). This satisfies:

$$\partial_j R_{i,n} = \begin{cases} (\mathrm{id} \times d^j)_* R_{i-1,n-1}, & j < i \quad \text{(both give the face } [0 \cdots (j-1)(j+1) \cdots i\bar{i} \cdots \bar{n}]) \\ \partial_i R_{i-1,n}, & j = i \quad \text{(both give the face } [0 \cdots \cdots (i-1)\bar{i} \cdots \bar{n}]) \\ \partial_{i+1} R_{i+1,n}, & j = i+1 \quad \text{(both give the face } [0 \cdots i\overline{(i+1)} \cdots \bar{n}]) \\ (\mathrm{id} \times d^{j-1})_* R_{i,n-1}, & j > i+1 \quad \text{(both give the face } [0 \cdots i\bar{i} \cdots \overline{(j-2)}\bar{j} \cdots \bar{n}]) \end{cases}$$

If we set $\iota_{1,n} := \sum_{i=0}^{n}(-1)^i R_{i,n}$, then we get

$$
\begin{aligned}
\partial\iota_{1,n} &= \sum_{i=0}^{n}(-1)^i \partial R_{i,n} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j}\partial_j R_{i,n} \\
&= \sum_{0\leq j<i\leq n}(-1)^{i+j}(\mathrm{id}\times d^j)_* R_{i-1,n-1} \\
&\quad + \sum_{i=0}^{n}(-1)^{2i}\partial_i R_{i,n} + \sum_{i=0}^{n}(-1)^{2i+1}\partial_{i+1}R_{i,n} \\
&\quad + \sum_{\substack{0\leq i\leq n \\ i+1<j\leq n+1}}(-1)^{i+j}(\mathrm{id}\times d^{j-1})_* R_{i,n-1} \\
&= \sum_{0\leq j\leq i<n}(-1)^{i+j+1}(\mathrm{id}\times d^j)_* R_{i,n-1} \\
&\quad + \partial_0 R_{0,n} + \sum_{i=1}^{n}\partial_i R_{i,n} - \sum_{i=0}^{n-1}\partial_{i+1}R_{i,n} - \partial_{n+1}R_{n,n} \\
&\quad + \sum_{0\leq i<j\leq n}(-1)^{i+j+1}(\mathrm{id}\times d^j)_* R_{i,n-1} \\
&= \partial_0 R_{0,n} - \partial_{n+1}R_{n,n} - \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}(\mathrm{id}\times d^j)_* R_{i,n-1} \\
&= (d^0\times \mathrm{id})_*\iota_{0,n} - (d^1\times\mathrm{id})_*\iota_{0,n} - \sum_{j=0}^{n}(-1)^j(\mathrm{id}\times d^j)_*\iota_{1,n-1},
\end{aligned}
$$

as required. □

Abstract proof of Proposition 6.1.3. We'll see below in Exercise 6.4 that $\tilde{H}_*(\Delta^n) = 0$, without assuming homotopy invariance. Thus if $\sigma \in S_k(\Delta^n)$ for $k > 0$ is a cycle, it must also be a boundary. This means that to show there exists *some* chain with a prescribed boundary it's enough to show that this hypothetical boundary is a cycle. We can apply this to inductively choose chains ι_{n+m} as the degree $n+m$ increases: supposing we have found $\iota_{k,l}$ for all k, l with $k+l < n+m$, then we want $\iota_{n,m}$ (for $n, m > 0$) to be a chain whose boundary is

$$\sum_{i=0}^{n}(-1)^i(d^i\times\mathrm{id})_*\iota_{n-1,m} + (-1)^n\sum_{j=0}^{m}(-1)^j(\mathrm{id}\times d^j)_*\iota_{n,m-1}.$$

To show such a chain exists, we just need to check this is a cycle, which we leave as an exercise. □

The idea of this proof is called the "method of acyclic models", the "acyclic model" being $S_\bullet(\Delta^n)$; we will encounter this again several times further on in the course.

Exercise 6.2. Suppose the formula

$$\partial\iota_{k,l} = \sum_{i=0}^{k}(-1)^i(d^i\times\mathrm{id})_*\iota_{k-1,l} + (-1)^k\sum_{j=0}^{l}(-1)^j(\mathrm{id}\times d^j)_*\iota_{k,l-1}$$

holds for $\iota_{n-1,m}$ and $\iota_{n,m-1}$. Show that then the chain

$$\sum_{i=0}^{n}(-1)^i(d^i\times\mathrm{id})_*\iota_{n-1,m} + (-1)^n\sum_{j=0}^{m}(-1)^j(\mathrm{id}\times d^j)_*\iota_{n,m-1}$$

is a cycle.

Warning 6.1.5. Note that the chains $\iota_{n,m}$ in Proposition 6.1.3 are not *unique*, and hence neither are the maps $\mu_{n,m}$. However, one can show that all choices give the same maps in homology.

Remark 6.1.6. We can also define exterior products of relative singular chains: Given pairs (X, A) and (Y, B) we can define a map

$$\mu_{n,m} \colon S_n(X, A) \times S_m(Y, B) \to S_{n+m}(X \times Y, X \times B \cup A \times Y)$$

by taking $\mu_{n,m}([\sigma], [\tau])$ to be the image of $\mu_{n,m}(\sigma, \tau)$. This is well-defined: for $\alpha \in S_n(A)$ and $\beta \in S_m(B)$ we have

$$\mu_{n,m}(\sigma + \alpha, \tau + \beta) = \mu_{n,m}(\sigma, \tau) + \mu_{n,m}(\alpha, \tau) + \mu_{n,m}(\sigma, \beta) + \mu_{n,m}(\alpha, \beta),$$

where all but the first term lie in $S_{n+m}(X \times B \cup A \times Y)$. Hence $[\mu_{n,m}(\sigma + \alpha, \tau + \beta)] = [\mu_{n,m}(\sigma, \tau)]$ in $S_{n+m}(X \times Y, X \times B \cup A \times Y)$.

6.2 *Chain Homotopies*

The other ingredient in the proof of homotopy invariance is an algebraic notion of homotopy, namely *chain homotopies* between chain maps.

Definition 6.2.1. Let $f, g \colon A_\bullet \to B_\bullet$ be two chain maps. A *chain homotopy* h from f to g consists of homomorphisms $h_n \colon A_{n-1} \to B_n$ for every n, such that

$$f_n(a) - g_n(a) = \partial h_{n+1}(a) + h_n(\partial a). \tag{6.1}$$

We say that two chain maps are *(chain) homotopic* if there exists a chain homotopy between them.

Exercise 6.3.

(i) Show that being chain homotopic is an equivalence relation on the set of chain maps $A_\bullet \to B_\bullet$.

(ii) Show the chain homotopies are compatible with compositions: if $f, g \colon A_\bullet \to B_\bullet$ are chain homotopic, then so are ϕf and ϕg for any chain map $\phi \colon B_\bullet \to B'_\bullet$, and likewise for $f\psi$ and $g\psi$ for any $\psi \colon A'_\bullet \to A_\bullet$.

(iii) Define the *homotopy category* of chain complexes, where the objects are chain complexes and the set of morphisms from $A_\bullet$ to $B_\bullet$ is the set of chain homotopy classes of chain maps.

Remark 6.2.2. If a is a cycle ($\partial a = 0$) then the chain homotopy gives $f(a) - g(b) = \partial h_n(a)$, exhibiting the two cycles $f(a)$ and $g(a)$ as agreeing up to a specified boundary. We should think of this as an algebraic analogue of a homotopy in topology: just as a homotopy between continuous maps $f, g \colon X \to Y$ species a path from $f(x)$ to $g(x)$ for all x, a chain homotopy between $f, g \colon A_\bullet \to B_\bullet$ specifies an algebraic "path" between $f(a)$ and $g(a)$ in the form of a boundary. We will see later (after we've talked about tensor products) that we can reformulate the definition of chain homotopy in a way that makes the two notions look more similar than they do currently. To justify the preicse form of (6.1) note that for a boundary ∂a we must have

$$f(\partial a) - g(\partial a) = \partial h_n(\partial a)$$

but the left-hand side is also $\partial(f(a) - g(a))$ so it is natural that the term $h_n(\partial a)$ appear.

Proposition 6.2.3. *If $f, g\colon A_\bullet \to B_\bullet$ are homotopic chain maps, then the induced maps in homology are the same, i.e.*

$$H_n(f) = H_n(g)\colon H_n(A) \to H_n(B).$$

Proof. If h is a chain homotopy from f to g, then for a homology class $[a]$ represented by a cycle a, we have

$$f_n(a) - g_n(a) = \partial h_{n+1}(a)$$

since $\partial a = 0$, and so

$$H_n(f)[a] = [f_n(a)] = [g_n(a) + \partial h_{n+1}(a)] = [g_n(a)] = H_n(g)[a]. \qquad \square$$

Definition 6.2.4. A chain map $f\colon A_\bullet \to B_\bullet$ is a *chain homotopy equivalence* if there exists a chain map $g\colon B_\bullet \to A_\bullet$ such that gf is chain homotopic to $\mathrm{id}_{A_\bullet}$ and fg is chain homotopic to $\mathrm{id}_{B_\bullet}$; g is then called a *chain homotopy inverse* of f.

By the same argument as in Remark 4.4.5, we have:

Corollary 6.2.5. *if $f\colon A_\bullet \to B_\bullet$ is a chain homotopy equivalence with chain homotopy inverse g, then $f_*\colon H_*(A) \to H_*(B)$ is an isomorphism with inverse g_*.* $\square$

In other words, homology takes chain homotopy equivalences to isomorphisms, i.e. chain homotopy equivalences are *quasi-isomorphisms* in the following sense:

Definition 6.2.6. A chain map $f\colon A_\bullet \to B_\bullet$ is a *quasi-isomorphism* if the induced map in homology

$$f_*\colon H_*(A) \to H_*(B)$$

is an isomorphism of graded abelian groups.

6.3 *Proof of Homotopy Invariance*

We are now ready to prove homotopy invariance of singular homology. Recall that what we want to prove is the following:

Theorem 6.3.1. *If $f, g\colon (X, A) \to (Y, B)$ are homotopic maps of pairs, then the induced maps on singular homology agree, i.e.*

$$f_* = g_*\colon H_n(X, A) \to H_n(Y, B).$$

We will deduce this from Proposition 6.2.3, so what we'll actually prove is the following:

Proposition 6.3.2. *If $f, g\colon (X, A) \to (Y, B)$ are homotopic maps of pairs, then the induced maps on singular chains*

$$f_*, g_*\colon S_\bullet(X, A) \to S_\bullet(Y, B)$$

are chain homotopic.

Proof. We'll show that any homotopy of pairs $h\colon X \times \Delta^1 \to Y$ induces a chain homotopy. To do so we use the exterior multiplication maps (extended to relative chains as in Remark 6.1.6)

$$\mu_{1,m}\colon S_1(\Delta^1) \times S_m(X, A) \to S_{m+1}(\Delta^1 \times X, \Delta^1 \times A).$$

These satisfy $\partial\mu_{1,m}(\sigma, \tau) = \mu_{0,m}(\partial\sigma, \tau) - \mu_{1,m-1}(\sigma, \partial\tau)$. In particular, we can take σ to be the chain ι_1 corresponding to the identity of Δ^1, with $\partial\iota_1 = [1] - [0]$. If we define $\nu_{m+1}(\tau) := \mu_{1,m}(\iota_1, \tau)$ then we get

$$\partial\nu_{m+1}(\tau) = \{1\} \times \tau - \{0\} \times \tau - \nu_m(\partial\tau),$$

where we used property (iii) in Theorem 6.1.1 to identify $\mu_{0,m}(\partial\iota_1, \tau)$.

Since h is a homotopy of pairs, we can think of it as a map of pairs $(\Delta^1 \times X, \Delta^1 \times A) \to (Y, B)$ and so it induces a chain map $h_*\colon S_\bullet(\Delta^1 \times X, \Delta^1 \times A) \to S_\bullet(Y, B)$. Then the composites $h_*\nu_m\colon S_m(X, A) \to S_{m+1}(Y, B)$ give a chain homotopy: we have

$$\partial h_*\nu_{m+1}(\tau) = g_*(\tau) - f_*(\tau) - h_*\nu_m(\partial\tau),$$

which is precisely the identity required of a chain homotopy between g_* and f_*. □

6.4 Locality and Excision

Our next goal is to prove the excision property for singular homology:

Theorem 6.4.1. *Let (X, A) be a subspace pair and $U \subseteq A$ a subspace such that $\overline{U} \subseteq A^\circ$. Then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism in homology*

$$H_*(X \setminus U, A \setminus U) \xrightarrow{\sim} H_*(X, A).$$

It is notationally convenient to formulate this slightly differently: if $B := X \setminus U$ then the condition is that $A^\circ \cup B^\circ = X$ so that the interiors of A and B form an open cover of X, and in this case the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ should induce an isomorphism

$$H_*(B, A \cap B) \xrightarrow{\sim} H_*(X, A).$$

A key ingredient in the proof is the *locality property* of singular homology, which essentially says that to compute the homology of X it is enough to look only at simplices whose images are contained in either A or B. We will now formulate this more precisely, and then use it to prove excision; we will give the proof of locality in the next section.

Definition 6.4.2. A chain $\gamma \in S_n(X)$ is called *small* (with respect to the cover $\{A, B\}$) if it is a linear combination of simplices $\sigma\colon \Delta^n \to X$ such that the image $\sigma(\Delta^n)$ is contained either in A or in B. The small chains form a subgroup of $S_n(X)$ which we denote $S'_n(X)$. Since the boundary of small chain is clearly again small, we get a subcomplex $S'_\bullet(X) \subseteq S_\bullet(X)$. The subcomplex $S_\bullet(A)$ is contained in $S'_\bullet(X)$, and we define $S'_\bullet(X, A)$ as the quotient $S'_\bullet(X)/S_\bullet(A)$.

Theorem 6.4.3 (Locality)**.** *The inclusion $S'_\bullet(X) \to S_\bullet(X)$ induces isomorphisms in homology.*

Remark 6.4.4. More generally, we can define small chains with respect to any open cover, and the analogue of Theorem 6.4.3 still holds.

We have a commutative square of chain complexes

$$\begin{array}{ccc} S_\bullet(A \cap B) & \hookrightarrow & S_\bullet(B) \\ \downarrow & & \downarrow \\ S_\bullet(A) & \hookrightarrow & S'_\bullet(X), \end{array}$$

where all four maps are injective; in particular, there is an induced map of quotient complexes

$$S_\bullet(B, A \cap B) \to S'_\bullet(X, A).$$

Lemma 6.4.5. *The chain map $S_\bullet(B, A \cap B) \to S'_\bullet(X, A)$ is an isomorphism.*

Proof. We need to check we have an isomorphism $S_n(B, A \cap B) \xrightarrow{\sim} S'_n(X, A)$ for every n. To see that this is injective, suppose $[\alpha] \in S_n(B, A \cap B)$ is in the kernel, where α is a representative in $S_n(B)$. Then when we view α as an element in $S'_n(X)$ it maps to 0 in $S'_n(X, A)$, which means it is in the image of $S_n(A)$. But then the chain α lies in both $S_n(A)$ and $S_n(B)$, which means it is a linear combination of simplices whose images lie in both A and B, and hence in $A \cap B$, i.e. α lies in the subgroup $S_n(A \cap B)$. This implies $[\alpha] = 0$ in $S_n(B, A \cap B)$ as required.

To see that the map is surjective, consider $[\gamma] \in S'_n(X, A)$, represented by $\gamma \in S'_n(X)$. Since γ is small, we can write it as a sum $\gamma = \alpha + \beta$ where $\alpha \in S_n(A)$ and $\beta \in S_n(B)$. But then $[\gamma] = [\beta]$ in $S'_n(X)$ and here $[\beta]$ is in the image of $S_n(B, A \cap B)$. □

Proof of Theorem 6.4.1. From the inclusions $S_\bullet(A) \subseteq S'_\bullet(X) \subseteq S_\bullet(X)$ we get a canonical morphism of quotient complexes $S'_\bullet(X, A) \to S_\bullet(X, A)$, which lives in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_\bullet(A) & \longrightarrow & S'_\bullet(X) & \longrightarrow & S'_\bullet(X, A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_\bullet(A) & \longrightarrow & S_\bullet(X) & \longrightarrow & S_\bullet(X, A) & \longrightarrow & 0 \end{array}$$

where the rows are short exact sequences. We get an associated diagram of long exact sequences in homology:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(S'_\bullet(X)) & \longrightarrow & H_n(S'_\bullet(X, A)) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(S'_\bullet(X)) & \longrightarrow & \cdots \\ & & \| & & \downarrow\wr & & \downarrow & & \| & & \downarrow\wr & & \\ \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow & \cdots \end{array}$$

Here the isomorphisms follow from Theorem 6.4.3. We can then apply the 5-Lemma to conclude that $H_n(S'_\bullet(X, A)) \to H_n(X, A)$ is an isomorphism.

The chain map $S_\bullet(B, A \cap B) \to S_\bullet(X, A)$ factors as

$$S_\bullet(B, A \cap B) \to S'_\bullet(X, A) \to S_\bullet(X, A),$$

where the first map is an isomorphism by Lemma 6.4.5, and we just saw the second induces isomorphisms in homology. □

6.5 *Barycentric Subdivision*

We now want to prove the locality property, Theorem 6.4.3. To do so we will use a construction called *barycentric subdivision*. This will give as natural chain maps

$$\mathrm{bs}^X \colon S_\bullet(X) \to S_\bullet(X)$$

and a chain homotopy ρ^X from bs^X to the identity, i.e.

$$\mathrm{bs}^X(\alpha) - \alpha = \partial\rho^X(\alpha) + \rho^X(\partial\alpha)$$

(where we drop the superscript X from the notation when it is clear from context). These will have the following key properties:

(1) for every chain $\alpha \in S_\bullet(X)$ there exists an integer k such that $\mathrm{bs}^n(X)$ is small for $n \geq k$.

(2) for every small chain α the chain $\rho(\alpha)$ is also small.

Before we turn to the definition, let's see that this suffices to prove locality:

Proof of Theorem 6.4.3. We must show that the homomorphism

$$H_n(S'_\bullet(X)) \to H_n(X)$$

is an isomorphism for every n. To see that it is surjective, consider a class $[\alpha] \in H_n(X)$ represented by a cycle $\alpha \in S_n(X)$. The chain homotopy ρ implies that $[\alpha] = [\mathrm{bs}(\alpha)]$ and so $[\alpha] = [\mathrm{bs}^k(\alpha)]$ for every k, and by assumption $\mathrm{bs}^k(\alpha)$ is small for k sufficiently large, so $[\alpha]$ is in the image of $H_n(S'_\bullet(X))$. To see that the map is injective, suppose $[\alpha] = 0$ in $H_n(X)$ for some $\alpha \in S'_n(X)$. Then $\alpha = \partial\beta$ where $\beta \in S_{n+1}(X)$ need not be small. But then $\mathrm{bs}^k(\alpha) = \partial\,\mathrm{bs}^k(\beta)$ and for k sufficiently large we know $\mathrm{bs}^k(\beta)$ is also small, so $[\mathrm{bs}^k(\alpha)] = 0$ also in $H_n(S'_\bullet(X))$. Moreover, property (2) implies that ρ restricts to a chain homotopy between bs and the identity also on $S'_\bullet(X)$, so that $[\alpha] = [\mathrm{bs}^k(\alpha)]$ also holds in $H_n(S'_\bullet(X))$. Thus $[\alpha] = 0$ also here, as required. □

Now we need to show such maps bs^X and ρ^X actually exist. To define bs^X we will use the following procedure:

- Define a chain $\beta_n \in S_n(\Delta^n)$; this will be $\mathrm{bs}^{\Delta^n}(\iota_n)$ where $\iota_n \in S_n(\Delta^n)$ denotes the chain corresponding to the identity map id_{Δ^n}.
- For $\sigma \colon \Delta^n \to X$, define $\mathrm{bs}^X(\sigma) := \sigma_*\beta_n$, and extend linearly to define bs^X on chains.

The chain β_n will represent a subdivision of Δ^n into smaller simplices, the *barycentric subdivision*, i.e. it will be a (signed) sum of the inclusions $\Delta^n \hookrightarrow \Delta^n$ of these smaller simplices; geometrically this chain represents all of Δ^n.

Roughly speaking, the barycentric subdivision of Δ^n is obtained by first taking the subdivision of the faces of Δ^n and then adding a new vertex at the "barycentre" (or centre of mass) of Δ^n, with edges connecting this to all the vertices in the subdivision of the boundary, and so on for higher-dimensional faces.

To give a more precise definition, we introduce the *cone construction* of simplices:

Definition 6.5.1. Suppose $K \subseteq \mathbb{R}^d$ is a convex set (so for any points $x, y \in K$, the line segment between x and y is also contained in K). Given a singular simplex $\alpha\colon \Delta^n \to K$ and a point $p \in K$ we define $\operatorname{cone}_p \alpha\colon \Delta^{n+1} \to K$ to be the continuous map given by

$$\operatorname{cone}_p(\alpha)(t_0, \ldots, t_{n+1}) = \begin{cases} t_0 p + (1-t_0)\alpha(t'_0, \ldots, t'_n) & t_0 \neq 1, \\ p, & t_0 = 1, \end{cases}$$

where $t'_i = t_{i+1}/(1-t_0)$ (so that $(t'_0, \ldots, t'_n)$ lies in $\Delta^n \subseteq \mathbb{R}^{n+1}$) We can extend this linearly to obtain a homomorphism

$$\operatorname{cone}_p\colon S_n(K) \to S_{n+1}(K).$$

Lemma 6.5.2. $\partial \operatorname{cone}_p(\alpha) = \alpha - \operatorname{cone}_p(\partial\alpha)$.

Proof. From the definition we see that if $\alpha\colon \Delta^n \to K$ is a simplex, then $\partial_0 \operatorname{cone}_p \alpha = \alpha$, while $\partial_i \operatorname{cone}_p \alpha = \operatorname{cone}_p(\partial_{i-1}\alpha)$ for $i > 0$. Hence

$$\partial \operatorname{cone}_p(\alpha) = \sum_{i=0}^{n+1} (-1)^i \partial_i \operatorname{cone}_p(\alpha) = \alpha - \sum_{i=0}^{n} (-1)^i \operatorname{cone}_p(\partial_i \alpha),$$

where the sum is precisely $\operatorname{cone}_p(\partial\alpha)$. □

Exercise 6.4. Use the cone construction to show directly (without using homotopy invariance) that for any convex subset $K \subseteq \mathbb{R}^n$ we have $H_*(K) = 0$, $* > 0$.

Definition 6.5.3. Viewing $\Delta^n \subseteq \mathbb{R}^{n+1}$ as the subset of points $t = (t_0, \ldots, t_n)$ with $t_i \geq 0$ and $\sum_i t_i = 1$, the *barycenter* of Δ^n is the point $z = (1/(n+1), \ldots, 1/(n+1))$. We inductively define classes $\beta_n \in S_n(\Delta^n)$ and homomorphisms $\operatorname{bs}_n^X\colon S_n(X) \to S_n(X)$ as follows:

- set $\beta_0 := \iota_0$,
- given β_n, for $\sigma\colon \Delta^n \to X$ define $\operatorname{bs}_n^X(\sigma) := \sigma_*\beta_n$ and extend linearly in σ,
- set $\beta_{n+1} := \operatorname{cone}_z(\operatorname{bs}_n(\partial\iota_{n+1}))$.

Note that this gives $\operatorname{bs}_0^X := \operatorname{id}$.

Lemma 6.5.4. $\mathrm{bs}^X\colon S_\bullet(X)\to S_\bullet(X)$ *is a chain map, and it is natural in* X*, i.e. given a continuous map* $f\colon X\to Y$ *we have a commutative square*

$$\begin{array}{ccc} S_\bullet(X) & \xrightarrow{\mathrm{bs}^X} & S_\bullet(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ S_\bullet(Y) & \xrightarrow{\mathrm{bs}^Y} & S_\bullet(Y). \end{array}$$

Proof. To prove naturality it suffices to check that the two maps agree at a simplex $\sigma\colon\Delta^n\to X$. We have

$$f_*\,\mathrm{bs}^X(\sigma)=f_*(\sigma_*\beta_n)=(f\sigma)_*\beta_n=\mathrm{bs}^Y(f_*\sigma),$$

as required. We now prove that bs^X is a chain map for all X by checking that $\partial\,\mathrm{bs}^X_n=\mathrm{bs}^X_{n-1}\,\partial$ by induction on n. For $\sigma\colon\Delta^n\to X$ we have

$$\partial\,\mathrm{bs}^X_n(\sigma)=\partial\sigma_*\beta_n=\sigma_*(\partial\beta_n),$$

and Lemma 6.5.2 gives

$$\partial\beta_n=\partial\,\mathrm{cone}_z(\mathrm{bs}^{\Delta^n}_{n-1}\,\partial\iota_n)=\mathrm{bs}^{\Delta^n}_{n-1}\,\partial\iota_n-\mathrm{cone}_z(\partial\,\mathrm{bs}^{\Delta^n}_{n-1}\,\partial\iota_n).$$

Applying the inductive hypothesis to $\mathrm{bs}^{\Delta^n}_{n-1}$, we see that the last term vanishes and so $\partial\beta_n=\mathrm{bs}^{\Delta^n}_{n-1}\,\partial\iota_n$, which means

$$\partial\,\mathrm{bs}^X_n(\sigma)=\sigma_*(\mathrm{bs}^{\Delta^n}_{n-1}\,\partial\iota_n)=\mathrm{bs}^X_{n-1}(\sigma_*\partial\iota_n)=\mathrm{bs}^X_{n-1}(\partial\sigma).$$

To start the induction, note that as $\mathrm{bs}^X_0=\mathrm{id}$, for $n=1$ we get $\partial\beta_1=\partial\iota_1+\mathrm{cone}_z(\partial^2\iota_1)=\mathrm{bs}_0(\partial\iota_1)$. □

Fact 6.5.5. *If* (Y,d) *is a compact metric space and* $\mathcal{U}=\{U_i : i\in I\}$ *is an open cover of* Y*, then there exists* $\lambda>0$ *such that any subset of* Y *of diameter* $<\lambda$ *lies in one of the subsets* U_i*. Such a* λ *is called a* Lebesgue number *of* $\mathcal{U}$*.*

For a proof, you could try the Wikipedia article on "Lebesgue's number lemma".

Lemma 6.5.6. $(\mathrm{bs}^X)^k\alpha$ *is small for* k *sufficiently large.*

Proof. It's enough to consider a simplex $\alpha\colon\Delta^n\to X$. Then $\alpha^{-1}A^\circ$ and $\alpha^{-1}B^\circ$ give an open cover of Δ^n. This is a compact metric space, so there exists a Lebesgue number λ. But from the definition of $\mathrm{bs}^{\Delta^n}_n$ we see that there is a real number $\epsilon<1$ such that the simplices in $(\mathrm{bs}^{\Delta^n}_n)^k(\iota_n)$ all have diameter $<\epsilon^k$. For k sufficiently large we have $\epsilon^k<\lambda$ and so each simplex in the chain $(\mathrm{bs}^{\Delta^n}_n)^k(\iota_n)$ is contained in one of the open sets in the cover. In other words, $(\mathrm{bs}^X)^k\alpha=\alpha_*(\mathrm{bs}^{\Delta^n}_n)^k(\iota_n)$ is small. □

Now we want to define the chain homotopy $\rho^X_n\colon S_nX\to S_{n+1}X$. We proceed by the same method as before:

- we first define a chain $R_n\in S_{n+1}(\Delta^n)$ that will be $\rho^{\Delta^n}_n(\iota_n)$, starting with taking $R_0\in S_1(\Delta^0)$ to be the unique simplex $\Delta^1\to\Delta^0$,
- for $\sigma\colon\Delta^n\to X$ we set $\rho^X_n(\sigma):=\sigma_*R_n$ and extend linearly in σ.

For ρ_n^X to be a chain homotopy between bs^X and id we certainly need

$$\partial R_n = \partial \rho_n^{\Delta^n}(\iota_n) = -\rho_{n-1}(\partial \iota_n) + \mathrm{bs}_n^{\Delta^n} \iota_n - \iota_n.$$

In fact *any* R_n with this property will give a chain homotopy:

Lemma 6.5.7. *Given chains R_n with the prescribed boundaries, the resulting homomorphisms ρ_n^X are a natural chain homotopy.*

The proof is similar to that of Lemma 6.5.4 and is left as an exercise:

Exercise 6.5. Suppose we have chains $R_n \in S_{n+1}(\Delta^n)$ with R_0 the unique simplex $\Delta^1 \to \Delta^0$, and define $\rho_n^X : S_n(X) \to S_{n+1}(X)$ by $\rho_n^X(\sigma) := \sigma_* R_n$ for $\sigma \colon \Delta^n \to X$ and extending linearly in σ. Show (by induction on n) that if the chains R_n satisfy

$$\partial R_n = -\rho_{n-1}(\partial \iota_n) + \mathrm{bs}_n^{\Delta^n} \iota_n - \iota_n$$

then the homomorphisms ρ_n^X are a natural chain homotopy between bs^X and id.

So we just need to find *some* choice of R_n's. But we know $H_n(\Delta^n) = 0$, so if the right-hand side is a cycle it is a boundary, and hence a suitable R_n must exist. So we just have to compute

$$\partial \left(-\rho_{n-1}(\partial \iota_n) + \mathrm{bs}_n^{\Delta^n} \iota_n - \iota_n\right) = -\rho_{n-1}(\partial^2 \iota_n) - \mathrm{bs}_{n-1}^{\Delta^n}(\partial \iota_n) + \partial \iota_n + \partial\, \mathrm{bs}_n^{\Delta^n}(\iota_n) - \partial \iota_n = 0,$$

where we assumed we already knew ρ_{n-1} satisfied the chain homotopy equation. So we can proceed by induction to define R_n, since for $n = 0$ we have $\partial R_0 = 0$ by the definition of R_0.

We are left with making the following observation:

Lemma 6.5.8. *If $\alpha \in S_n(X)$ is small then $\rho_n^X \alpha \in S_{n+1}(X)$ is also small.*

Proof. It suffices to consider a small simplex $\alpha \colon \Delta^n \to X$. Then $\rho_n^X \alpha = \alpha_*(\rho_n^{\Delta^n} \iota_n)$, which means all simplices in the chain $\rho_n^X \alpha$ are given by composing with α and hence have image contained in $\alpha(\Delta^n)$. That means these simplices are all small too, hence so is $\rho_n^X \alpha$. □

7
Tensor Products and Homology with Coefficients

In this chapter we will define variants of singular homology with *coefficients* in any abelian group M using chain complexes $S_\bullet(X; M)$; an element of $S_n(X; M)$ should be a "linear combination" $\sum a_i \sigma_i$ with $\sigma_i \colon \Delta^n \to X$, but now with the coefficients a_i living in M. To make this precise we first need to introduce the *tensor product* of abelian groups in §7.1 before we define homology with coefficients in §7.2. We then need to discuss a little homological algebra, specifically the *torsion product* of abelian groups, in §7.3, which we use in §7.4 to prove the *universal coefficient theorem*, which describes the relation between homology with coefficients in M and the singular homology (with coefficients in $\mathbb{Z}$) we have studied so far. In §7.5 we show that we can also extend the cellular chains on a cell complex to compute homology with coefficients.

7.1 Tensor Products of Abelian Groups

Definition 7.1.1. If A, B, C are abelian groups, a *bilinear map*

$$\phi \colon A \times B \to C$$

is a map of sets that satisfies

$$\phi(a + a', b) = \phi(a, b) + \phi(a', b), \quad \phi(a, b + b') = \phi(a, b) + \phi(a, b')$$

for all $a, a' \in A, b, b' \in B$. A *tensor product* of A and B is an abelian group $A \otimes B$ together with a *universal* bilinear map $u \colon A \times B \to A \otimes B$: for every bilinear map $\phi \colon A \times B \to C$ there exists a *unique* homomorphism $f \colon A \otimes B \to C$ such that $\phi = f \circ u$, i.e.

$$\begin{array}{ccc} A \times B & \xrightarrow{u} & A \otimes B \\ & \searrow^{\phi} & \Big\downarrow{\scriptstyle \exists!} \\ & & C. \end{array}$$

As with any definition of an object by a universal property, if the tensor product exists it is unique up to unique isomorphism.

Lemma 7.1.2. *For any abelian groups A, B, their tensor product exists.*

Proof. Let $F = \mathbb{Z}(A \times B)$ be the free abelian group on the set $A \times B$. Then by the universal property of F, a morphism of sets $\phi \colon A \times B \to C$

where C is an abelian group corresponds to a unique homomorphism $\phi'\colon F \to C$. Moreover, ϕ is a bilinear map if and only if the subgroup R of F generated by $(a+a',b)-(a,b)-(a',b)$ and $(a,b+b')-(a,b)-(a,b')$ for all $a,a' \in A$, $b,b' \in B$, is contained in the kernel of ϕ'. Setting $T := F/R$, then by the universal property of the quotient we get a correspondence between bilinear maps $A \times B \to C$ and homomorphisms $T \to C$, given by composing with the map $u\colon A \times B \to F \to T$ where the first map is the inclusion of the generators and the second is the quotient map. This shows that (T,u) is a tensor product of A and B. □

Remark 7.1.3. For $a \in A, b \in B$, we write $a \otimes b \in A \otimes B$ for $u(a,b)$. From the explicit construction we see that an element of $A \otimes B$ can be written as a finite sum $\sum a_i \otimes b_i$ of elements of this form. Since u is bilinear, we have

$$(a+a') \otimes b = a \otimes b + a' \otimes b,$$
$$a \otimes (b+b') = a \otimes b + a \otimes b',$$
$$(na) \otimes b = n(a \otimes b) = a \otimes nb,\ n \in \mathbb{Z}.$$

Warning 7.1.4. A general element of $A \otimes B$ can *not* be written as $a \otimes b$.

The following exercises give some examples of tensor products:

Exercise 7.1. For integers n,m, show that $\mathbb{Z}/n \otimes \mathbb{Z}/m \cong \mathbb{Z}/r$ where $r = \gcd(n,m)$ is the greatest common divisor of n and m. In particular, if p and q are distinct primes, then $\mathbb{Z}/p \otimes \mathbb{Z}/q \cong 0$.

Exercise 7.2. For sets S,T, show that there is a canonical isomorphism

$$\mathbb{Z}S \otimes \mathbb{Z}T \cong \mathbb{Z}(S \times T).$$

We now state the basic formal properties of the tensor product; proving them using the universal property is left as an exercise.

Proposition 7.1.5.

(i) *For abelian groups A,B, there is a canonical isomorphism*

$$A \otimes B \xrightarrow{\sim} B \otimes A$$

taking $a \otimes b$ to $b \otimes a$.

(ii) *For abelian groups A,B,C there is a canonical isomorphism*

$$A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C.$$

(iii) *For abelian groups $A_i, i \in I$ and B, there is a canonical isomorphism*

$$\bigoplus_{i \in I} A_i \otimes B \xrightarrow{\sim} \left(\bigoplus_{i \in I} A_i\right) \otimes B.$$

(iv) *For an abelian group A, there are canonical isomorphisms*

$$\mathbb{Z} \otimes A \xrightarrow{\sim} A,$$
$$0 \otimes A \xrightarrow{\sim} 0.$$

(v) *For homomorphisms $f\colon A \to A'$, $g\colon B \to B'$, there is a canonical homomorphism*

$$f \otimes g\colon A \otimes B \to A' \otimes B'$$

taking $a \otimes b$ to $f(a) \otimes g(b)$. Moreover this is compatible with composition (so $\otimes$ is a functor $\mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$).

(vi) *For abelian groups A, A', B, B' and a homomorphism $f\colon A \to A'$ the tensor product $f \otimes 0$ with the zero map $0\colon B \to B'$ is the zero map $0\colon B \otimes A \to B' \otimes A'$.*

(vii) *For homomorphisms $f, g\colon A \to A'$, $h\colon B \to B'$, we have $(f+g) \otimes h = f \otimes h + g \otimes h$ as homomorphisms $A \otimes B \to A' \otimes B'$.*

Exercise 7.3. Prove the formal properties of $\otimes$ using the universal property.

Lemma 7.1.6 (Tensor product preserves cokernels)**.** *For M an abelian group and an exact sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

(so that $C \cong B/\operatorname{im} f$) the sequence

$$A \otimes M \xrightarrow{f \otimes \mathrm{id}} B \otimes M \xrightarrow{g \otimes \mathrm{id}} C \otimes M \to 0$$

is also exact.

Proof. The cokernel of $f \otimes \mathrm{id}$ has the universal property that any homomorphism $\phi\colon B \otimes M \to N$ such that $\phi \circ (f \otimes \mathrm{id}) = 0$ factors uniquely through it. We thus need to check that $g \otimes \mathrm{id}$ has this universal property.

A homomorphism $\phi\colon B \otimes M \to N$ corresponds to a bilinear map $\phi'\colon B \times M \to N$, and $\phi \circ (f \otimes \mathrm{id}) = 0$ holds if and only if $\phi' \circ (f \times \mathrm{id}) = 0$. But then we can define a bilinear map $\psi\colon C \times M \to N$ by

$$\psi(gx, y) = \phi'(x, y),$$

which is well-defined since $gx = gx'$ means $x = x' + f(a)$ and then

$$\phi'(x, y) = \phi'(x', y) + \phi'(f(a), y) = \phi'(x', y).$$

Moreover, ψ is clearly the unique bilinear map that factors ϕ' through $g \times \mathrm{id}$, which means that it corresponds to the unique homomorphism $C \otimes M \to N$ that factors ϕ through $g \otimes \mathrm{id}$. □

Remark 7.1.7. This implies in particular that if $f\colon A \to B$ is surjective, then so is $f \otimes \mathrm{id}\colon A \otimes M \to B \otimes M$ for any M. However, if f is injective, the map $f \otimes \mathrm{id}$ need not be injective. For example, consider the injective map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ (given by multiplication by 2) — if we tensor this with $\mathbb{Z}/2$ we get $\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$, which is certainly not injective. This implies that in general the functor $- \otimes M$ does not preserve short exact sequences. We now consider two special situations where exactness *is* preserved:

Lemma 7.1.8. *If $F = \mathbb{Z}S$ is a free abelian group, then $- \otimes F$ preserves short exact sequences.*

Proof. Since $F \cong \bigoplus_{s \in S} \mathbb{Z}$ and we know $\otimes$ preserves direct sums, we have $A \otimes F \cong \bigoplus_{s \in S} A$. For a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0,$$

tensoring with F therefore gives

$$0 \to \bigoplus_{s \in S} A \xrightarrow{\bigoplus_{s \in S} i} \bigoplus_{s \in S} B \xrightarrow{\bigoplus_{s \in S} q} \bigoplus_{s \in S} C \to 0.$$

This is exact since a direct sum of exact sequences is always exact by Exercise 3.4. □

Lemma 7.1.9. *If* $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$ *is a splittable short exact sequence, then for any abelian group the sequence*

$$0 \to A \otimes M \xrightarrow{i \otimes \mathrm{id}} B \otimes M \xrightarrow{q \otimes \mathrm{id}} C \otimes M \to 0$$

is exact, and is again splittable.

Proof. Choosing a splitting $s \colon C \to B$ gives an isomorphism of the exact sequence with

$$0 \to A \to A \oplus C \to C \to 0.$$

Since $- \otimes M$ preserves direct sums, tensoring this with M we get

$$0 \to A \otimes M \to A \otimes M \oplus C \otimes M \to C \otimes M \to 0,$$

which is obviously again exact, and this is isomorphic to the tensor product of the original sequence with M, with the isomorphism given by the splitting $s \otimes \mathrm{id}$. □

As an important special case, we have:

Lemma 7.1.10. *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence where* C *is a free abelian group, then*

$$0 \to A \otimes M \to B \otimes M \to C \otimes M$$

is a (splittable) short exact sequence for every M*.*

Proof. Any such short exact sequence is splittable, by Exercise 4.3. □

Remark 7.1.11. Everything we've done in this section works exactly the same over an arbitrary commutative ring R: If M, N, K are R-modules, we can define an R-bilinear map $\phi \colon M \times N \to K$ to be a morphism of sets such that

$$\phi(m + m', n) = \phi(m, n) + \phi(m', n), \quad \phi(m, n + n') = \phi(m, n) + \phi(m, n'), \quad \phi(rm, n) = r\phi(m, n) = \phi(m, rn),$$

for $m, m' \in M, n, n' \in N, r \in R$. There exists a universal R-bilinear map $M \times N \to M \otimes_R N$, and the relative tensor product $\otimes_R$ has the same formal properties as those we prove for $\otimes = \otimes_{\mathbb{Z}}$. Note that if k is a field and V is a k-module (k-vector space), then $\otimes_k V$ always preserves short exact sequences (since every k-module is free).

Exercise 7.4. Show that $M \otimes_R N$ is the quotient of $M \otimes N$ by the subgroup generated by elements of the form $rm \otimes n - m \otimes rn$ for $r \in R$, $m \in M$, $n \in N$.

Exercise 7.5.

(i) Let R be an (associative, unital) ring. Show that an R-module is the same as an abelian group M and a homomorphism $\alpha\colon R \otimes M \to M$ such that the square

$$\begin{array}{ccc} R \otimes R \otimes M & \xrightarrow{\mathrm{id}_R \otimes \alpha} & R \otimes M \\ \big\downarrow{\scriptstyle \mu \otimes \mathrm{id}_M} & & \big\downarrow{\scriptstyle \mu} \\ R \otimes M & \xrightarrow{\alpha} & M \end{array}$$

and the triangle

$$\begin{array}{ccccc} M & \xrightarrow{\sim} & \mathbb{Z} \otimes M & \xrightarrow{\eta \otimes \mathrm{id}_M} & R \otimes M \\ & \diagdown\!\!\diagdown & & & \big\downarrow{\scriptstyle \alpha} \\ & & & & M \end{array}$$

commute, where the homomorphism $\mu\colon R \otimes R \to R$ is given by multiplication in R and $\eta\colon \mathbb{Z} \to R$ is given by the unit of R (i.e. $\eta(1) = 1$).

(ii) Show that an R-module homomorphism $\phi\colon M \to N$ is the same as a homomorphism of abelian groups such that the square

$$\begin{array}{ccc} R \otimes M & \xrightarrow{\mathrm{id} \otimes \phi} & R \otimes N \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{\phi} & N \end{array}$$

commutes.

(iii) Show that if M is an abelian group and R is a ring, then $R \otimes M$ has a natural R-module structure. [Hint: Use the multiplication in R.]

(iv) Show that if M is an abelian group and N is an R-module, there is a natural correspondence between R-module homomorphisms $R \otimes M \to N$ and homomorphisms of abelian groups $M \to N$.

(v) If S is a set and R is a ring, show that $R \otimes \mathbb{Z}S$ has the universal property of the *free* R-module RS on S: R-module homomorphisms $RS \to M$ correspond to functions $S \to M$.

Exercise 7.6. If k is a field and V, W are k-vector spaces, with bases $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$, respectively, show that $x_i \otimes y_j$ is a basis for $V \otimes_k W$. If V and W are finite-dimensional, conclude that

$$\dim(V \otimes_k W) = \dim V \cdot \dim W.$$

7.2 *Homology with Coefficients*

Definition 7.2.1. Let $C_\bullet$ be a chain complex and M an abelian group. We define a chain complex $C_\bullet \otimes M$ given levelwise by the tensor products $C_n \otimes M$, with boundary maps

$$\partial \otimes \mathrm{id}_M\colon C_n \otimes M \to C_{n-1} \otimes M.$$

By Proposition 7.1.5 we have

$$(\partial \otimes \mathrm{id}_M)^2 = \partial^2 \otimes \mathrm{id}_M = 0 \otimes \mathrm{id}_M = 0$$

so this is a chain complex. This gives a functor $- \otimes M\colon \mathsf{Ch} \to \mathsf{Ch}$.

Lemma 7.2.2. *There is a natural map $H_k(C) \otimes M \to H_k(C \otimes M)$.*

Proof. By definition, we have an exact sequence

$$B_k(C) \to Z_k(C) \to H_k(C) \to 0,$$

and so an exact sequence

$$B_k(C) \otimes M \to Z_k(C) \otimes M \to H_k(C) \otimes M \to 0$$

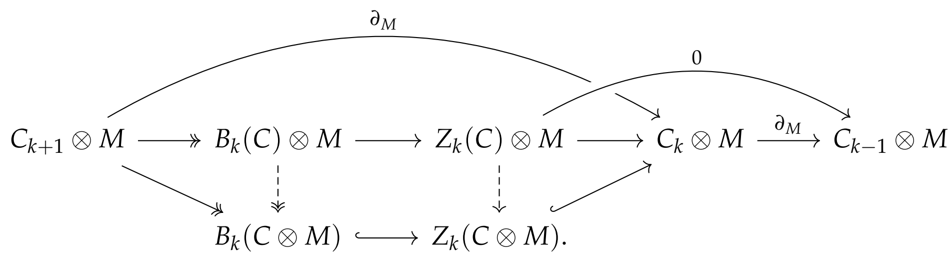
by Lemma 7.1.6. To construct a natural map $H_k(C) \otimes M \to H_k(C \otimes M)$ it is therefore enough to show that the boundary maps

$$\partial_M = \partial \otimes \mathrm{id}_M \colon C_n \otimes M \to C_{n-1} \otimes M$$

induce commutative squares

$$\begin{array}{ccc} B_k(C) \otimes M & \longrightarrow & Z_k(C) \otimes M \\ \downarrow & & \downarrow \\ B_k(C \otimes M) & \longrightarrow & Z_k(C \otimes M). \end{array}$$

This is true because ∂_M factors through $B_k(C) \otimes M$ and $Z_k(C) \otimes M$, so that we have a commutative diagram



Here the dashed arrow $B_k(C) \otimes M \to B_k(C \otimes M)$ exists since $C_{k+1} \otimes M \to B_k(C) \otimes M$ is surjective by Lemma 7.1.6, and so factors uniquely through the image $B_k(C \otimes M)$, while the dashed arrow $Z_k(C) \otimes M \to Z_k(C \otimes M)$ exists since the composite $Z_k \otimes M \to C_{k-1} \otimes M$ is zero, and so factors uniquely through the kernel $Z_k(C \otimes M)$. □

Exercise 7.7. Show that for $C_\bullet$ a chain complex, the natural map

$$H_k(C) \otimes \mathbb{Z} \to H_k(C \otimes \mathbb{Z}) \cong H_k C$$

is an isomorphism.

Remark 7.2.3. However, since tensoring does not preserve short exact sequences, this natural map is typically *not* an isomorphism.

Exercise 7.8. Show that a chain homotopy h between chain maps $f, g \colon C_\bullet \to D_\bullet$ induces a chain homotopy between $f \otimes M, g \otimes M \colon C_\bullet \otimes M \to D_\bullet \otimes M$ for any abelian group M.

Definition 7.2.4. For a topological space X and an abelian group M we define $S_\bullet(X; M) := S_\bullet(X) \otimes M$, with homology $H_n(X; M) := H_n(S_\bullet(X; M))$. This is the *singular homology of X with coefficients in M*. Similarly, for a subspace pair (X, A) we define $S_\bullet(X, A; M) := S_\bullet(X, A) \otimes M$ giving the relative singular homology of (X, A) with coefficients in M as

$$H_n(X, A; M) := H_n(S_\bullet(X, A; M)).$$

Remark 7.2.5. Note that $H_n(X, A; \mathbb{Z})$ is just the usual singular homology we have been calling $H_n(X, A)$ so far.

Remark 7.2.6. Since $S_n(X)$ is the free abelian group on $\mathsf{Sing}_n(X)$, the abelian group $S_n(X; M)$ is isomorphic to $\bigoplus_{\sigma \in \mathsf{Sing}_n(X)} M$. We can thus think of elements of $S_n(X; M)$ as finite sums $\sum_{i=1}^{n} m_i \sigma_i$ with $m_i \in M$ and $\sigma_i \in \mathsf{Sing}_n(X)$. If R is a commutative ring, then $S_n(X; M) \otimes R$ is isomorphic to the free R-module $R\,\mathsf{Sing}_n(X)$ on the set $\mathsf{Sing}_n(X)$.

Remark 7.2.7. Homology with M-coefficients is a composite of functors

$$\mathsf{Pair} \xrightarrow{S_\bullet} \mathsf{Ch} \xrightarrow{-\otimes M} \mathsf{Ch} \xrightarrow{H_n} \mathsf{Ab},$$

and so is functorial in continuous maps of pairs.

Remark 7.2.8. If R is a ring and M is an R-module, then $H_*(X; M)$ also has a natural R-module structure. This is because we can lift $- \otimes M$ to a functor $\mathsf{Ab} \to \mathsf{Mod}_R$, where Mod_R denotes the category of R-modules.

Remark 7.2.9. The coefficients we are interested in are typically the field $\mathbb{Q}$ (and sometimes $\mathbb{R}$ and $\mathbb{C}$) and the finite fields $\mathbb{F}_p = \mathbb{Z}/p$. This is because homology with field coefficients is in some ways better-behaved than with integral coefficients, and it can often be easier in practice to compute the homology "one prime at a time". Although we cannot exactly recover $H_*(X)$ from $H_*(X; \mathbb{Q})$ and $H_*(X; \mathbb{F}_p)$, in some ways these homologies taken together contains "the same information" about X as the integral homology groups. (We will see one example of this below in Theorem 7.4.8.)

We use the convention that $\mathbb{F}_p$ denotes $\mathbb{Z}/p$ when we think of it as a ring (or field), while we write $\mathbb{Z}/p$ when we only consider the additive structure.

Lemma 7.2.10. *If (X, A) is a subspace pair and M is an abelian group, then there is a long exact sequence*

$$\cdots \to H_n(A; M) \to H_n(X; M) \to H_n(X, A; M) \xrightarrow{\partial} H_{n-1}(A; M) \to \cdots$$

with the first two maps coming from the inclusions $A \hookrightarrow X$ and $(X, \emptyset) \hookrightarrow (X, A)$.

Proof. Tensoring the short exact sequence

$$0 \to S_\bullet(A) \to S_\bullet(X) \to S_\bullet(X, A) \to 0$$

with M, we get

$$0 \to S_\bullet(A; M) \to S_\bullet(X; M) \to S_\bullet(X, A; M) \to 0,$$

which is again a short exact sequence by Lemma 7.1.10, since the abelian group $S_n(X, A)$ is free (on the set $\mathsf{Sing}_n(X) \setminus \mathsf{Sing}_n(A)$) for every n. We therefore get a long exact sequence in homology of the required form. □

Remark 7.2.11. We also have Mayer–Vietoris sequences with coefficients in M. We can use these long exact sequences to redo many of the computations we did with $\mathbb{Z}$-coefficients. In particular, we have

$$\tilde{H}_*(S^n; M) \cong \begin{cases} M, & * = n, \\ 0, & * \neq n. \end{cases}$$

Proposition 7.2.12. *Singular homology with coefficents in an abelian group M satisfies the Eilenberg–Steenrod axioms of Definition 4.4.3, except that we have*

$$H_*(*; M) \cong \begin{cases} M, & * = 0, \\ 0, & * \neq 0. \end{cases}$$

Proof. It remains to check that $H_*(–; M)$ satisfies the homotopy, excision, and additivity axioms.

Recall that we proved in Theorem 6.3.1 that a homotopy between maps of pairs $f, g\colon (X, A) \to (Y, B)$ induces a chain homotopy between the chain maps $f_*, g_*\colon S_\bullet(X, A) \to S_\bullet(Y, B)$. Tensoring with an abelian group M preserves chain homotopies by Exercise 7.8, so that we also get a chain homotopy between $f_*, g_*\colon S_\bullet(X, A; M) \to S_\bullet(Y, B; M)$, which implies the homotopy axiom.

To prove excision, note that the key properties of barycentric subdivision are preserved by tensoring with M, so that locality, Theorem 6.4.3, also holds for singular chains with M-coefficients. Our proof of excision in Theorem 6.4.1 therefore goes through also for $H_*(–; M)$.

The additivity axiom is also clear, since tensoring preserves direct sums. Finally, we recall that $S_\bullet(*)$ is given by $\mathbb{Z}$ in degrees ≥ 0, with the identity and 0 alternating as boundary maps, and 0 in negative degrees. Tensoring this with M we get the chain complex

$$\cdots \to M \xrightarrow{\text{id}} M \xrightarrow{0} M \to \cdots \xrightarrow{0} M \to 0 \to \cdots,$$

with homology $H_0(*; M) = M$ and $H_*(*; M) = 0$ for $* \neq 0$. □

Definition 7.2.13. If $\phi\colon M \to M'$ is a homomorphism of abelian groups, then ϕ induces a natural chain map

$$S_\bullet(X, A) \otimes M \xrightarrow{\text{id}\otimes\phi} S_\bullet(X, A) \otimes M',$$

and so natural homomorphisms in homology

$$\phi_*\colon H_n(X, A; M) \to H_n(X, A; M').$$

Proposition 7.2.14. *Suppose $0 \to M' \xrightarrow{i} M \xrightarrow{q} M'' \to 0$ is a short exact sequence of abelian groups. Then for any subspace pair (X, A) there is a long exact sequence*

$$\cdots \to H_n(X, A; M') \xrightarrow{i_*} H_n(X, A; M) \xrightarrow{q_*} H_n(X, A; M'') \to H_{n-1}(X, A; M') \to \cdots.$$

Proof. Since the abelian group $S_n(X, A)$ is free for all n, if we tensor the short exact sequence with $S_\bullet(X, A)$ we get a short exact sequence of chain complexes

$$0 \to S_\bullet(X, A; M') \xrightarrow{\text{id}\otimes i} S_\bullet(X, A; M) \xrightarrow{\text{id}\otimes q} S_\bullet(X, A; M'') \to 0$$

by Lemma 7.1.8. This gives the required long exact sequence in homology. □

Exercise 7.9. For an integer n, let us write $n\colon \mathbb{Z} \to \mathbb{Z}$ for the homomorphism given by multiplication with n.

(i) For any abelian group M, use the universal property of $\otimes$ to show that under the natural isomorphism $M \cong M \otimes \mathbb{Z}$ the homomorphism $\mathrm{id} \otimes n\colon M \otimes \mathbb{Z} \to M \otimes \mathbb{Z}$ corresponds to the homomorphism $M \to M$ given by multiplication with n.

(ii) Use the natural isomorphism from Exercise 7.7 to show that the natural map $m_*\colon H_n(X, A; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$ induced by $m\colon \mathbb{Z} \to \mathbb{Z}$ on coefficients is again given by multiplication with m.

Corollary 7.2.15. *Let (X, A) be a subspace pair. Then for any integer m there are short exact sequences*

$$0 \to H_n(X, A) \otimes \mathbb{Z}/m \to H_n(X, A; \mathbb{Z}/m) \to \mathrm{Tor}(\mathbb{Z}/m, H_{n-1}(X, A)) \to 0,$$

where $\mathrm{Tor}(\mathbb{Z}/m, H_{n-1}(X, A))$ denotes the m-torsion subgroup of $H_{n-1}(X, A)$ (i.e. the subgroup of elements x such that $mx = 0$).

Proof. Applying Proposition 7.2.14 to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0,$$

we get a long exact sequence

$$\cdots \to H_n(X, A) \xrightarrow{m} H_n(X, A) \to H_n(X, A; \mathbb{Z}/n) \to H_{n-1}(X, A) \to \cdots,$$

and so short exact sequences

$$0 \to \mathrm{coker}\left(H_n(X, A) \xrightarrow{m} H_n(X, A)\right) \to H_n(X, A; M) \to \ker\left(H_{n-1}(X, A) \xrightarrow{m} H_{n-1}(X, A)\right) \to 0.$$

Here we can identify

$$\mathrm{coker}\left(H_n(X, A) \xrightarrow{m} H_n(X, A)\right) \cong H_n(X, A) \otimes \mathbb{Z}/m,$$

$$\ker\left(H_{n-1}(X, A) \xrightarrow{m} H_{n-1}(X, A)\right) \cong \mathrm{Tor}(\mathbb{Z}/m, H_{n-1}(X, A)),$$

using Exercise 7.9, which gives the required short exact sequences. □

Example 7.2.16. Let us compute $H_*(\mathbb{RP}^n; \mathbb{F}_p)$ where p is a prime. Recall that

$$H_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2, & i \text{ odd}, i < n, \\ 0, & i \text{ even} > 0, \\ \mathbb{Z}, & i = 0 \text{ or } i = n \text{ odd}. \end{cases}$$

If $p = 2$ the short exact sequences from Corollary 7.2.15 take the form

$$0 \to \mathbb{Z}/2 \to H_i(\mathbb{RP}^n; \mathbb{F}_2) \to 0 \to 0, \quad (i = 0 \text{ or } i \leq n \text{ odd}),$$

$$0 \to 0 \to H_i(\mathbb{RP}^n; \mathbb{F}_2) \to \mathbb{Z}/2 \to 0, \quad (i \text{ even} > 0),$$

so that we get

$$H_*(\mathbb{RP}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{Z}/2, & 0 \leq * \leq n, \\ 0, & * > n. \end{cases}$$

On the other hand, if p is an odd prime, we get

$$H_*(\mathbb{RP}^n;\mathbb{F}_p) \cong \begin{cases} \mathbb{Z}/p, & * = 0 \text{ or } * = n \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

In particular, we see that homology with different coefficients can look quite different. For example, if n is even and p is an odd prime then the $\mathbb{F}_p$-homology of $\mathbb{RP}^n$ is the same as that of a point.

Example 7.2.17. Sometimes homology with coefficients can detect more than integral homology. For example, consider a continuous map $f\colon S^n \to S^n$ of degree d, and form a cell complex X by attaching an $(n+1)$-cell to S^n along f, so that we have a pushout square

$$\begin{array}{ccc} S^n & \hookrightarrow & D^{n+1} \\ \downarrow f & & \downarrow \\ S^n & \hookrightarrow & X. \end{array}$$

The space X is called the *Moore space* of $\mathbb{Z}/d$ in degree n. Its integral homology is given by

$$\tilde{H}_*(X) \cong \begin{cases} \mathbb{Z}/d, & * = n, \\ 0, & \text{otherwise,} \end{cases}$$

as is easily seen using cellular homology, for example. We have a quotient map $q\colon X \to X/S^n \cong S^{n+1}$. In reduced integral homology this is the zero map, since $\tilde{H}_*(X)$ is 0 except in degree n and $\tilde{H}_*(S^{n+1})$ is 0 except in degree $n+1$. Thus integral homology cannot distinguish q from a constant map. On the other hand, from Corollary 7.2.15 we see that

$$\tilde{H}_*(X;\mathbb{Z}/d) \cong \begin{cases} \mathbb{Z}/d, & * = n, n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, in the long exact sequence for the pair (X, S^n) we have

$$\cdots \to 0 = H_{n+1}(S^n;\mathbb{Z}/d) \to H_{n+1}(X;\mathbb{Z}/d) \xrightarrow{q_*} \tilde{H}_{n+1}(S^{n+1};\mathbb{Z}/d) \to \cdots,$$

so that $q_*\colon \mathbb{Z}/d \to \mathbb{Z}/d$ is *injective* and so non-zero. Hence homology with $\mathbb{Z}/d$-coefficients sees that q is not homotopic to a constant map.

7.3 *Torsion Products*

Our next goal is to understand the relation between $H_*(X;M)$ and $H_*(X)$ for a general abelian group M. To do so we need a little bit of homological algebra related to tensor products. We will take advantage of the fact that homological algebra over $\mathbb{Z}$ is fairly simple to derive what we need without building any general machinery for derived functors.

Definition 7.3.1. Let A be an abelian group. A *free resolution* of A (of length 2) is a short exact sequence

$$0 \to F_1 \to F_0 \to A \to 0,$$

where F_0 and F_1 are free abelian groups.

Lemma 7.3.2. *Every abelian group has a free resolution of length 2.*

We need the following important fact about abelian groups:

Fact 7.3.3. *Every subgroup of a free abelian group is free.*

Proof of Lemma 7.3.2. Let A be an abelian group. Then there exists a surjection $s\colon F \to A$ with F free; for example we can always take $F := \bigoplus_{a\in A} \mathbb{Z}$, the free abelian group on the *set* A, with its canonical homomorphism to A. We have a short exact sequence

$$0 \to \ker s \to F \to A \to 0,$$

where $\ker s$ is free by Fact 7.3.3. □

Definition 7.3.4. Let A and B be abelian groups, and let

$$0 \to F_1 \xrightarrow{i} F_0 \to A \to 0$$

be a free resolution. The *torsion product* $\mathrm{Tor}(A, B)$ is the kernel of

$$i \otimes \mathrm{id}\colon F_1 \otimes B \to F_0 \otimes B.$$

Remark 7.3.5. Let $F_\bullet$ denote the chain complex $\cdots \to 0 \to F_1 \xrightarrow{i} F_0$. Then $H_0(F_\bullet \otimes B) \cong A \otimes B$ by Lemma 7.1.6, while $H_1(F_\bullet \otimes B) \cong \mathrm{Tor}(A, B)$.

Warning 7.3.6. The analogue of Fact 7.3.3 is *false* for R-modules over a general commutative ring R (though it does hold whenever R is a principal ideal domain). This means that an R-module M need not have a free resolution of length 2; it *is* always possible to find a free resolution, in the form of an exact sequence

$$\cdots \to F_2 \to F_2 \to F_1 \to F_0 \to M \to 0,$$

with the F_i's all free, but this may have to be infinitely long. We can then define a sequence of higher Tor functors by

$$\mathrm{Tor}_i^R(M, N) := H_i(F_\bullet \otimes_R N),$$

which can potentially be non-trivial for all $i \geq 0$.

We will see in a moment that $\mathrm{Tor}(A, B)$ is well-defined, but let us first mention some examples:

Example 7.3.7. If F is a free abelian group, then $0 \to F \xrightarrow{\mathrm{id}} F$ is a free resolution, so that $\mathrm{Tor}(F, B) = 0$ for any abelian group B since this is the kernel of $0 \cong 0 \otimes B \to F \otimes B$. We also have that $\mathrm{Tor}(A, F) = 0$ for any abelian group A, since tensoring with F preserves short exact sequences by Lemma 7.1.8.

Example 7.3.8. For an integer m, a free resolution of $\mathbb{Z}/m$ is given by the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0.$$

For an abelian group B, the torsion product $\mathrm{Tor}(\mathbb{Z}/m, B)$ is therefore isomorphic the kernel of the homomorphism $B \to B$ given by multiplication with m, i.e. the subgroup of m-torsion elements in B. This is the reason for the name "torsion product".

Exercise 7.10. Show that for integers n, m we have $\mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/r$, where $r = \gcd(n, m)$.

Definition 7.3.9. An abelian group B is *flat* if $-\otimes B$ preserves short exact sequences. By definition, this implies $\mathrm{Tor}(A, B) = 0$ for all A.

Fact 7.3.10. *An abelian group A is flat if and only if A is torsion-free, i.e. for every $a \in A$ if $a \neq 0$ then $na \neq 0$ for all $n \neq 0$ in $\mathbb{Z}$.*

Example 7.3.11. Since $\mathbb{Q}$ is torsion-free, we see that $\mathrm{Tor}(A, \mathbb{Q}) = 0$ for all A. The same holds for any field of characteristic 0, such as $\mathbb{R}$ or $\mathbb{C}$.

Proposition 7.3.12. *Suppose $0 \to F_1 \xrightarrow{i} F_0 \xrightarrow{q} A \to 0$ is a free resolution, and*

$$0 \to B_1 \xrightarrow{j} B_0 \xrightarrow{p} C \to 0$$

is any short exact sequence. Given a homomorphism $f\colon A \to C$ there exists an extension of f to a commutative diagram

$$\begin{array}{ccccc} F_1 & \xrightarrow{i} & F_0 & \xrightarrow{q} & A \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle f} \\ B_1 & \xrightarrow{j} & B_0 & \xrightarrow{p} & C. \end{array}$$

The maps $\phi_\bullet$ can be viewed as a chain map $F_\bullet \to B_\bullet$ such that $H_0(\phi_\bullet) = f$. Given another such chain map $\phi'_\bullet\colon F_\bullet \to B_\bullet$, there exists a chain homotopy between $\phi_\bullet$ and $\phi'_\bullet$,

$$\begin{array}{ccc} F_1 & \overset{\phi_1}{\underset{\phi'_1}{\rightrightarrows}} & B_1 \\ {\scriptstyle q}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ F_0 & \overset{\phi_0}{\underset{\phi'_0}{\rightrightarrows}} & B_0. \end{array}$$

Proof. By assumption F_0 is a free abelian group $\mathbb{Z}S$ for some set S. Since p is surjective, we can choose preimages in B_0 of the images of S under $fq\colon F_0 \to C$. This determines a homomorphism $\phi_0\colon F_0 \to B_0$ such that $p\phi_0 = fq$. Since $F_1 = \ker q$ and $B_1 = \ker p$ the map ϕ_0 restricts to a homomorphism $\phi_1\colon F_1 \to B_1$. This shows that a lift exists. Given another lift (ϕ'_0, ϕ'_1) we see that $p(\phi_0 - \phi'_0) = 0$ so that since $B_1 = \ker p$ there exists a unique homomorphism $h\colon F_0 \to B_1$ such that $jh = \phi_0 - \phi'_0$, which is precisely the equation required of a chain homotopy. (We also have $hi = \phi_1 - \phi'_1$ since

$$jhi = \phi_0 i - \phi'_0 i = j\phi_1 - j\phi'_1$$

and j is injective.) □

Corollary 7.3.13. *Suppose $F_\bullet$ and $F'_\bullet$ are two free resolutions of an abelian group A, viewed as chain complexes. Then $F_\bullet$ and $F'_\bullet$ are chain homotopy equivalent.*

Proof. We can choose lifts $\alpha\colon F_\bullet \to F'_\bullet$ and $\beta\colon F'_\bullet \to F_\bullet$ of $\mathrm{id}\colon A \to A$. Then $\beta\alpha$ and $\mathrm{id}_{F_\bullet}$ are two lifts of id_A and hence are chain homotopic, and similarly $\alpha\beta$ is chain homotopic to $\mathrm{id}_{F'_\bullet}$. □

Corollary 7.3.14. *For abelian groups A, B their torsion product* $\mathrm{Tor}(A, B)$ *is well-defined and independent of the choice of free resolution of A.*

Proof. Suppose we have two free resolutions $F_\bullet$ and $F'_\bullet$ of A. Then $F_\bullet$ and $F'_\bullet$ are chain homotopic. Since $- \otimes B$ preserves chain homotopies, this implies that $F_\bullet \otimes B$ and $F'_\bullet \otimes B$ are chain homotopic, and hence have the same homology. Thus we get an isomorphism between $\mathrm{Tor}(A, B)$ computed using these two resolutions. □

Proposition 7.3.15 (Horseshoe Lemma). *Given a short exact sequence of abelian groups*

$$0 \to A' \xrightarrow{i} A \xrightarrow{q} A'' \to 0,$$

and free resolutions

$$0 \to F'_1 \xrightarrow{j'} F'_0 \xrightarrow{p'} A' \to 0,$$

$$0 \to F''_1 \xrightarrow{j''} F''_0 \xrightarrow{p''} A'' \to 0,$$

we can choose a free resolution $F_\bullet$ of A such that there is a short exact sequence of chain complexes

$$0 \to F'_\bullet \to F_\bullet \to F''_\bullet \to 0.$$

Proof. Since F''_0 is free and q is surjective, we can choose $s\colon F''_0 \to A$ such that $qs = p''$. Set $F_0 := F'_0 \oplus F''_0$ and define $p\colon F_0 \to A$ by $p(x, y) = ip'(x) + s(y)$. Then p is surjective: Given a in A we know $qa = p''(y)$ for some y since p'' is surjective. Then $q(a - s(y)) = 0$ so $a - s(y) = i(b)$ for some $b \in A'$, and since p' is surjective there exists x such that $b = p'(x)$. Hence

$$a = ip'(x) + s(y) = p(x, y).$$

Let $F_1 := \ker p$ with $j\colon F_1 \to F_0$ the inclusion. If $p(x, y) = 0$ then $qp(x, y) = p''y = 0$, so $y \in F''_1$, and if $x \in F'_1$ then $p(x, 0) = ip'(x) = 0$, so we get a commutative diagram

$$\begin{array}{ccccc} F'_1 & \longrightarrow & F_1 & \longrightarrow & F''_1 \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j''} \\ F'_0 & \longrightarrow & F_0 & \longrightarrow & F''_0 \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} \\ A' & \xrightarrow{i} & A & \xrightarrow{q} & A''. \end{array}$$

The abelian group F_1 is free by Fact 7.3.3. The top row is certainly exact at F'_1, since the map $F'_1 \to F_1$ is a restriction of an injective map, and it is exact at F_1 since the kernel of $F_1 \to F''_1$ is the elements $(x, 0) \in F_0$ such that $p(x, 0) = ip'x = 0$, which is precisely the image of F'_1. It remains to see that $F_1 \to F''_1$ is surjective. For $y \in F''_1$ we have $qs(y) = p''(y) = 0$ so $s(y) = i(a)$ for some unique $a \in A'$. Since F''_1 is free and p' is surjective we can choose a map $\gamma\colon F''_1 \to F'_0$ such that $ip'\gamma(y) = s(y)$; then $p(-\gamma(y), y) = 0$ so $(-\gamma(y), y)$ is in F_1 and maps to y. □

Note that since F''_1 is free, there exists a splitting of F_1 as $F'_1 \oplus F''_1$. One such splitting is given by $(-\gamma, \mathrm{id})$. However, this splitting is probably *not* compatible with the splitting of F_0, and indeed it may well be the case that no compatible splitting exists, since this would require that for $y \in F''_1$ we have $p(0, y) = s(y) = 0$.

Corollary 7.3.16. *Suppose we have a short exact sequence of abelian groups* $0 \to A' \to A \to A'' \to 0$*. Then for any abelian group* B *there is a long exact sequence*

$$0 \to \mathrm{Tor}(A', B) \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A'', B) \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0.$$

Proof. By Proposition 7.3.15 we can lift the short exact sequence to a short exact sequence of free resolutions

$$0 \to F'_\bullet \to F_\bullet \to F''_\bullet \to 0.$$

Tensoring with B gives

$$0 \to F'_\bullet \otimes B \to F_\bullet \otimes B \to F''_\bullet \otimes B \to 0,$$

and this is again a short exact sequence of chain complexes since tensoring with free abelian groups preserves short exact sequences by Lemma 7.1.8. This gives a long exact sequence in homology of the required form. □

We also have an exact sequence in the second variable:

Proposition 7.3.17. *If* A *is an abelian group and* $0 \to B' \to B \to B'' \to 0$ *is a short exact sequence of abelian groups, then there is a long exact sequence*

$$0 \to \mathrm{Tor}(A, B') \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, B'') \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0.$$

Proof. Let $F_\bullet$ be a free resolution of A, viewed as a two-term chain complex. Then

$$0 \to F_\bullet \otimes B' \to F_\bullet \otimes B \to F_\bullet \otimes B'' \to 0$$

is a short exact sequence of chain complexes by Lemma 7.1.8. This has a long exact sequence in homology, which has the required form by Remark 7.3.5. □

This implies that we can also compute Tor using the second variable:

Corollary 7.3.18 (Tor is symmetric)**.** *Suppose* A *and* B *are abelian groups and* $0 \to G_1 \to G_0 \to B \to 0$ *is a free resolution. Then* $\mathrm{Tor}(A, B)$ *is isomorphic to the kernel of* $A \otimes G_1 \to A \otimes G_0$.

Proof. Applying Proposition 7.3.17 and the computation of Example 7.3.7 we get a long exact sequence

$$0 \to \mathrm{Tor}(A, B) \to A \otimes G_1 \to A \otimes G_0 \to A \otimes B \to 0,$$

which implies that $\mathrm{Tor}(A, B)$ is the kernel of the subsequent map. □

7.4 *The Universal Coefficient Theorem*

We now want to apply our work on Tor in topology, to get the following result:

Theorem 7.4.1 (Universal Coefficient Theorem). *Let (X, A) be a subspace pair and M an abelian group. There are natural short exact sequences*

$$0 \to H_n(X, A) \otimes M \to H_n(X, A; M) \to \operatorname{Tor}(H_{n-1}(X, A), M) \to 0.$$

In fact, this is just a special case (for $S_\bullet(X, A)$) of an algebraic result:

Proposition 7.4.2. *Suppose $C_\bullet$ is a chain complex of abelian groups such that C_n is free for all n. Then for any abelian group M there are natural short exact sequences*

$$0 \to H_n(C) \otimes M \to H_n(C \otimes M) \to \operatorname{Tor}(H_{n-1}(C), M) \to 0.$$

Proof. Let $B_n := B_n(C)$ and $Z_n := Z_n(C)$ be the boundaries and cycles in C. By the definition of $H_n(C)$ we have short exact sequences

$$0 \to B_n \xrightarrow{j_n} Z_n \to H_n(C) \to 0,$$

where j_n denotes the inclusion $B_n \hookrightarrow Z_n$. Note that the abelian groups B_n and Z_n are free, since they are subgroups of C_n, so this is a free resolution of $H_n(C)$.

We also have a short exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial'} B_{n-1} \to 0,$$

where ∂' denotes the boundary map $C_n \xrightarrow{\partial} C_{n-1}$ viewed as a map to its image B_{n-1}. Since B_{n-1} is free, this is a splittable short exact sequence. Moreover, if we define chain complexes $Z_\bullet$ and $B'_\bullet$ by taking the groups Z_n and $B'_n = B_{n-1}$ with 0 differential, then we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \xrightarrow{\partial'} B'_\bullet \to 0,$$

since the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial'} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial'} & B_{n-2} & \longrightarrow & 0 \end{array}$$

commutes. Since this is a levelwise splittable short exact sequence, if we tensor with M we get again a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes M \to C_\bullet \otimes M \to B'_\bullet \otimes M \to 0,$$

which induces a long exact sequence in homology. Since the differentials in $Z_\bullet$ and $B'_\bullet$ are all zero, we have $H_n(Z_\bullet \otimes M) = Z_n \otimes M$ and $H_n(B'_\bullet \otimes M) = B_{n-1} \otimes M$, and this long exact sequence looks like

$$\ldots B_n \otimes M \xrightarrow{\delta_{n+1}} Z_n \otimes M \to H_n(C_\bullet \otimes M) \to B_{n-1} \otimes M \xrightarrow{\delta_n} Z_{n-1} \otimes M \to \cdots,$$

where we use δ_n for the connecting boundary map in the long exact sequence to avoid confusion with that in $C_\bullet$. Around $H_n(C_\bullet \otimes M)$ we get a short exact sequence

$$0 \to \operatorname{coker} \delta_{n+1} \to H_n(C_\bullet \otimes M) \to \ker \delta_n \to 0.$$

To compute $\delta_n(x)$ for $x \in B_{n-1} \otimes M$ we should lift x to $C_n \otimes M$ along $\partial' \otimes \mathrm{id}$, i.e. choose y such that $(\partial' \otimes \mathrm{id})(y) = x$, then apply the boundary map in $C_\bullet \otimes M$, which just gives x again, but viewed as an element in $C_{n-1} \otimes M$; finally we should take the unique preimage of x in $Z_{n-1} \otimes M$. Thus the boundary map δ_n is just $j_{n-1} \otimes \mathrm{id}\colon B_{n-1} \otimes M \to Z_{n-1} \otimes M$. Since j_{n-1} gives a free presentation of $H_{n-1}(C)$, we have

$$\ker \delta_n = \operatorname{Tor}(H_{n-1}(C), M), \quad \operatorname{coker} \delta_n = H_{n-1}(C) \otimes M.$$

Plugging this into the previous short exact sequence now gives the result. □

Remark 7.4.3. With some more work it can be shown that the short exact sequences in the universal coefficient theorem can always be split, although the splitting is not natural. This means that we have non-canonical isomorphisms

$$H_n(X; M) \cong H_n(X) \otimes M \oplus \operatorname{Tor}(H_{n-1}(X), M).$$

Thus the abelian groups $H_n(X; M)$ are algebraically determined by the integral homology groups, so that the integral coefficients are "universal".

Since torsion-free abelian groups are flat, we have the following special case:

Corollary 7.4.4. *If M is a torsion-free abelian group, then there is a natural isomorphism*

$$H_n(X, A) \otimes M \xrightarrow{\sim} H_n(X, A; M).$$

Remark 7.4.5. In particular, we always have $H_n(X, A; \mathbb{Q}) \cong H_n(X, A) \otimes \mathbb{Q}$. The same holds for $\mathbb{R}$ or $\mathbb{C}$: in each case we get a vector space over the relevant field of dimension $\operatorname{rk} H_n(X, A)$.

Remark 7.4.6. Suppose $H_n(X, A)$ is a finitely generated abelian group for all n and let p be a prime. By the classification of finitely generated abelian groups we can write

$$H_n(X, A) \cong F_n \oplus P_n \oplus T_n,$$

where $F_n \cong \mathbb{Z}^{r_n}$ is free, P_n consists of the torsion of order p^i for all $i > 0$, and T_n consists of the torsion of order prime to p. Then we have

$$F_n \otimes \mathbb{Z}/p \cong (\mathbb{Z}/p)^{r_n}, \qquad \operatorname{Tor}(F_n, \mathbb{Z}/p) = 0,$$

$$T_n \otimes \mathbb{Z}/p = \operatorname{Tor}(T_n, \mathbb{Z}/p) = 0,$$

$$P_n \otimes \mathbb{Z}/p \cong \operatorname{Tor}(P_n, \mathbb{Z}/p) \cong (\mathbb{Z}/p)^{q_n}$$

where P_n is a sum of q_n groups of the form $\mathbb{Z}/p^i$. The universal coefficient theorem implies

$$H_n(X,A;\mathbb{Z}/p) \cong F_n \otimes \mathbb{Z}/p \oplus P_n \otimes \mathbb{Z}/p \oplus \mathrm{Tor}(P_{n-1},\mathbb{Z}/p) \cong (\mathbb{Z}/p)^{r_n+q_n+q_{n-1}}.$$

Corollary 7.4.7. *Let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs. If $f_*\colon H_n(X,A) \to H_n(Y,B)$ is an isomorphism for all n, then so is $f_*\colon H_n(X,A;M) \to H_n(Y,B;M)$ for any abelian group M.*

Proof. From the universal coefficient theorem we have morphisms of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(X,A)\otimes M & \longrightarrow & H_n(X,A;M) & \longrightarrow & \mathrm{Tor}(H_{n-1}(X,A),M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_*\otimes \mathrm{id}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \mathrm{Tor}(f_*,M)} & & \\
0 & \longrightarrow & H_n(Y,B)\otimes M & \longrightarrow & H_n(Y,B;M) & \longrightarrow & \mathrm{Tor}(H_{n-1}(Y,B),M) & \longrightarrow & 0.
\end{array}$$

By assumption the maps $f_* \otimes \mathrm{id}$ and $\mathrm{Tor}(f_*,M)$ are isomorphisms, hence the middle map is also an isomorphism by the 5-Lemma. □

This result has a converse of sorts:

Theorem 7.4.8. *Let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs. If $f_*\colon H_n(X,A;M) \to H_n(Y,B;M)$ is an isomorphism for all n when $M = \mathbb{Q}$ and $\mathbb{F}_p$ for all primes p, then so is $f_*\colon H_n(X,A) \to H_n(Y,B)$.*

Here is part of the proof, which follows from the universal coefficient theorem:

Proposition 7.4.9. *Suppose (X,Y) is a subspace pair such that $H_*(X,Y;M) = 0$ for $M = \mathbb{Q}$ and $M = \mathbb{F}_p$ for all primes p. Then $H_*(X,Y) = 0$.*

Proof. We abbreviate $H_n := H_n(X,Y)$. For every prime p the universal coefficient theorem gives short exact sequences

$$0 \to H_n \otimes \mathbb{F}_p \to H_n(X,Y;\mathbb{F}_p) \to \mathrm{Tor}(H_{n-1},\mathbb{F}_p) \to 0.$$

If $H_n(X,Y;\mathbb{F}_p) = 0$ for all n then this implies $H_n \otimes \mathbb{F}_p = 0$ and $\mathrm{Tor}(H_n,\mathbb{F}_p) = 0$ for all n. Since we have a short exact sequence

$$0 \to \mathrm{Tor}(H_n,\mathbb{F}_p) \to H_n \xrightarrow{p} H_n \to H_n \otimes \mathbb{F}_p \to 0,$$

this implies that multiplication by p on H_n is an isomorphism, i.e. H_n is uniquely p-divisible. Since this holds for all primes p, H_n is a rational vector space. But then $H_n \cong H_n \otimes \mathbb{Q} \cong H_n(X,Y;\mathbb{Q})$, so if the rational homology of (X,Y) is also 0 we have $H_n = 0$. □

We use the following algebraic facts: an abelian group A has a $\mathbb{Q}$-vector space structure (which is unique if it exists) if and only if A is uniquely p-divisible for all primes p, and in this case $A \cong A \otimes \mathbb{Q}$.

To complete the proof of Theorem 7.4.8 we need a new topological construction (the *mapping cone*); we leave this as an exercise:

Exercise 7.11. For a continuous map $f\colon X \to Y$, the *mapping cone* $M(f)$ of M is defined as the pushout

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
CX & \longrightarrow & M(f),
\end{array}$$

where CX is the cone on X, as in exercise 5 from week 4.

(i) Show that $H_i(M(f), Y) \cong \tilde{H}_{i-1}(X)$ and the boundary map

$$H_i(M(f), Y) \to H_{i-1}(Y)$$

corresponds to $f_*\colon H_{i-1}(X) \to H_{i-1}(Y)$ (for $i > 1$). [Hint: $M(f)/Y \cong CX/X \cong \Sigma X$ plus Exercise 4.10; to identify the map use the naturality of the boundary map for $(CX, X) \to (M(f), Y)$.]

(ii) From (i) the long exact sequence for the pair $(M(f), Y)$ looks like

$$\cdots \to H_i(M(f)) \to H_{i-1}(X) \xrightarrow{f_*} H_{i-1}(Y) \to H_{i-1}(M(f)) \to \cdots$$

for $i > 1$. Use this to prove that $f_*\colon H_*(X) \to H_*(Y)$ is an isomorphism if and only if $\tilde{H}_*(M(f)) = 0$.

(iii) Complete the proof that f_* is an isomorphism in integral homology if it is an isomorphism in homology with $\mathbb{Q}$- and $\mathbb{F}_p$-coefficients for all primes p.

7.5 Cellular Homology with Coefficients

Suppose X is a cell complex, with Γ_n the set of n-cells. Then for any abelian group M we have

$$H_*(X_k, X_{k-1}; M) \cong \tilde{H}_*(\bigvee_{\alpha\in\Gamma_k} S^k; M) \cong \bigoplus_{\alpha\in\Gamma_k} M \cong \mathbb{Z}\Gamma_k \otimes M.$$

We can therefore redo the construction of the cellular chain complex of X with M-coefficients, giving $C^{\text{cell}}_\bullet(X; M)$ where

$$C^{\text{cell}}_n(X; M) := H_n(X_n, X_{n-1}; M).$$

The same proof as before then shows that

$$H_*(C^{\text{cell}}_\bullet(X; M)) \cong H_*(X; M).$$

Proposition 7.5.1. *We have an isomorphism of chain complexes*

$$C^{\text{cell}}_\bullet(X; M) \cong C^{\text{cell}}_\bullet(X) \otimes M.$$

We know both sides are levelwise isomorphic, so it suffices to check the differentials on both sides agree. Since we computed these in terms of degrees of maps of spheres, it suffices to make the following observation:

Lemma 7.5.2. *Let $f\colon S^n \to S^n$ be a map of degree d. Then*

$$f_*\colon \tilde{H}_n(S^n; M) \to \tilde{H}_n(S^n; M)$$

corresponds under the isomorphism $\tilde{H}_n(S^n; M) \cong M$ to the homomorphism $M \to M$ given by multiplication with d.

Proof. By Lemma 7.2.2 we have a commutative square

$$\begin{array}{ccc} \tilde{H}_n(S^n) \otimes M & \xrightarrow{\sim} & \tilde{H}_n(S^n; M) \\ \downarrow{\scriptstyle f_*\otimes \mathrm{id}_M} & & \downarrow{\scriptstyle f_*} \\ \tilde{H}_n(S^n) \otimes M & \xrightarrow{\sim} & \tilde{H}_n(S^n; M), \end{array}$$

where the universal coefficient theorem implies that the horizontal maps are isomorphisms, and Exercise 7.9 tells us that the left vertical homomorphism is given by multiplication with d. □

Example 7.5.3. We use cellular homology with $\mathbb{F}_2$-coefficients to compute $H_*(\mathbb{RP}^n; \mathbb{F}_2)$. Recall that the cellular chain complex of $\mathbb{RP}^n$ was

$$\cdots \to 0 \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0 \to \cdots,$$

with $\mathbb{Z}$ in each degree i with $0 \leq i \leq n$, and the differential in degree i given by multiplication with $1 + (-1)^i$. Tensoring with $\mathbb{Z}/2$ we get

$$\cdots \to 0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}/2 \to 0 \to \cdots,$$

since $1 + (-1)^{i+1}$ is always 0 mod 2, so that

$$H_*(\mathbb{RP}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{Z}/2, & 0 \leq * \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

8
Cohomology

In this chapter we are going to define a "dual" variant of homology, called *cohomology*. At first sight it may be hard to see the point of doing this, especially since our first main result will be a "universal coefficient theorem" that implies cohomology groups contain essentially the same information as homology groups. However, later on we will see that cohomology has additional structure that is not present on homology: the singular cohomology groups of a space form a *graded commutative ring*. This additional structure makes cohomology a more powerful invariant than homology.

We begin with a bit of algebra in §8.1, where we look at the abelian groups $\mathrm{Hom}(A, B)$ of homomorphisms between abelian groups A and B. Then we apply this to define singular cohomology groups in §8.2. We then look at a little more homological algebra in §8.3, where we define Ext of abelian groups, which we apply to prove the universal coefficient theorem for cohomology in §8.4. Finally, in §8.5 we show that there is a cohomological version of cellular homology, which computes the singular cohomology groups of a cell complex.

8.1 Hom of Abelian Groups

We begin with a brief algebraic interlude, to introduce the basic properties of Hom for abelian groups.

Definition 8.1.1. For abelian groups A, B, let $\mathrm{Hom}(A, B)$ denote the set of homomorphisms $A \to B$. This is itself an abelian group, if for $f, g \colon A \to B$ we define $f + g$ to be the homomorphism given by $(f + g)(a) = f(a) + g(a)$.

The following exercises give some basic examples of this construction:

Exercise 8.1. For M an abelian group, show that $\mathrm{Hom}(\mathbb{Z}/n, M)$ is the group of n-torsion elements in M; in particular $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/r$ where $r = \gcd(n, m)$.

Definition 8.1.2. Suppose S is a set and M an abelian group. Then we write M^S for the set of functions $S \to M$, which we think of as an abelian group via pointwise addition, i.e. $(f + g)(s) = f(s) + g(s)$.

Exercise 8.2. If S is a set and M an abelian group, then we have canonical isomorphisms
$$\mathrm{Hom}(\mathbb{Z}S, M) \cong M^S \cong \prod_{s \in S} M,$$
of abelian groups.

We now state some formal properties of Hom; proving these is also left as an exercise.

Proposition 8.1.3.

(i) For abelian groups $A_i, i \in I$ and B, there is a canonical isomorphism
$$\mathrm{Hom}(\bigoplus_{i \in I} A_i, B) \xrightarrow{\sim} \prod_{i \in I} \mathrm{Hom}(A_i, B).$$

(ii) For an abelian group A, there are canonical isomorphisms
$$\mathrm{Hom}(\mathbb{Z}, A) \cong A, \quad \mathrm{Hom}(0, A) \cong \mathrm{Hom}(A, 0) \cong 0.$$

(iii) For homomorphisms $f: A \to A'$, $g: B \to B'$, there are canonical homomorphisms
$$f^*: \mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B), \quad g_*: \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B'),$$
given by composition with f and g. These are compatible with composition, so Hom *is a functor* $\mathsf{Ab}^{\mathrm{op}} \times \mathsf{Ab} \to \mathsf{Ab}$.

(iv) For abelian groups A, A', B, B' the homomorphisms
$$0^*: \mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B), \quad 0_*: \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B'),$$
determined by the zero maps $A \to A'$, $B \to B'$, are both the respective zero homomorphisms.

(v) For homomorphisms $f, g: A \to A'$, $h, k: B \to B'$, we have identities $(f+g)^* = f^* + g^*$ and $(h+k)_* = h_* + k_*$ of homomorphisms $\mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B)$ *and* $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$, *respectively.*

Exercise 8.3. Prove the basic formal properties of Hom.

In (iii), $\mathsf{Ab}^{\mathrm{op}}$ denotes the *opposite* of the category Ab, in the following sense:

Definition 8.1.4. If $\mathcal{C}$ is a category, its *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, but morphisms go the opposite way, i.e. $\mathcal{C}^{\mathrm{op}}(x, y) = \mathcal{C}(y, x)$, with composition given by composition in $\mathcal{C}$. Thus a functor $F: \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ gives for every morphism $f: x \to y$ in $\mathcal{C}$ a morphism $F(f): F(y) \to F(x)$; in somewhat old-fashioned terminology this is called a *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$.

Lemma 8.1.5 (Hom is left exact)**.** *For M an abelian group and exact sequences*
$$A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$
$$0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C',$$

(so that $C \cong \operatorname{coker} g$ *and* $A' \cong \ker g'$*), the sequences*

$$0 \to \operatorname{Hom}(C, M) \xrightarrow{g^*} \operatorname{Hom}(B, M) \xrightarrow{f^*} \operatorname{Hom}(A, M),$$

$$0 \to \operatorname{Hom}(M, A') \xrightarrow{f'_*} \operatorname{Hom}(M, B') \xrightarrow{g'_*} \operatorname{Hom}(M, C')$$

are exact.

Proof. To see that g^* is injective, suppose $\phi \in \operatorname{Hom}(C, M)$ satisfies $g^*\phi = 0$. Then $\phi(g(b)) = 0$ for all $b \in B$. Since g is surjective, this implies that $\phi(c) = 0$ for all $c \in C$, i.e. $\phi = 0$. Since $gf = 0$ we have $f^*g^* = (gf)^* = 0$, so $\operatorname{im} g^* \subseteq \ker f^*$. Now if $\phi \in \operatorname{Hom}(B, M)$ is in $\ker f^*$, then $\phi(f(a)) = 0$ for $a \in A$. In other words, $\operatorname{im} f \subseteq \ker \phi$, so ϕ factors as $B \to B/\operatorname{im} f \xrightarrow{\phi'} M$. But by exactness we can identify the map $B \to B/\operatorname{im} f$ with g, so $\phi = \phi' g = g^*\phi'$, and so $\phi \in \operatorname{im} g^*$. This proves the first sequence is exact.

Now we consider the second sequence. To see that f'_* is injective, suppose $\phi \in \operatorname{Hom}(M, A')$ and $f'_*\phi = f' \circ \phi = 0$. Then $f'(\phi(a)) = 0$ for all $a \in A'$, and so $\phi = 0$ since f' is injective. We have $g'_*f'_* = (g'f')_* = 0$, so $\operatorname{im} f'_* \subseteq \ker g'_*$. If $\phi \colon M \to B'$ is in $\ker g'_*$ then $g'(\phi(b)) = 0$ for all $b \in B'$, so that ϕ factors as $M \xrightarrow{\phi'} \ker g' \to B'$. But by exactness we can identify the inclusion $\ker g' \to B'$ with $f' \colon A' \to B'$ so $\phi = f'_*\phi'$, i.e. $\phi \in \operatorname{im} f'_*$ as required. □

Remark 8.1.6. As special cases, we see that if $f \colon A \to B$ is surjective, then $f^* \colon \operatorname{Hom}(B, M) \to \operatorname{Hom}(A, M)$ is injective, while if f is injective then $f_* \colon \operatorname{Hom}(M, A) \to \operatorname{Hom}(M, B)$ is injective. The dual properties are false, however: if we apply $\operatorname{Hom}(\mathbb{Z}/2, -)$ to the surjective map $\mathbb{Z} \to \mathbb{Z}/2$ we get $0 \to \mathbb{Z}/2$, which is not surjective, while if we apply $\operatorname{Hom}(-, \mathbb{Z}/2)$ to the injective map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ we get $\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$, which is not surjective. In particular, we see that Hom does not preserve short exact sequences in either variable. However, just as for the tensor product, there are special cases where exactness *is* preserved:

Proposition 8.1.7. *For any abelian group* M*, the functor* $\operatorname{Hom}(-, M)$ *preserves splittable short exact sequences.*

Proof. Suppose $0 \to A \to B \to C \to 0$ is a splittable short exact sequence. Choosing a splitting determines an isomorphism to the the trivial short exact sequence $0 \to A \to A \oplus C \to C \to 0$. Since Hom takes direct sums to products (and finite products and finite direct sums are the same thing), we have

$$\operatorname{Hom}(A \oplus C, M) \cong \operatorname{Hom}(A, M) \oplus \operatorname{Hom}(C, M),$$

so that the sequence

$$0 \to \operatorname{Hom}(C, M) \to \operatorname{Hom}(A \oplus C, M) \to \operatorname{Hom}(A, M) \to 0$$

is exact. But this is isomorphic to $\operatorname{Hom}(-, M)$ of the original sequence, so this is also exact. □

Remark 8.1.8. The same is true in the second variable, though we will not need it.

Corollary 8.1.9. *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of abelian groups such that* C *is free, then*

$$0 \to \operatorname{Hom}(C, M) \to \operatorname{Hom}(B, M) \to \operatorname{Hom}(A, M) \to 0$$

is a short exact sequence for any abelian group M*.*

Lemma 8.1.10. *If* $F = \mathbb{Z}S$ *is a free abelian group, then* $\operatorname{Hom}(\mathbb{Z}S, -)$ *preserves short exact sequences.*

Proof. Since we have $\operatorname{Hom}(\mathbb{Z}S, M) \cong M^S \cong \prod_{s \in S} M$ this boils down to the fact that an arbitrary product of short exact sequences is again a short exact sequence; checking this is left as an exercise. □

Remark 8.1.11. It is not true that $\operatorname{Hom}(-, B)$ preserves short exact sequences if B is free. However, it can be shown that this is true if B is *divisible*, meaning that for any b in B and every integer $n \neq 0$ we can write $b = nb'$ for some b'. In particular, $\operatorname{Hom}(-, \mathbb{Q})$ preserves short exact sequences.

Exercise 8.4.

(i) Show that for abelian groups A, B, C there is a natural bijection between the sets of homomorphisms $A \otimes B \to C$ and $A \to \operatorname{Hom}(B, C)$.

(ii) Show that this bijection is moreover an isomorphism of abelian groups

$$\operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(A, \operatorname{Hom}(B, C)).$$

(iii) Show that composition of homomorphisms of abelian groups gives a homomorphism

$$\operatorname{Hom}(A, B) \otimes \operatorname{Hom}(B, C) \to \operatorname{Hom}(A, C).$$

Exercise 8.5. If $m \colon \mathbb{Z} \to \mathbb{Z}$ is the map given by multiplication with m, show that $m^* \colon \operatorname{Hom}(\mathbb{Z}, M) \to \operatorname{Hom}(\mathbb{Z}, M)$ corresponds under the isomorphism $\operatorname{Hom}(\mathbb{Z}, M) \cong M$ to the map $M \to M$ given by multiplication with m.

8.2 *Singular Cohomology*

Definition 8.2.1. If $(C_\bullet, \partial)$ is a chain complex of abelian groups, then for any abelian group M we define a new chain complex $\operatorname{Hom}(C, M)_\bullet$ by setting

$$\operatorname{Hom}(C, M)_k := \operatorname{Hom}(C_{-k}, M),$$

and defining the boundary map $\operatorname{Hom}(C, M)_{k+1} \to \operatorname{Hom}(C, M)_k$ to be ∂^* for $\partial \colon C_{-k} \to C_{-k-1}$. This determines a functor $\operatorname{Hom}(-, M) \colon \mathsf{Ch}^{\mathrm{op}} \to \mathsf{Ch}$.

Exercise 8.6. Show that a chain homotopy between chain maps $f, g \colon C_\bullet \to D_\bullet$ induces a natural chain homotopy between $f^*, g^* \colon \operatorname{Hom}(D, M) \to \operatorname{Hom}(C, M)$ for any abelian group M.

Definition 8.2.2. For a subspace pair (X, A) and an abelian group M, we define the *singular n-cochains* of (X, A) with coefficients in M to be the abelian group $S^n(X, A; M) := \operatorname{Hom}(S_n(X, A), M)$. The *singular cochains* thus form the chain complex $\operatorname{Hom}(S_\bullet(X, A), M)$ with $S^n(X, A; M) = \operatorname{Hom}(S_\bullet(X, A), M)_{-n}$; we will often just refer to this chain complex as $S^\bullet(X,A;M)$ but keep in mind that this chain complex really lives in *negative* degrees. The *singular cohomology groups* of (X,A) with coefficients in M are the homology groups of $S^\bullet(X,A;M)$, with the convention

$$H^n(X,A;M) := H_{-n}(\mathrm{Hom}(S_\bullet(X,A),M)).$$

For integral coefficients we abbreviate $S^\bullet(X,A) = S^\bullet(X,A;\mathbb{Z})$ and $H^n(X,A) = H^n(X,A;\mathbb{Z})$. To avoid confusion with the boundary map for $S_\bullet(X)$ we will refer to the differential in $S^\bullet(X)$ as the *coboundary map* and denote it by δ.

Remark 8.2.3. A *cochain complex* is the same thing as a chain complex, except that the differentials go the other way, so a cochain complex consists of abelian groups C^n, $n \in \mathbb{Z}$, and homomorphisms $\partial\colon C^n \to C^{n+1}$ such that $\partial^2 = 0$. We can pass between chain complexes and cochain complexes by changing the signs of the degrees: if $(C^\bullet,\partial)$ is a cochain complex we get a chain complex with the same boundary maps if we set $C_n := C^{-n}$, and vice versa. Traditionally the singular cochains are viewed as a cochain complex in positive degrees. However, apart from the overhead of introducing an unnecessary new concept and having to convince ourselves that the tools we developed for chain complexes work the same way for cochain complexes (while keeping track of signs of degree changes), there are good reasons to want singular chains and singular cochains to be objects of the same category, as we'll see later on.

Remark 8.2.4. Since $\mathrm{Hom}(-,M)$ is a functor, $S^\bullet(-;M)$ is a functor $\mathsf{Pair}^{\mathrm{op}} \to \mathsf{Ch}$, and singular cohomology gives functors

$$H^n(-;M)\colon \mathsf{Pair}^{\mathrm{op}} \to \mathsf{Ab}.$$

For a continuous map $f\colon (X,A) \to (Y,B)$ we typically write

$$f^*\colon S^\bullet(X,A;M) \to S^\bullet(Y,B;M), \quad f^*\colon H^n(X,A;M) \to H^n(Y,B;M)$$

for the induced chain map $S^\bullet(f;M)$ and homomorphism $H^n(f;M)$.

Remark 8.2.5. Since $S_n(X) = \mathbb{Z}\,\mathrm{Sing}_n(X)$, the abelian group $S^n(X;M)$ is isomorphic to $M^{\mathrm{Sing}_n(X)}$, i.e. the set of functions $\mathrm{Sing}_n(X) \to M$ with pointwise addition. Since $\mathrm{Sing}_0(X)$ is the underlying set of points of X, we have in particular that $S^0(X;M)$ is the set of all functions M^X from X to M. We can give a more explicit description of the coboundary map in these terms: the maps $\partial_i\colon \mathrm{Sing}_{n+1}(X) \to \mathrm{Sing}_n(X)$ induce homomorphisms $\partial_i^*\colon M^{\mathrm{Sing}_n(X)} \to M^{\mathrm{Sing}_{n+1}(X)}$, and for $\phi \in M^{\mathrm{Sing}_n(X)}$ we have

$$\delta\phi = \sum_{i=0}^{n+1} (-1)^i \partial_i^* \phi,$$

i.e. $\delta\phi\colon \mathrm{Sing}_{n+1}(X) \to M$ is given by

$$(\delta\phi)(\sigma) = \sum_{i=0}^{n+1} (-1)^i \phi(\partial_i \sigma).$$

Exercise 8.7. Show that for $A \subseteq X$ the relative cochains $S^n(X, A)$ can be identified with the subgroup of $S^n(X) \cong \mathbb{Z}^{\mathrm{Sing}_n(X)}$ consisting of functions $f\colon \mathrm{Sing}_n(X) \to \mathbb{Z}$ such that $f(\alpha) = 0$ for $\alpha \in \mathrm{Sing}_n(A) \subseteq \mathrm{Sing}_n(X)$.

Example 8.2.6 (H^0)**.** The 0th cohomology group $H^0(X)$ is by definition the kernel of $\delta\colon S^0(X) \to S^1(X)$. Here $S^0(X) \cong \mathbb{Z}^X$ and $S^1(X) = \mathbb{Z}^{\mathrm{Sing}_1(X)}$, where $\mathrm{Sing}_1(X)$ is the set of all continuous paths $\Delta^1 \to X$. For $\phi\colon X \to \mathbb{Z}$, we see that

$$\delta\phi(\sigma) = \phi(\sigma(1)) - \phi(\sigma(0)),$$

so that ϕ is in $\ker\delta$ if and only if $\phi(x) = \phi(y)$ whenever there exists a path in X from x to y, i.e. when x and y are in the same path component. Equivalently, ϕ is in $\ker\delta$ precisely when it factors through the quotient $\pi_0(X)$, and so we get a natural isomorphism

$$H^0(X) \cong \mathbb{Z}^{\pi_0(X)} \cong \prod_{\pi_0(X)} \mathbb{Z}.$$

(If X has finitely many components then this is isomorphic to the free abelian group $\mathbb{Z}\pi_0(X) \cong \bigoplus_{\pi_0(X)} \mathbb{Z}$, which is $H_0(X)$, but if $\pi_0(X)$ is infinite the group $H^0(X)$ is substantially larger: if $\pi_0(X)$ is countable then $\mathbb{Z}\pi_0(X)$ is also countable, but $\mathbb{Z}^{\pi_0(X)}$ is uncountable.)

Example 8.2.7. Let's compute $H^*(*)$ directly from the definition. Here $\mathrm{Sing}_n(*)$ consists of the unique map $c_n\colon \Delta^n \to *$, so $S^n(*) = \mathbb{Z}$ for all $n \geq 0$. For $\phi \in S^n(*)$ the coboundary $\delta\phi$ is given by

$$\delta(\phi)(c_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \phi(\partial_i c_{n+1}) = \left(\sum_{i=0}^{n+1} (-1)^i\right) \phi(c_n) = \begin{cases} 0, & n \text{ even}, \\ \phi(c_n), & n \text{ odd}. \end{cases}$$

Thus the chain complex $S^\bullet(*)$ looks like

$$\cdots 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to \cdots,$$

with cohomology

$$H^*(*) = \begin{cases} \mathbb{Z}, & * = 0, \\ 0, & * \neq 0. \end{cases}$$

Proposition 8.2.8. *If (X, A) is a subspace pair and M an abelian group, then there is a long exact sequence*

$$\cdots \to H^n(X, A; M) \to H^n(X; M) \to H^n(A; M) \to H^{n+1}(X, A; M) \to \cdots$$

Proof. Since $S_\bullet(X, A)$ is a chain complex of free abelian groups, if we apply $\mathrm{Hom}(–, M)$ to the short exact sequence

$$0 \to S_\bullet(A) \to S_\bullet(X) \to S_\bullet(X, A) \to 0,$$

we get a short exact sequence of chain complexes

$$0 \to S^\bullet(X, A; M) \to S^\bullet(X; M) \to S^\bullet(A; M) \to 0.$$

This induces a long exact sequence in homology of the required form. (Remember that H^n is a homology group in degree $-n$, so the boundary map goes from $H^n(A; M) = H_{-n}(S^\bullet(A; M))$ to $H_{-n-1}(S^\bullet(X, A; M)) = H^{n+1}(X, A; M)$.) □

Remark 8.2.9. Similarly, there is a Mayer–Vietoris sequence for cohomology: if A, B are subspaces of X such that $A^\circ \cup B^\circ = X$, and we denote the inclusions as in the square

$$\begin{array}{ccc} A \cap B & \overset{j}{\hookrightarrow} & A \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle i} \\ B & \overset{i'}{\hookrightarrow} & X, \end{array}$$

then by applying Proposition 4.8.1 to the long exact sequences in cohomology for the pairs (X, B) and $(A, A \cap B)$ we get a long exact sequence

$$\cdots \to H^n(X) \xrightarrow{(i^*, i'^*)} H^n(A) \oplus H^n(B) \xrightarrow{j^* - j'^*} H^n(A \cap B) \xrightarrow{\Delta} H^{n+1}(X),$$

where Δ is the composite

$$H^n(A \cap B) \xrightarrow{\partial} H^{n+1}(A, A \cap B) \xleftarrow{\simeq} H^{n+1}(X, B) \to H^n(X).$$

Exercise 8.8. Use the Mayer-Vietoris sequence for cohomology to compute $\tilde{H}^*(S^n; M)$.

We have a dual version of the Eilenberg–Steenrod axioms:

Definition 8.2.10. An (ordinary) *cohomology theory* consists of

- functors $h^n \colon \mathsf{Pair}^{\mathrm{op}} \to \mathsf{Ab}$, $n \in \mathbb{Z}$ (we abbreviate $h^n(X) := h^n(X, \varnothing)$),
- natural coboundary maps $\delta \colon h^n(A) \to h^{n+1}(X, A)$, so that the squares

$$\begin{array}{ccc} h^n(A) & \xrightarrow{\delta} & h^{n+1}(X, A) \\ \downarrow{\scriptstyle h^n(f|_A)} & & \downarrow{\scriptstyle h^n(f)} \\ h^n(B) & \xrightarrow{\delta} & h^{n+1}(Y, B) \end{array}$$

  commute for every map $f \colon (X, A) \to (Y, B)$,

with the following properties:

(1) (Long exact sequences) For every pair (X, A), the sequence of maps

$$\cdots \to h^n(X, A) \to h^n(X) \to h^n(A) \xrightarrow{\delta} h^{n+1}(X, A) \to \cdots$$

induced by the maps of pairs $(A, \varnothing) \to (X, \varnothing) \to (X, A)$, is a long exact sequence.

(2) (Homotopy axiom) If $f, g \colon (X, A) \to (Y, B)$ are homotopic, then $h^n(f) = h^n(g)$ for all $n \in \mathbb{Z}$.

(3) (Excision axiom) For $(X, A) \in \mathsf{Pair}$, if $U \subseteq A$ is a subset such that $\overline{U} \subseteq A^\circ$, then the homomorphisms

$$h^n(X, A) \to h^n(X \setminus U, A \setminus U)$$

induced by the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$, are isomorphisms for all $n \in \mathbb{Z}$.

(4) (Additivity axiom) If $X = \coprod_{i \in I} X_i$ is a disjoint union, then the inclusions $X_i \hookrightarrow X$ induce an isomorphism

$$h^n(X) \xrightarrow{\sim} \prod_{i \in I} h^n(X_i)$$

for all n.

(5) (Dimension axiom) $h^n(*) = 0$ if $n \neq 0$.

Proposition 8.2.11. *The singular cohomology groups $H^*(-; M)$ are a cohomology theory for every abelian group M.*

Proof. Since $\mathrm{Hom}(-, M)$ preserves chain homotopies by Exercise 8.6, the chain homotopies constructed to prove homotopy invariance in homology also give homotopy invariance for cohomology. This also allows us to transfer the key properties of barycentric subdivision to cochains, so our proof of locality (and hence excision) also works here. We leave the proof of additivity as an exercise. □

Exercise 8.9.

(i) Show that if we have short exact sequences $0 \to A_i \xrightarrow{j_i} B_i \xrightarrow{q_i} C_i \to 0$ for all $i \in I$, then there is a short exact sequence

$$0 \to \prod_{i \in I} A_i \xrightarrow{\prod_{i \in I} j_i} \prod_{i \in I} B_i \xrightarrow{\prod_{i \in I} q_i} \prod_{i \in I} C_i \to 0.$$

(ii) Use (i) to prove the additivity axiom for cohomology: for a disjoint union $X = \coprod_{i \in I} X_i$ the inclusions $X_i \hookrightarrow X$ induce an isomorphism

$$H^*(X) \xrightarrow{\sim} \prod_i H^*(X_i).$$

Lemma 8.2.12. *For a chain complex $C_\bullet$ and an abelian group M, there is a natural map*

$$H_{-k}(\mathrm{Hom}(C, M)) \to \mathrm{Hom}(H_k C, M),$$

which takes a homology class $[\phi] \in H_{-k}(\mathrm{Hom}(C, M))$, represented by $\phi \colon C_k \to M$, to the homomorphism $H_k C \to M$ given by $[c] \mapsto \phi(c)$ when $[c]$ is represented by $c \in C_k$.

Proof. We must check that this definition gives a well-defined homomorphism, and that this is natural in chain maps $C_\bullet \to C'_\bullet$ and homomorhisms $M \to M'$. Let us first see that we have a well-defined homomorphism $Z_{-k}(\mathrm{Hom}(C, M)) \to \mathrm{Hom}(H_k C, M)$: For $\phi \in Z_{-k}(\mathrm{Hom}(C, M))$ and $[c] \in H_k C$ a homology class represented by $c \in Z_k(C)$, then

$$\phi(c + \partial c') = \phi(c) + \phi(\partial c') = \phi(c) + (\delta\phi)(c') = \phi(c),$$

since $\phi \in \ker \delta$. This is also clearly a homomorphism. Now if $\phi \in B_{-k}(\mathrm{Hom}(C, M))$, so that $\phi = \delta\psi$, then $\delta\psi(c) = \psi(\partial c) = 0$ for $c \in Z_k(C)$, so $B_{-k}(\mathrm{Hom}(C, M))$ is in the kernel of the homomorphism $Z_{-k}(\mathrm{Hom}(C, M)) \to \mathrm{Hom}(H_k(C), M)$ so this factors through the cokernel $H_{-k}(\mathrm{Hom}(C, M))$, as required. We leave the (easy) proof of naturality to the reader. □

Remark 8.2.13. In particular, we have natural maps

$$H^k(X, A; M) \to \mathrm{Hom}(H_k(X, A), M).$$

These are typically *not* isomorphisms, however. Instead, we have a "dual" version of the universal coefficient theorem: the cohomology groups $H^k(X, A; M)$ are determined by $\mathrm{Hom}(H_k(X, A), M)$ together with a group $\mathrm{Ext}(H_k(X, A), M)$, where Ext is another algebraic derived functor that we now introduce.

Exercise 8.10 (Change of coefficients in cohomology)**.** Let (X, A) be a subspace pair.

(i) Show that a homomorphism of abelian groups $\phi \colon M \to M'$ induces natural maps $\phi_* \colon H^*(X, A; M) \to H^*(X, A; M')$.

(ii) Let $0 \to M' \xrightarrow{i} M \xrightarrow{q} M'' \to 0$ be a short exact sequence of abelian groups. Show that this induces a long exact sequence in cohomology

$$\cdots \to H^n(X, A; M') \xrightarrow{i_*} H^n(X, A; M) \xrightarrow{q_*} H^n(X, A; M'') \to H^{n+1}(X, A; M') \to \cdots.$$

8.3 *Ext of Abelian Groups*

The functor Ext has a similar relationship to Hom as Tor does to the tensor product:

Definition 8.3.1. Let A, B, be abelian groups, and take a free resolution $0 \to F_1 \xrightarrow{i} F_0 \xrightarrow{q} A \to 0$ of A. Then we define $\mathrm{Ext}(A, B)$ to be the cokernel of $i^* \colon \mathrm{Hom}(F_0, B) \to \mathrm{Hom}(F_1, B)$.

Remark 8.3.2. If we think of the free resolution as a chain complex $F_\bullet$, then we have

$$H_0(\mathrm{Hom}(F_\bullet, M)) = \mathrm{Hom}(A, M),$$

$$H_{-1}(\mathrm{Hom}(F_\bullet, M)) = \mathrm{Ext}(A, M).$$

Lemma 8.3.3. *For abelian groups A and B,* $\mathrm{Ext}(A, B)$ *is well-defined, i.e. independent of the choice of free resolution.*

Proof. If $F_\bullet$ and $F'_\bullet$ are two free resolutions, then they are chain homotopy equivalent by Corollary 7.3.13. Then $\mathrm{Hom}(F_\bullet, M)$ and $\mathrm{Hom}(F'_\bullet, M)$ are also chain homotopy equivalent, since $\mathrm{Hom}(-, M)$ preserves chain homotopies by Exercise 8.6. The homology groups $H_{-1}(\mathrm{Hom}(F_\bullet, M))$ and $H_{-1}(\mathrm{Hom}(F'_\bullet, M))$ are therefore isomorphic. □

Example 8.3.4. If F is a free abelian group, then we can take $0 \to 0 \to F \xrightarrow{\mathrm{id}} F \to 0$ as a free resolution, giving

$$\mathrm{Ext}(F, B) = \mathrm{coker}(\mathrm{Hom}(F, B) \to \mathrm{Hom}(0, B)) = 0.$$

Example 8.3.5. For $\mathbb{Z}/m$ we can take $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0$ as a free resolution. Then $m^* \colon \mathrm{Hom}(\mathbb{Z}, B) \to \mathrm{Hom}(\mathbb{Z}, B)$ is the map $B \to B$ given by multiplication with m by Exercise 8.5, and so

$$\mathrm{Ext}(\mathbb{Z}/m, B) = \mathrm{coker}(B \xrightarrow{m} B) = B/mB.$$

Example 8.3.6. If D is a *divisible* abelian group (such as $\mathbb{Q}$) then $\mathrm{Hom}(-, D)$ preserves short exact sequences, and thus $\mathrm{Ext}(A, D) = 0$ for all A.

Proposition 8.3.7. *Suppose we have a short exact sequence of abelian groups $0 \to A' \to A \to A'' \to 0$. Then for any abelian group B there is an exact sequence*

$$0 \to \mathrm{Hom}(A'', B) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B) \to \mathrm{Ext}(A'', B) \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B) \to 0.$$

Proof. By Proposition 7.3.15 we can lift the short exact sequence to a short exact sequence of free resolutions

$$0 \to F'_\bullet \to F_\bullet \to F''_\bullet \to 0.$$

Applying $\mathrm{Hom}(-, B)$ gives a short exact sequence

$$0 \to \mathrm{Hom}(F'', B) \to \mathrm{Hom}(F, B) \to \mathrm{Hom}(F', B) \to 0$$

by Corollary 8.1.9. This gives the required exact sequence as its long exact sequence in homology. □

Proposition 8.3.8. *Suppose we have a short exact sequence of abelian groups $0 \to B' \to B \to B'' \to 0$. Then for any abelian group A there is an exact sequence*

$$0 \to \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B'') \to \mathrm{Ext}(A, B') \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B'') \to 0.$$

Proof. Let $F_\bullet$ be a free resolution of A. Then by Lemma 8.1.10 we have a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}(F, B') \to \mathrm{Hom}(F, B) \to \mathrm{Hom}(F, B'') \to 0,$$

which gives the required long exact sequence in homology. □

Remark 8.3.9. From this exact sequence we see that if we can find a short exact sequence

$$0 \to B \to I_0 \xrightarrow{i} I_{-1} \to 0,$$

where $\mathrm{Ext}(A, I_j) = 0$, then we can compute $\mathrm{Ext}(A, B)$ as the cokernel of $i_* \colon \mathrm{Hom}(A, I_0) \to \mathrm{Hom}(A, I_{-1})$. In fact, it can be shown that we can always find such a resolution by divisible abelian groups.

8.4 The Universal Coefficient Theorem for Cohomology

Theorem 8.4.1 (Universal Coefficient Theorem for Cohomology)**.** *For (X, A) a subspace pair and M an abelian group, there are natural short exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(X, A), M) \to H^n(X, A; M) \to \mathrm{Hom}(H_n(X, A), M) \to 0.$$

This is a special case of the following algebraic result:

Proposition 8.4.2. *Let $C_\bullet$ be a levelwise free chain complex and M an abelian group. Then there are natural short exact sequences*

$$0 \to \operatorname{Ext}(H_{n-1}(C), M) \to H_{-n}(\operatorname{Hom}(C, M)) \to \operatorname{Hom}(H_n(C), M) \to 0.$$

Proof. The proof is very similar to that of the universal coefficient theorem for homology. We abbreviate $B_n := B_n(C), Z_n := Z_n(C), H_n := H_n(C)$. Then we have short exact sequences

$$0 \to B_n \xrightarrow{j_n} Z_n \to H_n \to 0,$$

where B_n and Z_n are free (since they are subgroups of the free abelian group C_n), so this is a free presentation of H_n. We also have short exact sequences

$$0 \to Z_n \to C_n \xrightarrow{\partial'} B_{n-1} \to 0$$

where ∂' is the boundary map in $C_\bullet$ viewed as a map to its image. If $B'_n := B_{n-1}$ then we can regard this as a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B'_\bullet \to 0,$$

where $Z_\bullet$ and $B_\bullet$ have 0 differentials. Since the groups B'_n are free, this short exact sequence is levelwise splittable, and so we can apply $\operatorname{Hom}(-, M)$ to get a new short exact sequence

$$0 \to \operatorname{Hom}(B'_\bullet, M) \to \operatorname{Hom}(C_\bullet, M) \to \operatorname{Hom}(Z_\bullet, M) \to 0.$$

Since $B'_\bullet$ and $Z_\bullet$ have zero differential, this gives a long exact sequence in homology of the form

$$\cdots \to \operatorname{Hom}(B_{-n-1}, M) \to H_n(\operatorname{Hom}(C, M)) \to \operatorname{Hom}(Z_{-n}, M) \xrightarrow{\Delta_n} \operatorname{Hom}(B_{-n}, M) \to \cdots,$$

where we write Δ for the boundary map in the long exact sequence to reduce confusion. We therefore have short exact sequences

$$0 \to \operatorname{coker} \Delta_{n+1} \to H_n \operatorname{Hom}(C, M) \to \ker \Delta_n \to 0.$$

Unwinding the definition of Δ_n, we see that

$$\Delta_n = j_{-n}^* \colon \operatorname{Hom}(Z_{-n}, M) \to \operatorname{Hom}(B_{-n}, M).$$

Since j_{-n} gives a free presentation of H_{-n}, we get

$$\ker \Delta_n \cong \operatorname{Hom}(H_{-n}, M), \quad \operatorname{coker} \Delta_n \cong \operatorname{Ext}(H_{-n}, M),$$

as required. □

Remark 8.4.3. With a bit more work, it can be shown that the short exact sequences in the universal coefficient theorem are splittable (but the splittings are not natural). Thus, there are non-canonical isomorphisms

$$H^n(X, A; M) \cong \operatorname{Hom}(H_n(X, A), M) \oplus \operatorname{Ext}(H_{n-1}(X, A), M).$$

Corollary 8.4.4. *If a continuous map of pairs $f \colon (X, A) \to (Y, B)$ induces isomorphisms $f_* \colon H_*(X, A) \to H_*(Y, B)$ for all $*$, then $f^* \colon H^*(Y, B) \to H^*(X, A)$ are also isomorphisms.*

Proof. Apply the 5-Lemma to the map of short exact sequences from the universal coefficient theorem. □

Corollary 8.4.5. *If D is a divisible abelian group (such as $\mathbb{Q}$), then there is a natural isomorphism*

$$H^*(X, A; D) \xrightarrow{\sim} \mathrm{Hom}(H_*(X, A), D).$$

Remark 8.4.6. In particular, if $H_n(X, A)$ is a finite-rank abelian group, then $H_n(X, A; \mathbb{Q})$ and $H^n(X, A; \mathbb{Q})$ are $\mathbb{Q}$-vector spaces of the same dimension.

Example 8.4.7. Let us compute $H^*(\mathbb{RP}^n)$ using the short exact sequences

$$0 \to \mathrm{Ext}(H_{n-1}(\mathbb{RP}^n), \mathbb{Z}) \to H^n(\mathbb{RP}^n) \to \mathrm{Hom}(H_n(\mathbb{RP}^n), \mathbb{Z}) \to 0.$$

Recall that

$$H_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2, & i \text{ odd}, i < n, \\ 0, & i \text{ even} > 0 \text{ or} > n, \\ \mathbb{Z}, & i = 0 \text{ or } i = n \text{ odd}. \end{cases}$$

Thus we have short exact sequences

$$0 \to 0 \to H^0(\mathbb{RP}^n) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0,$$

$$0 \to \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}) \to H^i(\mathbb{RP}^n) \to 0 \to 0, \quad (i \text{ even} \leq n)$$

$$0 \to 0 \to H^i(\mathbb{RP}^n) \to \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) \to 0, \quad (i \text{ odd} < n)$$

$$0 \to 0 \to H^i(\mathbb{RP}^n) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0, \quad (i = n \text{ odd}).$$

Here $\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}) \cong \mathbb{Z}/2$ and $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0$, so we conclude

$$H^i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}, & i = 0 \text{ or } i = n \text{ odd}, \\ \mathbb{Z}/2, & 0 < i \leq n \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Remark 8.4.8. Suppose A is a finitely generated abelian group. Then by the classification of finitely generated abelian groups we can write $A \cong F \oplus T$ where $F \cong \mathbb{Z}^r$ is free and $T \cong \bigoplus_i \mathbb{Z}/m_i$ is a torsion group. We then have

$$\mathrm{Hom}(A, \mathbb{Z}) \cong \mathrm{Hom}(F, \mathbb{Z}) \oplus \mathrm{Hom}(T, \mathbb{Z}) \cong \bigoplus_r \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \bigoplus_i \mathrm{Hom}(\mathbb{Z}/m_i, \mathbb{Z}).$$

Here $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ while $\mathrm{Hom}(\mathbb{Z}/m_i, \mathbb{Z}) \cong 0$ so we have an isomorphism $\mathrm{Hom}(A, \mathbb{Z}) \cong F$. On the other hand (as it is easy to see that $\mathrm{Ext}(-, B)$ also takes direct sums to products)

$$\mathrm{Ext}(A, \mathbb{Z}) \cong \mathrm{Ext}(F, \mathbb{Z}) \oplus \mathrm{Ext}(T, \mathbb{Z}) \cong \bigoplus_r \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) \oplus \bigoplus_i \mathrm{Ext}(\mathbb{Z}/m_i, \mathbb{Z}).$$

Here $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$ while $\mathrm{Ext}(\mathbb{Z}/m_i, \mathbb{Z}) \cong \mathbb{Z}/m_i$, hence we have an isomorphism $\mathrm{Ext}(A, \mathbb{Z}) \cong T$. We can apply this to get a (non-canonical) description of the cohomology groups of a space X whose homology groups $H_n X$ are all finitely generated: If we write $H_n(X) \cong F_n \oplus T_n$ where $F_n \cong \mathbb{Z}^{r_n}$ is free and T_n is a torsion group, then

$$\mathrm{Hom}(H_n(X), \mathbb{Z}) \cong \mathrm{Hom}(F_n, \mathbb{Z}) \cong F_n,$$

while

$$\mathrm{Ext}(H_n(X), \mathbb{Z}) \cong \mathrm{Ext}(T_n, \mathbb{Z}) \cong T_n,$$

so the universal coefficient theorem gives (non-canonical) isomorphisms

$$H^n(X) \cong F_n \oplus T_{n-1}.$$

So in this case torsion in homology shifts up by one degree in cohomology, while the free groups stay put.

8.5 Cellular Cohomology

Suppose X is a cell complex, with Γ_k the set of k-dimensional cells. Then we have isomorphisms

$$H^*(X_k, X_{k-1}; M) \cong \tilde{H}^*(\bigvee_{\Gamma_k} S^k; M) \cong \prod_{\Gamma_k} \tilde{H}^*(S^k; M) \cong \begin{cases} M^{\Gamma_k}, & * = k, \\ 0, & * \neq k. \end{cases}$$

We can define a *cellular cohomology chain complex* by taking $C^k_{\mathrm{cell}}(X; M) := H^k(X_k, X_{k-1}; M)$ in degree $-k$, with differential the composite

$$\delta \colon H^k(X_k, X_{k-1}; M) \to H^k(X_k; M) \to H^{k+1}(X_{k+1}, X_k; M),$$

where the first map uses the inclusion $(X_k, \emptyset) \to (X_k, X_{k-1})$ and the second is the connecting homomorphism in the cohomology long exact sequence for the pair (X_{k+1}, X_k). Just as for cellular homology we get $\delta^2 = 0$, and by studying the long exact sequences for the pairs (X_k, X_{k-1}) we get:

Proposition 8.5.1. *$H_{-k}(C^\bullet_{\mathrm{cell}}(X; M)) \cong H^k(X; M)$.* □

We also have the chain complex $\mathrm{Hom}(C^{\mathrm{cell}}_\bullet(X), M)$, where we already understand the differential. Luckily, this is isomorphic to the cellular cochain complex:

Proposition 8.5.2. *If X is a cell complex, there is an isomorphism of chain complexes*

$$C^\bullet_{\mathrm{cell}}(X; M) \cong \mathrm{Hom}(C^{\mathrm{cell}}_\bullet(X), M).$$

For the proof we need the following observation, which is left as an exercise:

Exercise 8.11. Let (X, A) be a subspace pair and M an abelian group. Show that the natural maps $H^*(-; M) \to \mathrm{Hom}(H_*(-), M)$ fit in a commutative square

$$\begin{array}{ccc} H^n(A; M) & \xrightarrow{\delta} & H^{n+1}(X, A; M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(H_n(A); M) & \xrightarrow{\partial^*} & \mathrm{Hom}(H_{n+1}(X, A); M), \end{array}$$

where ∂ and δ denote the connecting maps in the homology and cohomology long exact sequences for (X, A), respectively.

Proof of Proposition 8.5.2. In degree $-k$, we have

$$C^k_{\mathrm{cell}}(X;M) = H^k(X_k, X_{k-1};M), \quad \mathrm{Hom}(C^{\mathrm{cell}}_k(X), M) = \mathrm{Hom}(H_k(X_k, X_{k-1});M).$$

We have a natural map

$$H^k(X_k, X_{k-1};M) \to \mathrm{Hom}(H_k(X_k, X_{k-1});M),$$

and by the universal coefficient theorem this is surjective with kernel $\mathrm{Ext}(H_{k-1}(X_k, X_{k-1}), M)$, which is 0 since $H_{k-1}(X_k, X_{k-1}) = 0$. Thus we have degreewise isomorphisms $C^k_{\mathrm{cell}}(X;M) \cong \mathrm{Hom}(C^{\mathrm{cell}}_k(X), M)$, and we need to show these are compatible with the differentials. To see this we will show that there is a commutative diagram

$$\begin{array}{ccccc} H^k(X_k, X_{k-1};M) & \longrightarrow & H^k(X_k;M) & \longrightarrow & H^{k+1}(X_{k+1}, X_k;M) \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ \mathrm{Hom}(H_k(X_k, X_{k-1}), M) & \longrightarrow & \mathrm{Hom}(H_k(X_k), M) & \longrightarrow & \mathrm{Hom}(H_{k+1}(X_{k+1}, X_k), M). \end{array}$$

Here the left-hand square commutes by naturality, since both maps are induced by the map of singular chain complexes from the inclusion $(X_k, \emptyset) \to (X_k, X_{k-1})$, while the right-hand square commutes by Exercise 8.11. □

Since we know the differential in $\mathrm{Hom}(C^{\mathrm{cell}}_\bullet(X), M)$, we get the following description of the differential in cellular cohomology:

Corollary 8.5.3. *For $\alpha \in \Gamma_{k+1}$ let $f_\alpha \colon S^k \to X_k$ be the attaching map for the corresponding cell, and for $\beta \in \Gamma_k$ let $q_\beta \colon X_k \to X_k/X_{k-1} \cong \bigvee_{\Gamma_k} S^k \to S^k$ be the projection to the sphere corresponding to β. Then the differential $C^k_{\mathrm{cell}}(X;M) \to C^{k+1}_{\mathrm{cell}}(X;M)$ corresponds to the map $M^{\Gamma_k} \to M^{\Gamma_{k+1}}$ that takes $f \colon \Gamma_k \to M$ to the function*

$$\alpha \in \Gamma_{k+1} \mapsto \sum_{\beta \in \Gamma_k} \deg(q_\beta f_\alpha) f(\beta).$$

Example 8.5.4. Let us compute $H^*(\mathbb{RP}^n)$ using cellular cohomology: Recall that this has a cell structure with a single cell e_i in degree i, $0 \leq i \leq n$, and that the cellular chain complex looks like

$$\cdots \to 0 \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \cdots,$$

with the non-zero groups in degrees $0, 1, \ldots, n$. Applying $\mathrm{Hom}(-, \mathbb{Z})$, we get

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \to \mathbb{Z} \to 0 \to \cdots,$$

with the non-zero groups in degrees $-n, -n+1, \ldots, 0$. We get cocycles

$$Z^i = \begin{cases} \mathbb{Z}, & i \text{ even}, -n \leq i \leq 0, \\ \mathbb{Z}, & i = -n \text{ odd}, \\ 0, & \text{otherwise}, \end{cases}$$

with coboundaries

$$B^i = \begin{cases} 0 & i = 0, \\ 2\mathbb{Z}, & i \text{ even}, -n \leq i < 0 \\ 0, & \text{otherwise.} \end{cases}$$

The cohomology is therefore given as

$$H^i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}, & i = 0 \text{ or } i = n \text{ odd}, \\ \mathbb{Z}/2, & 0 < i \leq n \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

9
Homology of Products

Our next goal is to understand the relationship between the homology of spaces X, Y and the homology of their cartesian product $X \times Y$. This will have two parts: First, we will define the tensor product of chain complexes in §9.1, where we also prove a result that describes the homology of a tensor product of levelwise free chain complexes. Then in §9.2 we will prove that the singular chain complex $S_\bullet(X \times Y)$ is chain homotopy equivalent to the tensor product $S_\bullet(X) \otimes S_\bullet(Y)$ (the Eilenberg–Zilber theorem) and use this to describe the homology of $X \times Y$ (the Künneth theorem). In §9.3 we consider an explicit formula for one of the maps in this chain homotopy equivalence, the Alexander–Whitney map. Next we look at the Künneth theorem for cohomology in §9.4, which requires a finiteness assumption on the spaces involved, and finally we consider relative versions of the Eilenberg–Zilber and Künneth theorems in §9.5.

9.1 *Tensor Products of Chain Complexes*

Definition 9.1.1. A *graded abelian group* A_* is a sequence of abelian groups A_n, $n \in \mathbb{Z}$, and a homomorphism of graded abelian groups $\phi\colon A_* \to B_*$ is just a sequence of homomorphisms $\phi_n\colon A_n \to B_n$. These assemble into a category grAb.

Remark 9.1.2. We can view homology as a functor $H_*\colon \mathsf{Ch} \to \mathsf{grAb}$.

Definition 9.1.3. If A_* and B_* are graded abelian groups, their *tensor product* $A_* \otimes B_*$ is the graded abelian group with

$$(A_* \otimes B_*)_n = \bigoplus_{p+q=n} A_p \otimes B_q.$$

Remark 9.1.4. By the universal property of the tensor product of abelian groups, a homomorphism of graded abelian groups $A_* \otimes B_* \to C_*$ corresponds to a family of bilinear maps $A_p \times B_q \to C_{p+q}$.

Definition 9.1.5. If $(C_\bullet, \partial_C)$ and $(D_\bullet, \partial_D)$ are chain complexes, their *tensor product* is the chain complex $C_\bullet \otimes D_\bullet$ with underlying graded abelian groups the corresponding tensor product of graded abelian groups, and with differential $\partial\colon (C \otimes D)_n \to (C \otimes D)_{n-1}$ determined on a generator $c \otimes d$ ($c \in C_p, d \in D_q, p+q=n$) by

$$\partial(c \otimes d) = \partial_C c \otimes d + (-1)^p c \otimes \partial_D d$$

We have

$$\partial^2(c \otimes d) = \partial_C^2 c \otimes d + (-1)^{p-1}\partial_C c \otimes \partial_D d + (-1)^p \partial_C c \otimes \partial_D d + (-1)^{2p} c \otimes \partial_D^2 d = 0,$$

so this is indeed a chain complex.

Remark 9.1.6. Other choices of sign are possible, but note that we need to put *some* non-trivial signs in order to get $\partial^2 = 0$.

Notation 9.1.7. If A is an abelian group, we write $A[n]$ for the chain complex with A in degree n and 0 elsewhere; we also denote the underlying graded abelian group by $A[n]$.

Remark 9.1.8. The associativity and unitality of the tensor product of abelian groups gives associativity and unitality for the tensor products of graded abelian groups and chain complexes: given graded abelian groups A_*, B_*, C_* we have canonical isomorphisms

$$A_* \otimes (B_* \otimes C_*) \cong (A_* \otimes B_*) \otimes C_*, \quad A_* \otimes \mathbb{Z}[0] \cong A_*,$$

while given chain complexes $A_\bullet, B_\bullet, C_\bullet$ we have canonical isomorphisms

$$A_\bullet \otimes (B_\bullet \otimes C_\bullet) \cong (A_\bullet \otimes B_\bullet) \otimes C_\bullet, \quad A_\bullet \otimes \mathbb{Z}[0] \cong A_\bullet.$$

Remark 9.1.9. The tensor product of chain complexes is also symmetric, but this requires changing some signs: we define the twist isomorphism

$$\tau \colon C_\bullet \otimes D_\bullet \xrightarrow{\sim} D_\bullet \otimes C_\bullet$$

on generators $c \otimes d$ for $c \in C_p$, $d \in D_q$ by

$$\tau(c \otimes d) = (-1)^{pq} d \otimes c.$$

Then τ is a chain map:

$$\begin{aligned} \tau\partial(c \otimes d) &= \tau(\partial c \otimes d + (-1)^p c \otimes \partial d) \\ &= (-1)^{(p-1)q} d \otimes \partial c + (-1)^p (-1)^{p(q-1)} \partial d \otimes c \\ &= (-1)^{pq}(\partial d \otimes c + (-1)^q d \otimes \partial c) \\ &= \partial\tau(c \otimes d). \end{aligned}$$

Note that we again *need* to put some non-trivial signs for τ to be a chain map. We define the twist isomorphism $A_* \otimes B_* \xrightarrow{\sim} B_* \otimes A_*$ in the same way, so that it is compatible with the forgetful functor $\mathsf{Ch} \to \mathsf{grAb}$.

Exercise 9.1. Let $I_\bullet$ denote the chain complex with $I_1 = \mathbb{Z}$, $I_0 = \mathbb{Z}\{[0], [1]\}$ and $I_n = 0$ otherwise, with differential $\partial \colon I_1 \to I_0$ given by $\partial(1) = [1] - [0]$. Show that a chain homotopy between chain maps $C_\bullet \to D_\bullet$ is the same thing as a chain map $C_\bullet \otimes I_\bullet \to D_\bullet$. (Thus if we think of $I_\bullet$ as an "algebraic interval", chain homotopies are an algebraic version of homotopies between continuous maps.)

Exercise 9.2. Let $C_\bullet$ be a chain complex.

(i) Prove that the functor $C_\bullet \otimes -$ preserves chain homotopies and levelwise splittable short exact sequences of chain complexes. [Hint: For chain homotopies you can use Exercise 9.1.]

(ii) If $C_\bullet$ is levelwise free, show that $C_\bullet \otimes -$ preserves all short exact sequences of chain complexes.

Exercise 9.3. Show that if two chain complexes differ only by the signs of the boundary maps, then they are isomorphic.

Exercise 9.4.

(i) Show that for M an abelian group and $C_\bullet, D_\bullet$ chain complexes, there is a natural bijection between chain maps $C_\bullet \to \mathrm{Hom}(D, M)_\bullet$ and chain maps $C_\bullet \otimes D_\bullet \to M[0]$. [With our sign convention for the differential in $\mathrm{Hom}(D, M)_\bullet$ this bijection involves some signs. Alternatively, we can define the differential $\delta\phi$ for $\phi\colon D_{-n} \to M$ to be given by $(\delta\phi)(d) = (-1)^{n+1}\phi(\partial d)$ (without changing the homology, by Exercise 9.3).]

(ii)* For chain complexes $C_\bullet, D_\bullet$, define a chain complex $\mathrm{Hom}(C, D)_\bullet$ so that there is a natural bijection between chain maps $C_\bullet \otimes D_\bullet \to E_\bullet$ and chain maps $C_\bullet \to \mathrm{Hom}(D_\bullet, E_\bullet)$. [Hint: Do it first for graded abelian groups and then figure out the differential. This again involves some signs, and if you want a sign convention that recovers our previous definition of $\mathrm{Hom}(D, M)$ as $\mathrm{Hom}(D, M[0])$ then the bijection between maps also needs some signs.]

Lemma 9.1.10. *If $C_\bullet, D_\bullet$ are chain complexes, then there is a natural map of graded abelian groups*

$$H_*(C) \otimes H_*(D) \to H_*(C \otimes D),$$

compatible with the associativity, unitality, and symmetry isomorphisms for the two tensor products.

Proof. On generators $[c] \in H_pC, [d] \in H_qD$ represented by $c \in C_p$ and $d \in D_q$, we want to assign $[c \otimes d] \in H_{p+q}(C \otimes D)$. It is clear from the formula for ∂ in $C \otimes D$ that this is again a cycle, and we have

$$(c + \partial c') \otimes d = c \otimes d + \partial c' \otimes d = c \otimes d + \partial(c' \otimes d)$$

since d is a cycle, and similarly in the other variable, so that this map is well-defined. We leave the rest of the (easy) argument to the reader. □

This natural map is typically *not* an isomorphism. However, in the case where one of the chain complexes is levelwise free we have an algebraic description of the homology of the tensor product. To state this, we need some notation:

Definition 9.1.11. Let A_* and B_* be graded abelian group. We define the graded abelian group $\mathrm{Tor}(A, B)_*$ by

$$\mathrm{Tor}(A, B)_n = \bigoplus_{p+q=n} \mathrm{Tor}(A_p, B_q).$$

Proposition 9.1.12. *Let $C_\bullet$ and $D_\bullet$ be chain complexes, and suppose $C_\bullet$ is levelwise free. Then there are natural short exact sequences*

$$0 \to (H_*(C) \otimes H_*(D))_n \to H_n(C \otimes D) \to \mathrm{Tor}(H_*(C), H_*(D))_{n-1} \to 0.$$

Note that this is a generalization of Proposition 7.4.2, which was the algebraic input to the universal coefficient theorem in homology. The proof will also be essentially the same, but we first need the following observation:

Lemma 9.1.13. *Suppose $C_\bullet$ is a levelwise free chain complex with zero differentials. Then for any chain complex $D_\bullet$, the natural map*

$$C_* \otimes H_*(D) \cong H_*(C) \otimes H_*(D) \to H_*(C \otimes D)$$

is an isomorphism.

Proof. Since the differentials in $C_\bullet$ are zero, the differential $\partial\colon (C_\bullet \otimes D_\bullet)_n \to (C_\bullet \otimes D_\bullet)_{n-1}$ is given on $c \otimes d$ with $c \in C_p$, $d \in D_q$ ($p+q = n$) by

$$\partial(c \otimes d) = (-1)^p c \otimes \partial d.$$

Since $C_\bullet$ is levelwise free, so that tensoring with C_n preserves short exact sequences, we therefore have isomorphisms

$$Z_n(C \otimes D) \cong \bigoplus_{p+q=n} C_p \otimes Z_q(D),$$

$$B_n(C \otimes D) \cong \bigoplus_{p+q=n} C_p \otimes B_q(D),$$

$$H_n(C \otimes D) \cong \bigoplus_{p+q=n} C_p \otimes H_q(D),$$

as required. □

Proof of Proposition 9.1.12. We abbreviate $Z_k := Z_k(C), B_k := B_k(C)$, and $H_k := H_k(C)$. Then we have short exact sequences

$$0 \to B_k \xrightarrow{j_k} Z_k \to H_k \to 0,$$

and since C_k is levelwise free this is a free presentation of H_k. As in the proof of Proposition 7.4.2 we also have a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \xrightarrow{\partial'} B'_\bullet \to 0,$$

where $B'_n = B_{n-1}$ and the chain complexes $Z_\bullet$ and $B'_\bullet$ have zero differentials, with ∂' denoting the differential in $C_\bullet$ viewed as a homomorphism to its image.

This short exact sequence is levelwise splittable since B'_n is a free abelian group (being a subgroup of the free abelian group C_{n-1}), so by Exercise 9.2 we can tensor with $D_\bullet$ and get another levelwise split short exact sequence

$$0 \to Z_\bullet \otimes D_\bullet \to C_\bullet \otimes D_\bullet \to B'_\bullet \otimes D_\bullet \to 0.$$

This gives a long exact sequence in homology, and by Lemma 9.1.13 this looks like

$$\cdots \xrightarrow{\Delta_{n+1}} (Z_* \otimes H_*(D))_n \to H_n(C \otimes D) \to (B_* \otimes H_*(D))_{n-1} \xrightarrow{\Delta_n} (Z_* \otimes H_*(D))_{n-1} \to \cdots,$$

where Δ_n is the boundary map in the long exact sequence. We therefore have short exact sequences

$$0 \to \operatorname{coker} \Delta_{n+1} \to H_n(C \otimes D) \to \ker \Delta_n \to 0.$$

Unwinding the definition of Δ_n as in the proof of Proposition 7.4.2 we see that

$$\Delta_{n+1} = (j \otimes \mathrm{id})_n \colon (B_* \otimes H_*(D))_n \to (Z_* \otimes H_*(D))_n.$$

Since j_p gives a free presentation of $H_p(C)$ we therefore have isomorphisms

$$\operatorname{coker} \Delta_{n+1} \cong (H_*(C) \otimes H_*(D))_n,$$
$$\ker \Delta_{n+1} \cong \operatorname{Tor}(H_*(C), H_*(D))_n,$$

which give the short exact sequences we want. □

Remark 9.1.14. With a bit more work it can be shown that the short exact sequences in Proposition 9.1.12 are always splittable, but the splittings are not natural. Thus there are non-canonical isomorphisms

$$H_n(C \otimes D) \cong (H_*(C) \otimes H_*(D))_n \oplus \operatorname{Tor}(H_*(C), H_*(D))_{n-1}.$$

The following consequence might seem trivial, but will be useful to us in the next section:

Corollary 9.1.15. *If X and Y are contractible topological spaces, then*

$$H_*(S_\bullet(X) \otimes S_\bullet(Y)) \cong \begin{cases} \mathbb{Z}, & * = 0, \\ 0, & * \neq 0 \end{cases}$$

Remark 9.1.16. The proof of Proposition 9.1.12 works the same over any principal ideal domain R, using the tensor product of R-modules. In particular, if k is a field, then (as any k-module is free and Tor vanishes) we have natural isomorphisms

$$H_*(C \otimes_k D) \cong H_*(C) \otimes_k H_*(D)$$

for arbitrary chain complexes $C_\bullet$ and $D_\bullet$ of k-vector spaces.

9.2 *The Eilenberg–Zilber and Künneth Theorems*

If X and Y are topological spaces, we can apply Proposition 9.1.12 to $S_\bullet(X)$ and $S_\bullet(Y)$ to get a description of $H_*(S(X) \otimes S(Y))$. We are now going to prove that the chain complex $S_\bullet(X) \otimes S_\bullet(Y)$ is chain homotopy equivalent to $S_\bullet(X \times Y)$, so that this actually gives a description of the homology of $X \times Y$.

Note that in degree 0 we have

$$(S_\bullet(X) \otimes S_\bullet(Y))_0 \cong S_0(X) \otimes S_0(Y) \cong \mathbb{Z}X \otimes \mathbb{Z}Y$$

There is a canonical isomorphism between $\mathbb{Z}X \otimes \mathbb{Z}Y$ and $\mathbb{Z}(X \times Y)$, and this gives a canonical isomorphism $S_0(X \times Y) \xrightarrow{\sim} (S_\bullet(X) \otimes S_\bullet(Y))_0$ defined on generators (x, y) with $x \in X, y \in Y$ by $(x, y) \mapsto x \otimes y$.

Proposition 9.2.1.

(i) *There exists a natural chain map $\phi\colon S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$ given by the canonical isomorphism in degree* 0.

(ii) *Any two such natural chain maps are naturally chain homotopic.*

Proof. We will use the method of acyclic models to show that such maps exist without having to give an explicit geometric or combinatorial construction.

For (i), we will inductively define $\phi_n^{X,Y} \colon S_n(X \times Y) \to (S_\bullet(X) \otimes S_\bullet(Y))_n$ assuming we have already defined natural maps $\phi_k^{X,Y}$ for $k < n$, such that $\partial\phi_k^{X,Y} = \phi_{k-1}^{X,Y}\partial$ for $k < n$. At $n = 0$ we start with the canonical isomorphism. If $(\sigma, \tau) \colon \Delta^n \to X \times Y$ is an n-simplex, then $(\sigma, \tau) = (\sigma \times \tau)_* \delta_n$, where δ_n is the diagonal $\Delta^n \to \Delta^n \times \Delta^n$. Thus we first consider the universal case of δ_n and then set $\phi_n^{X,Y}(\sigma, \tau) = (\sigma_* \otimes \tau_*)\phi_n^{\Delta^n,\Delta^n}(\delta_n)$ (which must be true for naturality to hold); extending this definition linearly we then get natural homomorphisms $\phi_n^{X,Y}$.

There are two cases to consider: If $n = 1$, then we want an element $\phi_1(\delta_1)$ such that

$$\partial\phi_1(\delta_1) = \phi_0(\partial\delta_1) = \phi_0(([1],[1]) - ([0],[0])) = [1] \otimes [1] - [0] \otimes [0].$$

We can take $\phi_1(\delta_1) = [0] \otimes \iota_1 + \iota_1 \otimes [1]$; then

$$\partial\phi_1(\delta_1) = [0] \otimes ([1] - [0]) + ([1] - [0]) \otimes [1] = -[0] \otimes [0] + [1] \otimes [1].$$

For $n > 1$, we want to find an element $\phi_n(\delta_n) \in (S_\bullet(\Delta^n) \otimes S_\bullet(\Delta^n))_n$ such that $\partial\phi_n(\delta_n) = \phi_{n-1}(\partial\delta_n)$. We know by Corollary 9.1.15 that $H_{n-1}(S_\bullet(\Delta^n) \otimes S_\bullet(\Delta^n)) = 0$, so the right-hand side is a boundary if and only if it is a cycle. We have $\partial\phi_{n-1}(\partial\delta_n) = \phi_{n-2}(\partial^2\delta_n) = 0$ since we know ϕ_{n-1} commutes with ∂, so some suitable class $\phi_n(\delta_n)$ exists.

It remains to check that the resulting homomorphisms $\phi_n^{X,Y}$ satisfy $\partial\phi_n^{X,Y} = \phi_{n-1}^{X,Y}\partial$. It's enough to check this on a singular simplex $(\sigma, \tau) \colon \Delta^n \to X \times Y$, for which we have

$$\partial\phi_n^{X,Y}(\sigma,\tau) = \partial(\sigma_* \otimes \tau_*)\phi_n(\delta_n) = (\sigma_* \otimes \tau_*)\partial\phi_n(\delta_n) = (\sigma_* \otimes \tau_*)\phi_{n-1}(\partial\delta_n) = \phi_{n-1}^{X,Y}((\sigma \times \tau)_*\partial\delta_n) = \phi_{n-1}^{X,Y}(\partial(\sigma,\tau).$$

This proves (i).

Now suppose we have two such natural chain maps ϕ and ϕ'. To prove (ii) we want to define natural maps

$$s_n^{X,Y} \colon S_n(X \times Y) \to (S_\bullet(X) \otimes S_\bullet(Y))_{n+1}$$

such that $\partial s_n + s_{n-1}\partial = \phi'_n - \phi_n$.

For $n = 0$ we know $\phi'_0 = \phi_0$ so we can take $s_0 = 0$ (since $s_{-1} = 0$ as $S_{-1}(X \times Y) = 0$). Suppose we have defined s_k for $k < n$. We again start by defining $s_n(\delta_n)$; this should satisfy

$$\partial s_n(\delta_n) = \phi'_n(\delta_n) - \phi_n(\delta_n) - s_{n-1}(\partial\delta_n).$$

Since $H_n(S_\bullet(\Delta^n) \otimes S_\bullet(\Delta^n)) = 0$ to see that such an element $s_n(\delta_n)$ exists it's enough to check that the right-hand side is a cycle:

$$\partial(\phi'_n(\delta_n) - \phi_n(\delta_n) - s_{n-1}(\partial\delta_n)) = \partial(\phi'_n - \phi_n)(\delta_n) + s_{n-2}(\partial^2\delta_n) - (\phi'_{n-1} - \phi_{n-1})(\partial\delta_n) = 0,$$

since $\phi' - \phi$ is a chain map. Now we as usual set $s_n^{X,Y}(\sigma,\tau) = (\sigma_* \otimes \tau_*)s_n(\delta_n)$ and extend linearly to obtain natural homomorphisms. We now just need to check this gives a chain homotopy:

$$(\partial s_n + s_{n-1}\partial)(\sigma,\tau) = \partial(\sigma_* \otimes \tau_*)(\partial s_n(\delta_n) + s_n(\partial\delta_n) = (\sigma_* \otimes \tau_*)(\phi'_n\delta_n - \phi_n\delta_n) = \phi'_n(\sigma,\tau) - \phi_n(\sigma,\tau).$$

This proves (ii). ∎

Proposition 9.2.2.

(i) There exists a natural chain map $\mu\colon S_\bullet(X)\otimes S_\bullet(Y)\to S_\bullet(X\times Y)$ *such that for* $x\in X, y\in Y, \sigma\colon \Delta^p\to X, \tau\colon\Delta^q\to Y$*, we have*

$$\mu(x\otimes\tau)=x\times\tau\colon \Delta^q\cong\Delta^0\times\Delta^q\to X\times Y,$$

$$\mu(\sigma\otimes y)=\sigma\times y\colon \Delta^p\cong\Delta^p\times\Delta^0\to X\times Y.$$

(ii) Any two such natural chain maps are chain homotopic.

For the first part we can use the exterior multiplication maps from Theorem 6.1.1:

Exercise 9.5. Check that the properties of the exterior multiplication maps $\mu_{n,m}\colon S_n(X)\times S_m(Y)\to S_{n+m}(X\times Y)$ imply that these fit together into a natural chain map

$$\mu\colon S_\bullet(X)\otimes S_\bullet(Y)\to S_\bullet(X\times Y).$$

Proof. We need to prove the second part of the statement. Assume we have two natural chain maps μ and μ' as in (i) (in fact we will only use that they agree in degree 0). Then we want to define maps $h_n\colon (S_\bullet(X)\otimes S_\bullet(Y))_n\to S_{n+1}(X\times Y)$ such that

$$\partial h_n+h_{n-1}\partial=\mu'_n-\mu_n.$$

If $n=0$ we have $\mu_0=\mu'_0$ so we can take $h_0=0$ (as h_{-1} is necessarily 0). Suppose we have already defined h_k for $k<n$. Then to define h_n we want to define maps $h_{p,q}\colon S_p(X)\otimes S_q(Y)\to S_{n+1}(X\times Y)$ where $p+q=n$. We start with the universal case $X=\Delta^p$, $Y=\Delta^q$, where we want to define $h_{p,q}(\iota_p\otimes\iota_q)$. This should satisfy

$$\partial h_{p,q}(\iota_p\otimes\iota_q)=\mu'_n(\iota_p\otimes\iota_q)-\mu_n(\iota_p\otimes\iota_q)-h_{n-1}(\partial(\iota_p\otimes\iota_q)).$$

Since $H_n(\Delta^p\times\Delta^q)=0$, it's enough to check that the right-hand side is a cycle:

$$\partial(\mu'_n-\mu_n-h_{n-1}\partial)(\iota_p\otimes\iota_q)=(\partial(\mu'_n-\mu_n)+h_{n-2}\partial^2-(\mu'_{n-1}-\mu_{n-1})\partial)(\iota_p\otimes\iota_q)=0.$$

Now for $\sigma\colon\Delta^p\to X, \tau\colon\Delta^q\to Y$, we set $h_n(\sigma\otimes\tau)=(\sigma\times\tau)_*h_n(\iota_p\otimes\iota_q)$ and extend linearly, and as usual the description of $\partial h_n(\iota_p\otimes\iota_q)$ implies that this is a chain homotopy. ☐

We spare the reader from the proof of the following statement, which again goes by exactly the same strategy:

Proposition 9.2.3.

(i) Any two natural chain maps $S_\bullet(X\times Y)\to S_\bullet(X\times Y)$ *given by the identity in degree 0 are naturally chain homotopic.*

(ii) Any two natural chain maps $S_\bullet(X)\otimes S_\bullet(Y)\to S_\bullet(X)\otimes S_\bullet(Y)$ *given by the identity in degree 0 are naturally chain homotopic.*

Remark 9.2.4. There does exist a general theorem on acyclic models of which all these results are a special case, but this is a bit beyond the scope of this course.

Theorem 9.2.5 (Eilenberg–Zilber)**.** *The chain complexes $S_\bullet(X) \otimes S_\bullet(Y)$ and $S_\bullet(X \times Y)$ are naturally chain homotopy equivalent.*

Proof. By Proposition 9.2.1 and Proposition 9.2.2 there are natural chain maps $\phi\colon S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$ and $\mu\colon S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(X \times Y)$. Now Proposition 9.2.3 implies that the composites $\mu\phi$ and $\phi\mu$ are naturally chain homotopic to the respective identities, so this is a natural chain homotopy equivalence. □

In particular, this implies that $H_*(X \times Y)$ is isomorphic to the homology of $S_\bullet(X) \otimes S_\bullet(Y)$. Applying Proposition 9.1.12 to this tensor product, we thus get:

Corollary 9.2.6 (Künneth Theorem)**.** *For topological spaces X and Y, there are natural short exact sequences*

$$0 \to (H_*(X) \otimes H_*(Y))_n \to H_n(X \times Y) \to \mathrm{Tor}(H_*(X), H_*(Y))_{n-1} \to 0.$$

Remark 9.2.7. By Remark 9.1.14 these short exact sequences are splittable, so that there are non-canonical isomorphisms

$$H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \oplus \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)).$$

Example 9.2.8. For spheres S^n and S^m ($n, m > 0$) we get

$$H_*(S^n \times S^m) \cong \begin{cases} \mathbb{Z}, & * = 0, n, m, n+m, \\ 0, & \text{otherwise}, \end{cases}$$

if $n \neq m$, while if $n = m$ we have

$$H_*(S^n \times S^n) \cong \begin{cases} \mathbb{Z}, & * = 0, 2n, \\ \mathbb{Z} \oplus \mathbb{Z}, & * = n, \\ 0, & \text{otherwise}, \end{cases}$$

Exercise 9.6. Use the Künneth Theorem to compute the homology of $\mathbb{RP}^2 \times \mathbb{RP}^2$.

Remark 9.2.9. If R is an arbitrary commutative ring, we can tensor the chain homotopy equivalence of the Eilenberg–Zilber Theorem with R to obtain a chain homotopy equivalence between $S_\bullet(X;R) \otimes_R S_\bullet(Y;R)$ and $S_\bullet(X \times Y;R)$. Since Proposition 9.1.12 works over any principal ideal domain we obtain a version of the Künneth theorem for $H_*(X \times Y;R)$ when R is a PID. In particular, if k is a field (so that Tor^k vanishes) we have a natural isomorphism

$$H_*(X \times Y;k) \cong H_*(X;k) \otimes_k H_*(Y;k).$$

For abelian groups A, B and a commutative ring R, there is a natural isomorphism $(A \otimes R) \otimes_R (B \otimes R) \cong (A \otimes B) \otimes R$. This gives a natural isomorphism $(S_\bullet(X) \otimes S_\bullet(Y)) \otimes R \cong S_\bullet(X;R) \otimes_R S_\bullet(Y;R)$.

9.3 *The Alexander–Whitney Map*

Later on we are going to use the chain homotopy equivalence between $S_\bullet(X \times Y)$ and $S_\bullet(X) \otimes S_\bullet(Y)$ to define product structures on cohomology. For some purposes it is convenient to have an explicit

There are also explicit definitions of chain maps $S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(X \times Y)$, in the other direction, but we will not need these.

formula for these products on the chain level, for which we need an explicit choice of the chain map $S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$. In this section, we will define the *Alexander–Whitney map*, which is one such chain map.

Definition 9.3.1. Let $\alpha_p^n\colon \Delta^p \hookrightarrow \Delta^n$ be the inclusion of the p-face with vertices $0, 1, \dots, p$, and let $\omega_q^n\colon \Delta^q \hookrightarrow \Delta^n$ be the inclusion of the q-face with vertices $n-q, \dots, n-1, n$.

Definition 9.3.2. The *Alexander–Whitney* map

$$\mathrm{aw}_n\colon S_n(X \times Y) \to (S_\bullet(X) \otimes S_\bullet(Y))_n$$

is defined on a singular simplex $(\sigma, \tau)\colon \Delta^n \to X \times Y$ by

$$\mathrm{aw}(\sigma,\tau) = \sum_{p+q=n} \sigma \circ \alpha_p^n \otimes \tau \circ \omega_q^n = (\sigma_* \otimes \tau_*)\left(\sum_{p+q=n} \alpha_p^n \otimes \omega_q^n\right)$$

Proposition 9.3.3. *The Alexander–Whitney maps are a natural family of chain maps.*

For the proof we need the following relations between the α's and ω's, which we leave to the reader to check:

Lemma 9.3.4.

$$d^i \circ \alpha_p^{n-1} = \begin{cases} \alpha_{p+1}^n \circ d^i, & i \le p \\ \alpha_p^n, & i > p, \end{cases}$$

$$\alpha_{p+1}^n \circ d_{p+1} = \alpha_p^n,$$

$$d^i \circ \omega_q^{n-1} = \begin{cases} \omega_q^n, & i \le n-1-q, \\ \omega_{q+1}^n \circ d^{i-n+1+q}, & i > n-1-q, \end{cases}$$

$$\omega_{q+1}^n \circ d^0 = \omega_q^n.$$

Proof. Naturality is clear from the definition, so it is enough to show that $\partial \mathrm{aw}(\sigma,\tau) = \mathrm{aw}(\partial(\sigma,\tau))$. Here

$$\mathrm{aw}(\partial(\sigma,\tau)) = \mathrm{aw}\left(\sum_i (-1)^i(\partial_i\sigma, \partial_i\tau)\right) = (\sigma_* \otimes \tau_*)\left(\sum_i (-1)^i \sum_{p+q=n-1} d^i \circ \alpha_p^{n-1} \otimes d^i \circ \omega_q^{n-1}\right),$$

$$\partial \mathrm{aw}(\sigma,\tau) = (\sigma_* \otimes \tau_*)\left(\sum_{p+q=n} \partial(\alpha_p^n \otimes \omega_q^n)\right).$$

It thus suffices to establish the identity

$$\sum_{p+q=n} \partial(\alpha_p^n \otimes \omega_q^n) = \sum_{i=0}^n (-1)^i \sum_{p+q=n-1} d^i \circ \alpha_p^{n-1} \otimes d^i \circ \omega_q^{n-1}.$$

Here we have

$$\begin{aligned} \partial(\alpha_p^n \otimes \omega_q^n) &= \partial\alpha_p^n \otimes \omega_q^n + (-1)^p \alpha_p^n \otimes \partial\omega_q^n \\ &= \sum_{i=0}^p (-1)^i \alpha_p^n \circ d^i \otimes \omega_q^n + \sum_{j=0}^q (-1)^{p+j} \alpha_p^n \otimes \omega_q^n \circ d^j. \end{aligned}$$

If $s + t = n - 1$ then the component of the left-hand side in our equation in $S_s(\Delta^n) \otimes S_t(\Delta^n)$ is

$$\sum_{i=0}^{s+1}(-1)^i \alpha_{s+1}^n \circ d^i \otimes \omega_t^n + \sum_{j=0}^{t+1}(-1)^{s+j}\alpha_s^n \otimes \omega_{t+1}^n \circ d^j,$$

while on the right-hand side it is

$$\sum_{i=0}^{n}(-1)^i d^i \circ \alpha_s^{n-1} \otimes d^i \circ \omega_t^{n-1}.$$

It suffices to check that we have

$$\sum_{i=0}^{s+1}(-1)^i \alpha_{s+1}^n \circ d^i \otimes \omega_t^n + \sum_{j=0}^{t+1}(-1)^{s+j}\alpha_s^n \otimes \omega_{t+1}^n \circ d^j = \sum_{i=0}^{n}(-1)^i d^i \circ \alpha_s^{n-1} \otimes d^i \circ \omega_t^{n-1},$$

which can be done using the identities from Lemma 9.3.4. □

9.4 (⋆) A Künneth Theorem for Cohomology

Although there is a natural chain map $S^\bullet(X) \otimes S^\bullet(Y) \to S^\bullet(X \times Y)$, this is in general *not* a chain homotopy equivalence. To get a Künneth formula in cohomology we therefore need to put some restrictions on the spaces involved.

Theorem 9.4.1. *If X and Y are finite type cell complexes, then there are natural short exact sequences*

$$0 \to (H^*(X) \otimes H^*(Y))_{-n} \to H^n(X \times Y) \to \mathrm{Tor}(H^*(X), H^*(Y))_{-n-1} \to 0.$$

These are splittable, so we have (non-canonical) isomorphisms

$$H^n(X \times Y) \cong \bigoplus_{p+q=n} H^p(X) \otimes H^q(Y) \oplus \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(X), H^j(Y)).$$

For chain complexes $C_\bullet, D_\bullet$ and abelian groups M, N we can also define a natural chain map

$$\mathrm{Hom}(C, M)_\bullet \otimes \mathrm{Hom}(D, N)_\bullet \to \mathrm{Hom}(C \otimes D, M \otimes N)_\bullet,$$

defined degreewise by the maps

$$\mathrm{Hom}(C_{-p}, M) \otimes \mathrm{Hom}(D_{-q}, N) \to \mathrm{Hom}(C_{-p} \otimes D_{-q}, N)$$

given by tensoring homomorphisms together.

Lemma 9.4.2. *Suppose $C_\bullet$ and $D_\bullet$ are chain complexes such that C_n and D_n are finitely generated free abelian groups for every n, and $C_n = D_n = 0$ for $n < 0$ (or, for sufficiently large negative n). Then the natural map*

$$\mathrm{Hom}(C, M)_\bullet \otimes \mathrm{Hom}(D, N)_\bullet \to \mathrm{Hom}(C \otimes D, M \otimes N)_\bullet,$$

is an isomorphism.

Proof. This essentially follows because everything in sight commutes with finite direct sums and the fact that if $A \cong \mathbb{Z}S$ is a finitely generated free abelian group then $\mathrm{Hom}(A, \mathbb{Z}) \cong \mathbb{Z}^S \cong \mathbb{Z}S$ is again free; we leave the details to the interested reader. □

If X and Y are finite type cell complexes, then the lemma applies to $C^{\text{cell}}_\bullet(X)$ and $C^{\text{cell}}_\bullet(Y)$, and so we have a natural isomorphism

$$\begin{aligned} C^\bullet_{\text{cell}}(X) \otimes C^\bullet_{\text{cell}}(Y) &\cong \operatorname{Hom}(C^{\text{cell}}(X), \mathbb{Z})_\bullet \otimes \operatorname{Hom}(C^{\text{cell}}(X), \mathbb{Z})_\bullet \\ &\cong \operatorname{Hom}(C^{\text{cell}}(X) \otimes C^{\text{cell}}(Y), \mathbb{Z})_\bullet. \end{aligned}$$

It is not hard to show that for any cell complex X there exists a (non-canonical) chain map $C^{\text{cell}}_\bullet(X) \to S_\bullet(X)$ that in homology gives the isomorpism between the cellular and singular homology of X. This is then a chain homotopy equivalence by the following result from homological algebra:

Fact 9.4.3. *Any* quasi-isomorphism *(meaning a chain map that gives isomorphisms on homology groups) between levelwise free chain complexes is a chain homotopy equivalence (i.e. there exists some choice of a chain homotopy inverse).*

Moreover, for any chain complex $C_\bullet$ the functor $- \otimes C_\bullet$ preserves chain homotopies, as does the functor $\operatorname{Hom}(-, \mathbb{Z})$. Thus we have chain homotopy equivalences between $C^\bullet_{\text{cell}}(X) \otimes C^\bullet_{\text{cell}}(Y)$ and $S^\bullet(X) \otimes S^\bullet(Y)$ and between $\operatorname{Hom}(C^{\text{cell}}(X) \otimes C^{\text{cell}}(Y), \mathbb{Z})_\bullet$ and $\operatorname{Hom}(S_\bullet(X) \otimes S_\bullet(Y), \mathbb{Z})_\bullet$. Applying the Eilenberg–Zilber theorem, the latter is also chain homotopy equivalent to $\operatorname{Hom}(S_\bullet(X \times Y), \mathbb{Z}[0]) \cong S^\bullet(X \times Y)$. We have thus proved:

Proposition 9.4.4. *If X and Y are finite type cell complexes, then there is a chain homotopy equivalence between $S^\bullet(X) \otimes S^\bullet(Y)$ and $S^\bullet(X \times Y)$.*

We can apply our algebraic Künneth theorem, Proposition 9.1.12, to the tensor product $C^\bullet_{\text{cell}}(X) \otimes C^\bullet_{\text{cell}}(Y)$. We obtain short exact sequences

The proof of Lemma 9.4.2 fails for a free abelian group on an infinite set, so we can't directly apply Proposition 9.1.12 to $S^\bullet(X)$.

$$0 \to (H^*(X) \otimes H^*(Y))_{-n} \to H_{-n}(C^\bullet_{\text{cell}}(X) \otimes C^\bullet_{\text{cell}}(Y)) \to \operatorname{Tor}(H^*(X), H^*(Y))_{-n-1} \to 0$$

which together with our chain homotopy equivalences give Theorem 9.4.1.

Remark 9.4.5. Just as in homology, the same proof goes through over any PID. In particular, if k is a field and X and Y are finite type cell complexes, then we have natural isomorphisms

$$H^*(X; k) \otimes_k H^*(Y; k) \cong H^*(X \times Y; k),$$

since the Tor term always vanishes over a field.

Remark 9.4.6. The hypothesis that X and Y are finite type cell complexes can be weakened: it is actually enough to assume that the homology groups of X and Y are finitely generated in each degree: If the homology of a chain complex $C_\bullet$ consists of finitely generated abelian groups and is bounded below then we can algebraically construct a quasi-isomorphic chain complex $C'_\bullet$ that is bounded below and given degreewise by finitely generated free abelian groups. We can apply Lemma 9.4.2 to such replacements of $S_\bullet(X)$ and $S_\bullet(Y)$ and proceed as before.

Remark 9.4.7. Another variant of the Künneth theorem for cohomology (which uses a slightly different algebraic argument) gives the same statement when X is a finite cell complex and Y is an arbitrary topological space.

9.5 (⋆) Relative Eilenberg–Zilber and Künneth Theorems

In this section we will briefly discuss versions of the Eilenberg–Zilber and Künneth theorems for relative homology. This requires a hypothesis on the subspace pairs (which will hold in all examples):

Definition 9.5.1. Let X be a topological space and let A, B be two subspaces of X. We say the pair A, B is *reasonable* if in the topological space $A \cup B$ we can find open sets U, V such that $A \subseteq U$, $B \subseteq V$, $A \cup B = U \cup V$, and the inclusions $A \hookrightarrow U, B \hookrightarrow V, A \cap B \hookrightarrow U \cap V$ all induce isomorphisms in homology.

Theorem 9.5.2 (Relative Eilenberg–Zilber)**.** *Suppose (X, A) and (Y, B) are subspace pairs such that $X \times B$ and $A \times Y$ are a reasonable pair of subspaces of $X \times Y$. Then there is a chain homotopy equivalence between $S_\bullet(X, A) \otimes S_\bullet(Y, B)$ and $S_\bullet(X \times Y, X \times B \cup A \times Y)$.*

Remark 9.5.3. Two important cases where the condition of the theorem holds are when A and B are open subsets of X and Y, and when one of A and B is empty. For example, we always have chain homotopy equivalences between $S_\bullet(X) \otimes S_\bullet(Y, B)$ and $S_\bullet(X \times Y, X \times B)$.

We start with the following easy observations, whose proofs we leave to the reader:

Lemma 9.5.4. *Suppose $C_\bullet$, $D_\bullet$ are chain complexes, with subcomplexes $C'_\bullet \subseteq C_\bullet$ and $D'_\bullet \subseteq D_\bullet$. If we have two chain maps $C_\bullet \to D_\bullet$ that both restrict to chain maps $C'_\bullet \to D'_\bullet$, and a chain homotopy between them that restricts to a chain homotopy on the subcomplexes, then there is an induced chain homotopy between the induced chain maps $C_\bullet/C'_\bullet \to D_\bullet/D'_\bullet$.*

Lemma 9.5.5. *Suppose A, B are subgroups of an abelian group M. Then there is a commutative diagram*

$$\begin{array}{ccccc}
A \cap B & \longrightarrow & A & \longrightarrow & A/A \cap B \\
\downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & M & \longrightarrow & M/B \\
\downarrow & & \downarrow & & \downarrow \\
B/A \cap B & \longrightarrow & M/A & \longrightarrow & M/A + B,
\end{array}$$

where all rows and columns are short exact sequences.

Notation 9.5.6. If A, B are both subspaces of a topological space X, then we write $S_\bullet(X, A + B)$ for the quotient $S_\bullet(X)/(S_\bullet(A) + S_\bullet(B))$.

Remark 9.5.7. In this situation Lemma 9.5.5 gives a commutative diagram

$$\begin{array}{ccccc} S_\bullet(A\cap B) & \longrightarrow & S_\bullet(A) & \longrightarrow & S_\bullet(A, A\cap B) \\ \downarrow & & \downarrow & & \downarrow \\ S_\bullet(B) & \longrightarrow & S_\bullet(X) & \longrightarrow & S_\bullet(X,B) \\ \downarrow & & \downarrow & & \downarrow \\ S_\bullet(B, A\cap B) & \longrightarrow & S_\bullet(X,A) & \longrightarrow & S_\bullet(X, A+B), \end{array}$$

where the rows and columns are short exact sequences.

Proposition 9.5.8. *Given subspace pairs (X,A) and (Y,B), there is a natural chain homotopy equivalence between $S_\bullet(X,A)\otimes S_\bullet(Y,B)$ and $S_\bullet(X\times Y, X\times B + A\times Y)$.*

Proof. We can apply Remark 9.5.7 to the subspaces $X\times B$ and $A\times Y$ of $X\times Y$ to get a commutative diagram

$$\begin{array}{ccccc} S_\bullet(A\times B) & \longrightarrow & S_\bullet(A\times Y) & \longrightarrow & S_\bullet(A\times Y, A\times B) \\ \downarrow & & \downarrow & & \downarrow \\ S_\bullet(X\times B) & \longrightarrow & S_\bullet(X\times Y) & \longrightarrow & S_\bullet(X\times Y, X\times B) \\ \downarrow & & \downarrow & & \downarrow \\ S_\bullet(X\times B, A\times B) & \longrightarrow & S_\bullet(X\times Y, A\times Y) & \longrightarrow & S_\bullet(X\times Y, A\times Y + X\times B) \end{array}$$

where the rows and columns are short exact sequences. Since tensoring with a levelwise free chain complex preserves short exact sequences, we also have such a commutative diagram of the form

$$\begin{array}{ccccc} S_\bullet(A)\otimes S_\bullet(B) & \longrightarrow & S_\bullet(A)\otimes S_\bullet(Y) & \longrightarrow & S_\bullet(A)\otimes S_\bullet(Y,B) \\ \downarrow & & \downarrow & & \downarrow \\ S_\bullet(X)\otimes S_\bullet(B) & \longrightarrow & S_\bullet(X)\otimes S_\bullet(Y) & \longrightarrow & S_\bullet(X)\otimes S_\bullet(Y,B) \\ \downarrow & & \downarrow & & \downarrow \\ S_\bullet(X,A)\otimes S_\bullet(B) & \longrightarrow & S_\bullet(X,A)\otimes S_\bullet(Y) & \longrightarrow & S_\bullet(X,A)\otimes S_\bullet(Y,B). \end{array}$$

Now Lemma 9.5.4 and the naturality of the Eilenberg–Zilberg chain homotopy equivalence implies that the corresponding terms in these diagrams are chain homotopy equivalent. In particular, we have a chain homotopy equivalence between $S_\bullet(X,A)\otimes S_\bullet(Y,B)$ and $S_\bullet(X\times Y, X\times B + A\times Y)$. □

Proposition 9.5.9. *If A, B are a reasonable pair of subspaces of a topological space X, then there is a chain homotopy equivalence between $S_\bullet(X, A+B)$ and $S_\bullet(X, A\cup B)$.*

Lemma 9.5.10. *Suppose A, B are a reasonable pair of subspaces of a topological space X. Then the inclusion*

$$S_\bullet(A) + S_\bullet(B) \to S_\bullet(A\cup B)$$

gives isomorphisms in homology.

Proof. Let $U \supseteq A, V \supseteq B$ the required pair of open sets in $A \cup B$. We will prove that both inclusions

$$S_\bullet(A) + S_\bullet(B) \to S_\bullet(U) + S_\bullet(V) \to S_\bullet(A \cup B)$$

give isomorphisms in homology. We can identify $S_\bullet(U) + S_\bullet(V)$ with the subgroup of $S_\bullet(A \cup B)$ of "small chains" with respect to the cover $\{U, V\}$, so the second map gives isomorphisms in homology by Theorem 6.4.3. To prove the same holds for the first inclusion, first consider the commutative diagram

$$\begin{array}{ccccc} S_\bullet(A \cap B) & \longrightarrow & S_\bullet(B) & \longrightarrow & S_\bullet(B, A \cap B) \\ \downarrow & & \downarrow & & \downarrow \\ S_\bullet(U \cap V) & \longrightarrow & S_\bullet(V) & \longrightarrow & S_\bullet(V, U \cap V), \end{array}$$

where both rows are short exact sequences. Applying the 5-Lemma to the corresponding map of long exact sequences in homology, our hypotheses imply that $S_\bullet(B, A \cap B) \to S_\bullet(V, U \cap V)$ gives isomorphisms in homology. Now we can apply the same argument to the commutative diagram

$$\begin{array}{ccccc} S_\bullet(A) & \longrightarrow & S_\bullet(A) + S_\bullet(B) & \longrightarrow & S_\bullet(B, A \cap B) \\ \downarrow & & \downarrow & & \downarrow \\ S_\bullet(U) & \longrightarrow & S_\bullet(U) + S_\bullet(V) & \longrightarrow & S_\bullet(V, U \cap V), \end{array}$$

to conclude that $S_\bullet(A) + S_\bullet(B) \to S_\bullet(U) + S_\bullet(V)$ gives isomorphisms in homology. □

Proof of Proposition 9.5.9. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_\bullet(A) + S_\bullet(B) & \longrightarrow & S_\bullet(X) & \longrightarrow & S_\bullet(X, A + B) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & S_\bullet(A \cup B) & \longrightarrow & S_\bullet(X) & \longrightarrow & S_\bullet(X, A \cup B) & \longrightarrow & 0 \end{array}$$

where the rows are short exact sequences. Here the left-most vertical map gives isomorphisms in homology by Lemma 9.5.10, so we can apply the 5-Lemma to the resulting map of homology long exact sequences to conclude that $S_\bullet(X, A + B) \to S_\bullet(X, A \cup B)$ is a quasi-isomorphism. We can identify the abelian groups $S_n(X, A + B)$ and $S_n(X, A \cup B)$ with the free abelian groups on $\mathrm{Sing}_n(X)/\mathrm{Sing}_n(A) \cup \mathrm{Sing}_n(B)$ and $\mathrm{Sing}_n(X)/\mathrm{Sing}_n(A \cup B)$, respectively, so this is a chain homotopy equivalence by Fact 9.4.3. □

Combining Proposition 9.5.9 and Proposition 9.5.8 we obtain Theorem 9.5.2. As a consequence, we get a relative version of the Künneth theorem:

Corollary 9.5.11 (Relative Künneth Theorem)**.** *Suppose (X, A) and (Y, B) are subspace pairs such that $X \times B$ and $A \times Y$ are a reasonable pair of subspaces of $X \times Y$. Then there are natural short exact sequences*

$$0 \to (H_*(X, A) \otimes H_*(Y, B))_n \to H_n(X \times Y, X \times B \cup A \times Y) \to (\mathrm{Tor}(H_*(X, A), H_*(Y, B)))_{n-1} \to 0.$$

This applies in particular if A and B are open subsets, and if A is empty, in which case we get:

Corollary 9.5.12. *Suppose X is a topological space and (Y, B) is a subspace pair. Then there are natural short exact sequences*

$$0 \to (H_*(X) \otimes H_*(Y, B))_n \to H_n(X \times Y, X \times B) \to (\mathrm{Tor}(H_*(X), H_*(Y, B)))_{n-1} \to 0.$$

Remark 9.5.13. The cohomological Eilenberg–Zilber theorem can also be extended to the relative case, assuming the relative homology groups are finitely generated in each degree.

10
The Ring Structure on Cohomology

In this chapter we introduce the *cup product*, which makes the cohomology $H^*(X;R)$ of a space X with coefficients in a (commutative) ring R into a graded (commutative) ring. In §10.1 we define the cup product (and the closely related *cross product*) as a pairing. Then in §10.2 we discuss (commutative) ring structures on graded abelian groups and chain complexes, and show that to get a ring structure on homology it's enough to have a ring structure "up to homotopy" on a chain complex. We use the method of acyclic models to obtain such a homotopy ring strucutre on singular cochains $S^\bullet(X;R)$ in §10.3. Finally, we compute the cup product structure on $H^*(\mathbb{RP}^n;\mathbb{F}_2)$ and $H^*(\mathbb{CP}^n;\mathbb{Z})$ in §10.4.

10.1 The Cross and Cup Products

We saw in Proposition 9.2.1 that there is a natural chain map

$$S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y).$$

We can use this to define a natural multiplication on the singular cochains $S^\bullet(X;R)$ where R is a ring, and hence on the homology $H^*(X;R)$. For this we need the following algebraic construction:

Exercise 10.1.

(i) Prove that there is a natural homomorphism of abelian groups

$$\mathrm{Hom}(A,M) \otimes \mathrm{Hom}(B,N) \to \mathrm{Hom}(A \otimes B, M \otimes N),$$

where A, B, M, N are abelian groups, given by tensoring homomorphisms.

(ii) Use this to define a natural chain map

$$\mathrm{Hom}(C,M)_\bullet \otimes \mathrm{Hom}(D,N)_\bullet \to \mathrm{Hom}(C \otimes D, M \otimes N)_\bullet,$$

where $C_\bullet, D_\bullet$ are chain complexes and M, N are abelian groups.

We then proceed by the following steps:

(1) As a special case of Exercise 10.1, for topological spaces X, Y and abelian groups M, N there is a natural chain map

$$S^\bullet(X;M) \otimes S^\bullet(Y;N) \to \mathrm{Hom}(S_\bullet(X) \otimes S_\bullet(Y), M \otimes N).$$

(2) Composing with the Eilenberg–Zilber maps, we get natural chain maps
$$S^\bullet(X;M) \otimes S^\bullet(Y;N) \to S^\bullet(X \times Y; M \otimes N).$$

(3) If R is a ring, we can view the multiplication as a homomorphism $R \otimes R \to R$; composing with this we get natural chain maps
$$S^\bullet(X;R) \otimes S^\bullet(Y;R) \to S^\bullet(X \times Y; R \otimes R) \to S^\bullet(X \times Y; R).$$

(4) For any space X, the diagonal $\Delta\colon X \to X \times X$ induces a natural chain map $\Delta^*\colon S^\bullet(X \times X; R) \to S^\bullet(X;R)$. Composing with this, we have a natural chain map
$$S^\bullet(X;R) \otimes S^\bullet(X;R) \to S^\bullet(X \times X; R) \xrightarrow{\Delta^*} S^\bullet(X;R).$$

(5) Using the maps from Lemma 9.1.10, on homology we get natural maps
$$H^*(X;R) \otimes H^*(Y;R) \to H_*(S^\bullet(X;R) \otimes S^\bullet(Y;R)) \to H^*(X \times Y; R),$$
$$H^*(X;R) \otimes H^*(X;R) \to H_*(S^\bullet(X;R) \otimes S^\bullet(X;R)) \to H^*(X \times X; R) \xrightarrow{\Delta^*} H^*(X;R).$$

This amounts to natural bilinear multiplication maps in cohomology
$$H^n(X;R) \times H^m(Y;R) \to H^{n+m}(X \times Y; R),$$
$$H^n(X;R) \times H^m(X;R) \to H^{n+m}(X;R).$$
The former is called the *cross product* and denoted
$$(\xi, \eta) \mapsto \xi \times \eta$$
for $\xi \in H^n(X;R), \eta \in H^m(Y;R)$, while the latter is called the *cup product* and denoted
$$(\alpha, \beta) \mapsto \alpha \smile \beta$$
for $\alpha \in H^n(X;R), \beta \in H^m(X;R)$. Note that by definition we have
$$\alpha \smile \beta = \Delta^*(\alpha \times \beta).$$

Remark 10.1.1. Note that if $R = \mathbb{Z}$ then the cross product map is exactly the map we used in the Künneth theorem for cohomology.

The cross and cup products in cohomology are well-defined, since by Proposition 9.2.1 any two Eilenberg–Zilber maps are chain homotopic, and so induce the same map in homology. To compute cup products it can be convenient to have an explicit formula, however. If we use the Alexander–Whitney map from Definition 9.3.2, we get the following formulae:

Proposition 10.1.2.

(i) If $[\xi] \in H^n(X;R), [\eta] \in H^m(Y;R)$ are cohomology classes represented by cocycles $\xi \in S^n(X;R) \cong R^{\mathrm{Sing}_n(X)}$, $\eta \in S^m(Y;R) \cong R^{\mathrm{Sing}_m(Y)}$, then $[\xi] \times [\eta]$ is represented by the cochain in $R^{\mathrm{Sing}_{n+m}(X \times Y)}$ given by
$$(\sigma, \tau) \mapsto \xi(\sigma \circ \alpha^{n+m}_n) \cdot \eta(\tau \circ \omega^{n+m}_m),$$
where the multiplication is in the ring R.

(ii) *If* $[\xi] \in H^n(X;R), [\xi'] \in H^m(X;R)$ *are cohomology classes represented by cocycles* $\xi \in S^n(X;R) \cong R^{\mathrm{Sing}_n(X)}$, $\xi' \in S^m(X;R) \cong R^{\mathrm{Sing}_m(X)}$, *then* $[\xi] \smile [\xi']$ *is represented by the cochain in* $R^{\mathrm{Sing}_{n+m}(X)}$ *given by*

$$\sigma \mapsto \xi(\sigma \circ \alpha_n^{n+m}) \cdot \xi'(\sigma \circ \omega_m^{n+m}).$$

Remark 10.1.3. For $S \subseteq \{0, \ldots, n\}$ let us write $\Delta^S \subseteq \Delta^n$ for the face (isomorphic to $\Delta^{|S|}$) of Δ^n whose vertices are the vertices in S. For $\sigma\colon \Delta^n \to X$ we can then denote the restriction of σ to this face by $\sigma|_{\Delta^S}$. With this notation the formula for the cup product using the Alexander–Whitney map can be written as

$$(\xi \smile \xi')(\sigma) = \xi(\sigma|_{\Delta\{0,\ldots,n\}}) \cdot \xi'(\sigma|_{\Delta\{n,n+1,\ldots,n+m\}}).$$

Example 10.1.4. For S^n, the cup product in positive degrees is trivial for degree reasons: if x is the generator of $H^n(S^n)$ then $x \smile x = 0$ since $H^{2n}(S^n) = 0$. However, we will see in Exercise 10.2 that $H^*(S^n \times S^m)$ has a non-trivial cup product: if x and y are the generators in degrees n and m, then $x \smile y$ is the generator in degree $n + m$.

Exercise 10.2.

(i) Show that the cross product $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ can be expressed in terms of the cup product by the formula

$$\xi \times \eta = p_X^* \xi \smile p_Y^* \eta$$

where p_X, p_Y are the projections from $X \times Y$ to X and Y. [Hint: Use the explicit formula for the cup and cross products.]

(ii) If R, R' are commutative rings, we can equip the tensor product $R \otimes R'$ with a commutative ring structure with the multiplication defined on generators by

$$(r_1 \otimes r_1') \cdot (r_2 \otimes r_2') = r_1 r_1' \otimes r_2 r_2'.$$

Check that the analogous construction for graded rings also makes sense. [Note that to get commutativity in the graded cases we need to add a sign.]

(iii) Show that the cross product map $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ is a ring homomorphism with respect to the tensor product of the cup product on X and Y and the cup product on $X \times Y$. [Hint: This amounts to checking the relation

$$(\xi \smile_X \xi') \times (\eta \smile_Y \eta') = (\xi \times \eta) \smile_{X \times Y} (\xi' \times \eta'),$$

for which you can use part (i) and naturality of cup products.]

(iv) Prove that if X and Y are finite type cell complexes and the integral cohomology groups of X are all free abelian groups, then the cross product map

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

is an isomorphism of rings. [Hint: Use the Künneth Theorem for cohomology.]

(v) Compute the ring structure on $H^*(S^n \times S^m)$.

Remark 10.1.5. We can also define relative versions of cross and cup products, though as with the relative Künneth theorem we need some mild assumptions:

- if (X, A) and (Y, B) are subspace pairs, such that $X \times B$ and $A \times Y$ are a reasonable pair of subspaces of $X \times Y$, then there is a relative cross product

$$H^*(X, A; R) \otimes H^*(Y, B; R) \to H^*(X \times Y, X \times B \cup A \times Y; R).$$

  (Since we do always have a chain homotopy equivalence between $S_\bullet(X, A) \otimes S_\bullet(Y, B)$ and $S_\bullet(X \times Y, X \times B + A \times Y)$, we can define a "cross product" with target the homology of the latter chain complex, but this may not agree with the relative homology in general.)

- if (X, A) is any subspace pair, we have a relative cup product

$$H^*(X, A; R) \otimes H^*(X, A; R) \to H^*(X, A; R).$$

  (This is because the diagonal of X always gives a map $S_\bullet(X, A) \to S_\bullet(X \times X, X \times A + A \times X)$.)

- more generally, if A, B are a reasonable pair of subspaces of X, then we have a relative cup product

$$H^*(X, A; R) \otimes H^*(X, B; R) \to H^*(X, A \cup B; R).$$

  (This uses that the diagonal of X gives a map $S_\bullet(X, A + B) \to S_\bullet(X \times X, A \times X + X \times B)$.)

10.2 Graded Rings

We are going to show that when R is a commutative ring, then the cup product makes the cohomology $H^*(X; R)$ into a commutative ring in an appropriate sense. To explain what we mean by this, let's start by looking at a diagrammatic reformulation of the usual definition of a (commutative) ring using the tensor product of abelian groups:

Definition 10.2.1. An (associative, unital) ring consists of an abelian group R together with homomorphisms $m \colon R \otimes R \to R$ (multiplication, corresponding to a bilinear map $R \times R \to R$) and $u \colon \mathbb{Z} \to R$ (corresponding to a unit element $u(1)$ in R), such that the following diagrams commute:

$$\begin{array}{ccc} R \otimes R \otimes R & \xrightarrow{m \otimes \mathrm{id}} & R \otimes R \\ \downarrow{\scriptstyle \mathrm{id} \otimes m} & & \downarrow{\scriptstyle m} \\ R \otimes R & \xrightarrow{m} & R, \end{array} \qquad \begin{array}{ccccc} \mathbb{Z} \otimes R & \xrightarrow{\cong} & R & \xleftarrow{\cong} & R \otimes \mathbb{Z} \\ \downarrow{\scriptstyle u \otimes \mathrm{id}} & & \| & & \downarrow{\scriptstyle \mathrm{id} \otimes u} \\ R \otimes R & \xrightarrow{m} & R & \xleftarrow[m]{} & R \otimes R, \end{array}$$

where the first expresses the associativity of the multiplication and the second that the multiplication on the left or right by the unit gives the identity. The ring is commutative if in addition the triangle

$$\begin{array}{ccc} R \otimes R & & \\ & \searrow^{m} & \\ \downarrow{\scriptstyle \tau} & & R \\ & \nearrow_{m} & \\ R \otimes R & & \end{array}$$

commutes, where τ is the natural symmetry isomorphism that swaps the factors in the tensor product.

Exercise 10.3. Convince yourself that the diagrammatic definition of a (commutative) ring agrees with the (equational) one you have seen before.

Remark 10.2.2. Here and elsewhere in this discussion we have ignored the fact that $R \otimes (R \otimes R)$ and $(R \otimes R) \otimes R$ are not equal, just canonically isomorphic — the commutative square describing associativity should really be a pentagon where this associativity isomorphism for $\otimes$ also appears.

We can make the same definition in any category with a tensor product and a symmetry isomorphism, such as graded abelian groups and chain complexes:

Definition 10.2.3. An (associative, unital) *graded ring* is a graded abelian group R_* together with maps $m\colon R_* \otimes R_* \to R_*$ and $u\colon \mathbb{Z}[0] \to R_*$ such that the analogues of the associtiavity and unit diagrams above commute. The graded ring R_* is commutative if in addition the commutativity diagram commutes.

More explicitly, this structure amounts to giving bilinear multiplication maps $R_n \times R_m \to R_{n+m}$ and a unit $e \in R_0$ such that we have $a(bc) = (ab)c$ and $ea = a = ae$. Note that because our preferred symmetry isomorphism for graded abelian groups has a sign, commutativity means that for elements $a \in R_n, b \in R_m$ we have

$$ab = (-1)^{nm} ba.$$

We can also make the analogous definition in chain complexes:

Definition 10.2.4. An (associative, unital) *differential graded ring* (or *dg-ring*) is a chain complex group $R_\bullet$ together with chain maps $m\colon R_\bullet \otimes R_\bullet \to R_\bullet$ and $u\colon \mathbb{Z}[0] \to R_\bullet$ such that the analogues of the associtiavity and unit diagrams above commute. The dg-ring $R_\bullet$ is commutative if in addition the commutativity diagram commutes.

This structure amounts to giving a (commutative) graded ring structure on the underlying graded abelian group R_* as above, such that the unit e is a cycle ($\partial e = 0$) and the product satisfies the "Leibniz formula":

$$\partial(ab) = (\partial a)b + (-1)^n a(\partial b)$$

for $a \in R_n$, $b \in R_m$.

A dg-ring structure on a chain complex $C_\bullet$ induces a graded ring structure on $H_*(C)$ (with multiplication given by the composite $H_*(C) \otimes H_*(C) \to H_*(C \otimes C) \to H_*(C)$), so we might hope that the ring structure on singular cohomology $H^*(X; R)$ arises from a dg-ring structure on $S^\bullet(X; R)$. However, this is not quite true, and we have to consider a slightly weaker structure:

Definition 10.2.5. A *homotopy dg-ring* is a chain complex group $R_\bullet$ together with chain maps $m\colon R_\bullet \otimes R_\bullet \to R_\bullet$ and $u\colon \mathbb{Z}[0] \to R_\bullet$

such that the analogues of the associtiavity and unit diagrams above commute up to chain homotopy, i.e. there exists some choice of chain homotopy between the composite chain maps. The homotopy dg-ring $R_\bullet$ is commutative if in addition the commutativity diagram commutes up to chain homotopy.

Lemma 10.2.6. *If $R_\bullet$ is a (commutative) homotopy dg-ring, then $H_*(R)$ inherits a (commutative) graded ring structure.*

Proof. Recall that for chain complexes $C_\bullet, D_\bullet$ we have a natural map $H_*(C) \otimes H_*(D) \to H_*(C \otimes D)$, and this is compatible with the associativity, unitality, and symmetry of $\otimes$. We define the multiplication on H_*R as the composite

$$H_*(R) \otimes H_*(R) \to H_*(R \otimes R) \to H_*(R),$$

where the second map is induced by the multiplication on $R_\bullet$, with unit

$$\mathbb{Z}[0] \xrightarrow{\sim} H_*(\mathbb{Z}[0]) \to H_*(R)$$

using the unit of $R_\bullet$. To prove associativity we consider the diagram

$$\begin{array}{ccccc}
H_*(R) \otimes H_*(R) \otimes H_*(R) & \longrightarrow & H_*(R) \otimes H_*(R \otimes R) & \longrightarrow & H_*(R) \otimes H_*(R) \\
\downarrow & & \downarrow & & \downarrow \\
H_*(R \otimes R) \otimes H_*(R) & \longrightarrow & H_*(R \otimes R \otimes R) & \longrightarrow & H_*(R \otimes R) \\
\downarrow & & \downarrow & & \downarrow \\
H_*(R) \otimes H_*(R) & \longrightarrow & H_*(R \otimes R) & \longrightarrow & H_*(R),
\end{array}$$

where the bottom right square commutes because chain homotopic maps give the same map in homology, and the other three squares commute by naturality. To prove the multiplication on H_*R is unital, we consider the diagram

$$\begin{array}{ccccc}
H_*R \otimes \mathbb{Z}[0] & \xrightarrow{\cong} & H_*(R \otimes \mathbb{Z}[0]) & \xrightarrow{\cong} & H_*(R) \\
\downarrow & & \downarrow & & \| \\
H_*(R) \otimes H_*(R) & \longrightarrow & H_*(R \otimes R) & \longrightarrow & H_*(R),
\end{array}$$

which commutes for the same reasons, as well as the corresponding diagram with $\mathbb{Z}[0]$ on the other side.

Finally, if $R_\bullet$ is homotopy commutative, we have the commutative diagram

$$\begin{array}{ccccc}
H_*(R) \otimes H_*(R) & \longrightarrow & H_*(R \otimes R) & \searrow & \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle H_*\tau} & & H_*(R) \\
H_*(R) \otimes H_*(R) & \longrightarrow & H_*(R \otimes R), & \nearrow &
\end{array}$$

which proves that $H_*(R)$ is commutative. $\square$

10.3 Homotopy Ring Structures on Cochains

In this section we will sketch an argument that any choice of Eilenberg–Zilber maps $S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$ can be used to make $S^\bullet(X; R)$ a (commutative) homotopy dg-ring when R is a (commutative) ring.

The following can be proved by using the method of acyclic models:

Proposition 10.3.1. *Let $\phi^{X,Y} \colon S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$ be a natural family of Eilenberg–Zilber chain maps.*

(i) For any three spaces X, Y, Z the square

Here and elsewhere in this section we are again suppressing the natural associativity isomorphism $(S_\bullet(X) \otimes S_\bullet(Y)) \otimes S_\bullet(Z) \cong S_\bullet(X) \otimes (S_\bullet(Y) \otimes S_\bullet(Z))$.

$$\begin{array}{ccc}
S_\bullet(X \times Y \times Z) & \xrightarrow{\phi^{X \times Y, Z}} & S_\bullet(X \times Y) \otimes S_\bullet(Z) \\
\downarrow{\scriptstyle \phi^{X, Y \times Z}} & & \downarrow{\scriptstyle \phi^{X,Y} \otimes \mathrm{id}} \\
S_\bullet(X) \otimes S_\bullet(Y \times Z) & \xrightarrow{\mathrm{id} \otimes \phi^{Y,Z}} & S_\bullet(X) \otimes S_\bullet(Y) \otimes S_\bullet(Z)
\end{array}$$

commutes up to a natural chain homotopy.

(ii) Let $u \colon S_\bullet(*) \to \mathbb{Z}[0]$ be the unique chain map that's the identity in degree 0. Then for any space X the diagram

$$\begin{array}{ccccc}
S_\bullet(X \times *) & \xrightarrow{\phi^{X,*}} & S_\bullet(X) \otimes S_\bullet(*) & \xrightarrow{\mathrm{id} \otimes u} & S_\bullet(X) \otimes \mathbb{Z}[0] \\
& \searrow{\scriptstyle \cong} & & \swarrow{\scriptstyle \cong} & \\
& & S_\bullet X & &
\end{array}$$

commutes up to a natural chain homotopy, as does the analogous diagram with $\mathbb{Z}[0]$ on the other side.

(iii) For any pair of spaces X, Y the square

$$\begin{array}{ccc}
S_\bullet(X \times Y) & \xrightarrow{\phi^{X,Y}} & S_\bullet(X) \otimes S_\bullet(Y) \\
\downarrow{\scriptstyle t_*} & & \downarrow{\scriptstyle \tau} \\
S_\bullet(Y \times X) & \xrightarrow{\phi^{Y,X}} & S_\bullet(Y) \otimes S_\bullet(X)
\end{array}$$

commutes up to a natural chain homotopy, where t denotes the natural isomorphism $X \times Y \xrightarrow{\sim} Y \times X$.

Applying $\mathrm{Hom}(-, R)$ and combining these diagrams with the natural maps $S^\bullet(X; R) \otimes S^\bullet(Y; R) \to \mathrm{Hom}(S_\bullet(X) \otimes S_\bullet(Y), R)$, we get:

Corollary 10.3.2. *Let R be a ring.*

(i) For any three spaces X, Y, Z the square

$$\begin{array}{ccc}
S^\bullet(X; R) \otimes S^\bullet(Y; R) \otimes S^\bullet(Z; R) & \longrightarrow & S^\bullet(X \times Y; R) \otimes S^\bullet(Z; R) \\
\downarrow & & \downarrow \\
S^\bullet(X; R) \otimes S^\bullet(Y \times Z; R) & \longrightarrow & S^\bullet(X \times Y \times Z; R)
\end{array}$$

commutes up to a natural chain homotopy.

(ii) *For any space X the diagram*

$$\begin{array}{ccccc} & & S^\bullet(X;R) & & \\ & \swarrow_{\cong} & & \searrow^{\cong} & \\ S^\bullet(X;R)\otimes \mathbb{Z}[0] & \longrightarrow & S^\bullet(X;R)\otimes S^\bullet(*;R) & \longrightarrow & S^\bullet(X\times *;R) \end{array}$$

commutes up to a natural chain homotopy, as does the analogous diagram with $\mathbb{Z}[0]$ *on the other side.*

(iii) *If R is commutative, then for any pair of spaces X, Y the square*

$$\begin{array}{ccc} S^\bullet(X;R)\otimes S^\bullet(Y;R) & \longrightarrow & S^\bullet(X\times Y;R) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle t^*} \\ S^\bullet(Y;R)\otimes S^\bullet(X;R) & \longrightarrow & S^\bullet(Y\times X;R) \end{array}$$

commutes up to a natural chain homotopy.

Corollary 10.3.3. *For any ring R and space X, we have a natural homotopy dg-ring structure on* $S^\bullet(X;R)$ *with multiplication given by*

$$S^\bullet(X;R)\otimes S^\bullet(X;R) \to S^\bullet(X\times X;R) \xrightarrow{\Delta^*} S^\bullet(X;R),$$

and unit given by

$$\mathbb{Z}[0] \xrightarrow{u^*} S^\bullet(*;R) \to S^\bullet(X;R).$$

If the ring R is commutative then this homotopy ring structure is also (homotopy) commutative.

Proof. To prove associativity, consider the diagram

$$\begin{array}{ccccc} S^\bullet(X;R)\otimes S^\bullet(X;R)\otimes S^\bullet(X;R) & \longrightarrow & S^\bullet(X\times X;R)\otimes S^\bullet(X;R) & \xrightarrow{\Delta^*\otimes \mathrm{id}} & S^\bullet(X;R)\otimes S^\bullet(X;R) \\ \downarrow & & \downarrow & & \downarrow \\ S^\bullet(X;R)\otimes S^\bullet(X\times X;R) & \longrightarrow & S^\bullet(X\times X\times X;R) & \xrightarrow{(\Delta\times \mathrm{id})^*} & S^\bullet(X\times X;R) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\Delta^*} & & \downarrow{\scriptstyle (\mathrm{id}\times\Delta)^*} & & \downarrow{\scriptstyle \Delta^*} \\ S^\bullet(X;R)\otimes S^\bullet(X;R) & \longrightarrow & S^\bullet(X\times X;R) & \xrightarrow{\Delta^*} & S^\bullet(X;R), \end{array}$$

where the top left square commutes up to a natural chain homotopy by Corollary 10.3.2(i), the top right and bottom left squares commute by naturality, and the bottom right square commutes because the square

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X\times X \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id}\times\Delta} \\ X\times X & \xrightarrow{\Delta\times \mathrm{id}} & X\times X\times X \end{array}$$

of topological spaces commutes. Next, to prove unitality we consider the diagram

$$\begin{array}{ccccc} S^\bullet(X;R)\otimes \mathbb{Z}[0] & & & & \\ \downarrow & \searrow^{\cong} & & & \\ S^\bullet(X;R)\otimes S^\bullet(*;R) & \longrightarrow & S^\bullet(X\times *;R) & & \\ \downarrow & & \downarrow & \searrow^{\cong} & \\ S^\bullet(X;R)\otimes S^\bullet(X;R) & \longrightarrow & S^\bullet(X\times X;R) & \xrightarrow{\Delta^*} & S^\bullet(X), \end{array}$$

where the top triangle commutes up to a natural chain homotopy by Corollary 10.3.2(ii), the bottom left square commutes by naturality, and the bottom right triangle commutes because the triangle

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ & {\scriptstyle\cong}\searrow & \downarrow \\ & & X \times * \end{array}$$

of topological spaces commutes. (And of course the case with $\mathbb{Z}[0]$ on the other side works the same.)

To prove commutativity we consider the diagram

$$\begin{array}{ccccc} S^\bullet(X;R) \otimes S^\bullet(X;R) & \longrightarrow & S^\bullet(X \times X;R) & & \\ & & & \overset{\Delta^*}{\searrow} & \\ \downarrow \tau & & \downarrow t^* & & S^\bullet(X;R) \\ & & & \underset{\Delta^*}{\nearrow} & \\ S^\bullet(X;R) \otimes S^\bullet(X;R) & \longrightarrow & S^\bullet(X \times X;R), & & \end{array}$$

where the left-hand square commutes up to a natural chain homotopy by Corollary 10.3.2(iii) and the right-hand triangle commutes because the triangle

$$\begin{array}{ccc} & & X \times X \\ & \overset{\Delta}{\nearrow} & \\ X & & \downarrow t \\ & \underset{\Delta}{\searrow} & \\ & & X \times X \end{array}$$

of topological spaces commutes. □

If we make a good choice of Eilenberg–Zilber maps, such as the Alexander–Whitney maps, then $S^\bullet(X;R)$ is a dg-ring in the strict sense (as can be seen easily from the formulae in Proposition 10.1.2):

Exercise 10.4. Show that if we define the cup product on the chain level using the Alexander–Whitney map, then $S^\bullet(X;R)$ is a (strictly) associative and unital dg-ring for any ring R.

Remark 10.3.4. However, using this specific choice of Eilenberg–Zilber map the dg-ring $S^\bullet(X;R)$ is still not commutative in the strict sense, but only up to chain homotopy. In fact, it is *impossible* to make the cup product commutative on the chain level in the strict sense.

The "failure" of strict commutativity actually turns out to encode further interesting structure on $H^*(X;R)$, namely *cohomology operations*, but this is beyond the scope of this course.

Combining Corollary 10.3.3 with Lemma 10.2.6, we get the following:

Corollary 10.3.5. *If R is a ring, then $H^*(X;R)$ has a natural (graded) ring structure, and if R is commutative then $H^*(X;R)$ is a graded commutative ring.* □

The graded ring structure on $H^*(X;R)$ restricts to an (ordinary) ring structure on $H^0(X;R)$ (commutative if R is commutative); the following exercise identifies this:

Exercise 10.5.

1. Show that under the isomorphism $H^0(X;R) \cong R^{\pi_0 X}$, the cup product in degree 0 corresponds to the pointwise multiplication of functions $\pi_0 X \to R$, with unit the constant function with value $1 \in R$ and product $(f \cdot g)(x) = f(x) \cdot g(x)$. [Hint: Use the explicit formula from the Alexander–Whitney map.]
2. By additivity for cohomology we have an isomorphism $H^i(X;R) \cong \prod_{t \in \pi_0(X)} H^i(X_t;R)$ where X_t denotes the path-component of X corresponding to $t \in \pi_0 X$. Show that under this isomorphism the cup product

$$H^0(X;R) \times H^i(X;R) \to H^i(X;R)$$

for $i > 0$ takes $f\colon \pi_0 X \to R$ and $(\alpha_t)_{t \in \pi_0 X}$ to $(f(t)\alpha_t)_t$ (in terms of the natural R-module structure on $H^i(X_t;R)$).

Exercise 10.6.

(i) If R_i, $i \in I$, are rings, then the cartesian product $\prod_{i \in I} R_i$ can be given a commutative ring structure with pointwise multiplication (i.e. $(r_i)_{i \in I} \cdot (r'_i)_{i \in I} = (r_i r'_i)_{i \in I}$). Check that this has the universal property of the product in the category of rings (i.e. given ring homomorphisms $\phi_i\colon R' \to R_i$ for each i, there exists a unique ring homomorphism $R' \to \prod_{i \in I} R_i$ that projects to ϕ_i in the ith coordinate). Also check the analogous statement holds for graded rings (where the cartesian product is taken degreewise).

(ii) Show that for topological spaces X_i, $i \in I$, the map

$$H^*(\coprod_{i \in I} X_i) \to \prod_{i \in I} H^*(X_i),$$

induced by the inclusions $X_i \hookrightarrow \coprod_{i \in I} X_i$, is an isomorphism of rings.

(iii) Compute the ring structure on $H^*(S^n \vee S^m)$. [Hint: The canonical map $S^n \amalg S^m \to S^n \vee S^m$ induces a ring homomorphism $H^*(S^n \vee S^m) \to H^*(S^n \amalg S^m)$; check that this is an isomorphism in degrees $* > 0$.]

Exercise 10.7 (∗). Let Σ_g be the orientable closed surface of genus g. There is a continuous map from Σ_g to a wedge of g tori that pinches the "necks" between the g holes to points,

$$q\colon \Sigma_g \to \bigvee_g (S^1 \times S^1)$$

Using that the induced map in cohomology is a ring homomorphism, compute the ring structure on $H^*(\Sigma_g)$. [Recall that in Exercise 4.8 you computed that

$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z}, & * = 0, 2 \\ \mathbb{Z}^{2g}, & * = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and use Exercise 10.6 and Exercise 10.2 to compute the ring structure for the wedge of tori.]

10.4 (⋆) The Cohomology Rings of $\mathbb{RP}^n$ and $\mathbb{CP}^n$

We now want to describe the cup product structure on the cohomology of projective spaces. To state the result in a nice way, we first introduce some terminology:

Definition 10.4.1. Let R be a commutative ring. The *graded polynomial ring* $R[x_1, \ldots, x_k]$ where the generator x_i has degree d_i, has as its nth graded piece $R[x_1, \ldots, x_k]_n$ the free R-module on the monomials $x_1^{i_1} \cdots x_k^{i_k}$ where $i_1 d_1 + \cdots + i_k d_k = n$. Multiplication is defined as you would expect, using the graded commutativity relation $x_i x_j = (-1)^{d_i d_j} x_j x_i$.

Remark 10.4.2. If R_* is a graded ring, we can use the graded multiplication to define an "underlying" (ungraded) ring structure on the direct sum $\bigoplus_n R_n$. For the graded polynomial ring $R[x_1, \ldots, x_k]$ as above, the underlying (ungraded) ring is the associative ring

$$R\langle x_1, \ldots, x_n \rangle / (x_i x_j = (-1)^{d_i d_j} x_j x_i)$$

generated by elements $x_1, \ldots, x_k$ subject to the graded commutativity relation (where the angle brackets denote the free associative ring on a set of generators)

Remark 10.4.3. If d_i is even for all i, then the x_i's commute and the underlying ring of $R[x_1, \ldots, x_k]$ is the ordinary polynomial ring on k generators. On the other hand, if d_i is odd for all i, we get the *exterior algebra* on k,

$$\Lambda_R(x_1, \ldots, x_k) = R\langle x_1, \ldots, x_k \rangle / (x_i x_j = -x_j x_i).$$

However, if $2 = 0$ in R (e.g. if $R = \mathbb{F}_2$) then the x_i's commute regardless of their degrees and the underlying ring is always a polynomial ring.

Theorem 10.4.4. *There are isomorphisms of graded rings*

$$H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1}), \quad H^*(\mathbb{RP}^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$$

where x is a generator in degree 1,

$$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1}), \quad H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[x]$$

where x is a generator in degree 2.

Remark 10.4.5. Recall that we have a cell structure on $\mathbb{RP}^n$ with a single cell in each dimension $\leq n$, such that the i-skeleton $\mathbb{RP}^n_i$ is the subspace $\mathbb{RP}^i \cong \mathbb{RP}^{\{0,\ldots,i\}}$; with $\mathbb{F}_2$-coefficients we saw that the corresponding cellular (co)chain complex has zero differentials, so that

$$H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{Z}/2, & 0 \leq * \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

In the case of $\mathbb{CP}^n$ we have a cell structure with a single cell in each *even* dimension $\leq 2n$, so that the integral cellular (co)chain complex has zero differentials, and

$$H^*(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z}, & * \text{ even}, 0 \leq * \leq 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 10.4.6. In the case of $\mathbb{RP}^n$ we can equivalently state this result as: if $x \in H^i(\mathbb{RP}^n; \mathbb{F}_2)$ and $y \in H^j(\mathbb{RP}^n; \mathbb{F}_2)$ are non-zero and $i + j \leq n$, then $x \smile y \in H^{i+j}(\mathbb{RP}^n; \mathbb{F}_2)$ is non-zero (recall that all of these groups are isomorphic to $\mathbb{Z}/2$). Similarly, for $\mathbb{CP}^n$ this amounts to saying that if $x \in H^i(\mathbb{CP}^n)$ and $y \in H^j(\mathbb{CP}^n)$ are generators and $i + j \leq n$, then $x \smile y$ is a generator of $H^{i+j}(\mathbb{CP}^n)$ (recall that all of these groups are isomorphic to $\mathbb{Z}$).

Notation 10.4.7. Let $\mathbf{n}$ denote the n-element set $\{1, \ldots, n\}$. For $S \subseteq \mathbf{n}$, write $\mathbb{R}^S$ for the closed subset of $\mathbb{R}^n$ consisting of vectors $(x_1, \ldots, x_n)$ such that $x_i = 0$ for $i \notin S$ and $\mathbb{R}^n_S$ for the closed subset of vectors $(x_1, \ldots, x_n)$ where $x_i = 0$ for $i \in S$ (so $\mathbb{R}^S = \mathbb{R}^n_{\mathbf{n} \setminus S}$, $\mathbb{R}^S \cong \mathbb{R}^{|S|}$ and $\mathbb{R}^n_S \cong \mathbb{R}^{n-|S|}$). Note that $\mathbb{R}^n_S \cap \mathbb{R}^n_T = \mathbb{R}^n_{S \cup T}$. We also let V_S denote the open subset $\mathbb{R}^n \setminus \mathbb{R}^n_S$; thus V_S is the set of vectors $(x_1, \ldots, x_n)$ such that $x_i \neq 0$ for at least one $i \in S$. We have $V_S \cup V_T = V_{S \cup T}$.

Proposition 10.4.8. *For $S \subseteq \mathbf{n}$ a subset of size i, the relative cup product*

$$H^i(\mathbb{R}^n, V_S) \otimes H^{n-i}(\mathbb{R}^n, V_{\mathbf{n} \setminus S}) \xrightarrow{\smile} H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

is an isomorphism (where $V_S \cup V_{\mathbf{n} \setminus S} = V_{\mathbf{n}} = \mathbb{R}^n \setminus \{0\}$).

Remark 10.4.9. The same proof works with coefficients in any PID R, if we replace the tensor product with a relative tensor product of R-modules.

Proof. The inclusion $(\mathbb{R}^S, \mathbb{R}^S \setminus \{0\}) \hookrightarrow (\mathbb{R}^n, V_S)$ is a deformation retract for any S, as is the inclusion $(D^m, D^m \setminus \{0\}) \hookrightarrow (\mathbb{R}^m, \mathbb{R}^m \setminus \{0\})$ for any m. But $\partial D^m \hookrightarrow D^m \setminus \{0\}$ is also a deformation retract, so there is a homotopy equivalence between $(\mathbb{R}^n, V_S)$ and $(D^{|S|}, \partial D^{|S|})$ and so $H^*(\mathbb{R}^n, V_S) \cong \tilde{H}^*(S^{|S|})$; in particular $H^i(\mathbb{R}^n, V_S) \cong \mathbb{Z}$ and the other cohomology groups are zero. Since these are finitely generated free abelian groups, we can apply the Künneth theorem for relative cohomology to conclude that the cross product map

$$H^i(\mathbb{R}^n, V_S) \otimes H^{n-i}(\mathbb{R}^n, V_{\mathbf{n} \setminus S}) \to H^n(\mathbb{R}^{2n}, V_S \times \mathbb{R}^n \cup \mathbb{R}^n \times V_{\mathbf{n} \setminus S})$$

is an isomorphism. The cup product we're interested in is obtained by composing this with the map in cohomology induced by the diagonal map

$$\Delta \colon (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to (\mathbb{R}^{2n}, V_S \times \mathbb{R}^n \cup \mathbb{R}^n \times V_{\mathbf{n} \setminus S}),$$

so to complete the proof it suffices to show that this map gives an isomorphism in cohomology. We will prove this by checking that this map is a deformation retract. The retraction $\rho \colon \mathbb{R}^{2n} \to \mathbb{R}^n$ is defined by

$$\rho(x, y)_i = \begin{cases} x_i, & i \in S, \\ y_i, & i \notin S; \end{cases}$$

note that the subset $V_S \times \mathbb{R}^n \cup \mathbb{R}^n \times V_{\mathbf{n} \setminus S}$ consists of those pairs (x, y) such that either some $x_i \neq 0$ with $i \in S$ *or* some $y_i \neq 0$ with $i \notin S$, so that $\rho(x, y) \neq 0$ for (x, y) in this subset. We clearly have $\rho\Delta = \mathrm{id}_{\mathbb{R}^n}$, and we can simply define a linear homotopy between $\mathrm{id}_{\mathbb{R}^{2n}}$ and $\Delta\rho$ by

$$h(x,y,t) = t(x,y) + (1-t)\Delta\rho(x,y);$$

if we write $h(x,y,t) = (x(t),y(t))$ then $x(t)_i = x_i$ for $i \in S$ and $y(t)_i = y_i$ for $i \notin S$ so this takes the subspace $V_S \times \mathbb{R}^n \cup \mathbb{R}^n \times V_{\mathbf{n}\setminus S}$ to itself, as required. □

Next we want an analogue of this result for projective space, which requires some more notation. We will only state this for $\mathbb{RP}^n$ for simplicity, but the same argument works for $\mathbb{CP}^n$.

Notation 10.4.10. Let $\mathbf{n}_+$ denote the set $\{0,\dots,n\}$. Recall that we can describe points of $\mathbb{RP}^n$ by projective coordinates as $(x_0 : x_1 : \cdots : x_n)$ with $x_i \in \mathbb{R}$ not all 0, where $(x_0 : x_1 : \cdots : x_n) = (\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n)$ for $\lambda \neq 0$ in $\mathbb{R}$. For $S \subseteq \mathbf{n}_+$, let $\mathbb{RP}^S$ denote the closed subset of $\mathbb{RP}^n$ containing those points $(x_0 : x_1 : \cdots : x_n)$ where $x_i = 0$ for $i \notin S$, and let $\mathbb{RP}^n_S := \mathbb{RP}^{\mathbf{n}_+\setminus S}$ denote the set of points $(x_0 : x_1 : \cdots : x_n)$ where $x_i = 0$ for $i \in S$. Then $\mathbb{RP}^S \cong \mathbb{RP}^{|S|-1}$ and we have $\mathbb{RP}^n_S \cap \mathbb{RP}^n_T = \mathbb{RP}^n_{S\cup T}$. We write $U_S := \mathbb{RP}^n \setminus \mathbb{RP}^n_S$ for the open subset consisting of points $(x_0 : x_1 : \cdots : x_n)$ such that $x_i \neq 0$ for some $i \in S$; then $U_S \cup U_T = U_{S\cup T}$.

The point $(x_0 : x_1 : \cdots : x_n)$ represents the line in $\mathbb{R}^{n+1}$ through the origin and the point $(x_0,\dots,x_n)$.

Remark 10.4.11. For the one-element set $\{i\}$, every point of $U_{\{i\}}$ has unique projective coordinates of the form $(x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n)$ with ith coordinate 1. This gives a homeomorphism between $U_{\{i\}}$ and $\mathbb{R}^n$. On the other hand, $\mathbb{RP}^n_{\mathbf{n}_+\setminus\{i\}}$ consists of the single point $(0 : \ldots : 0 : 1 : 0 \ldots : 0)$ with only the ith coordinate non-zero.

Corollary 10.4.12. *The relative cup product*

$$H^i(\mathbb{RP}^n, U_{\{0,\dots,i-1\}}; R) \otimes_R H^{n-i}(\mathbb{RP}^n, U_{\{i+1,\dots,n\}}; R) \xrightarrow{\smile} H^n(\mathbb{RP}^n, U_{\mathbf{n}_+\setminus\{i\}}; R)$$

is an isomorphism for any PID R.

Proof. By naturality we have a commutative square

$$\begin{array}{ccc} H^i(\mathbb{RP}^n, U_{\{0,\dots,i-1\}}; R) \otimes_R H^{n-i}(\mathbb{RP}^n, U_{\{i+1,\dots,n\}}; R) & \xrightarrow{\smile} & H^n(\mathbb{RP}^n, U_{\mathbf{n}_+\setminus\{i\}}; R) \\ \downarrow & & \downarrow \\ H^i(U_{\{i\}}, U_{\{i\}} \cap U_{\{0,\dots,i-1\}}; R) \otimes_R H^{n-i}(U_{\{i\}}, U_{\{i\}} \cap U_{\{i+1,\dots,n\}}; R) & \xrightarrow{\smile} & H^n(U_{\{i\}}, U_{\{i\}} \cap U_{\mathbf{n}_+\setminus\{i\}}; R), \end{array}$$

where the vertical morphisms are isomorphisms by excision (of $\mathbb{RP}^n_{\{i\}} = \mathbb{RP}^n \setminus U_{\{i\}}$). Now observe that under the homeomorphism $U_{\{i\}} \cong \mathbb{R}^n$ the bottom horizontal map corresponds to one of those we proved was an isomorphism in Proposition 10.4.8. □

We need one more observation before we can complete the proof:

Lemma 10.4.13. *If $S \subseteq \mathbf{n}_+$ is a subset of size i, then the homomorphism*

$$H^i(\mathbb{RP}^n, U_S; \mathbb{F}_2) \to H^i(\mathbb{RP}^n; \mathbb{F}_2)$$

is an isomorphism.

Proof. By definition, U_S is the subset of $\mathbb{RP}^n$ consisting of points $(x_0 : \cdots : x_n)$ where $x_i \neq 0$ for at least one $i \in S$. The subset $\mathbb{RP}^S$ consists of points where $x_i = 0$ for all $i \notin S$; since at least one x_i must be non-zero, we have $\mathbb{RP}^S \subseteq U_S$, so that the map we are interested in factors as

$$H^i(\mathbb{RP}^n, U_S; \mathbb{F}_2) \to H^i(\mathbb{RP}^n, \mathbb{RP}^S; \mathbb{F}_2) \to H^i(\mathbb{RP}^n; \mathbb{F}_2).$$

We claim that the inclusion $\mathbb{RP}^S \hookrightarrow U_S$ is a deformation retract, so that the first morphism in this factorization is an isomorphism. To see this we simply define a retraction $\rho \colon U_S \to \mathbb{RP}^S$ and a homotopy h between ρ and the identity by

$$\rho(x)_i = \begin{cases} x_i, & i \in S, \\ 0, & i \notin S \end{cases} \qquad h(x,t)_i = \begin{cases} x_i, & i \in S, \\ tx_i, & i \notin S. \end{cases}$$

It therefore suffices to prove that $H^i(\mathbb{RP}^n, \mathbb{RP}^S; \mathbb{F}_2) \to H^i(\mathbb{RP}^n; \mathbb{F}_2)$ is an isomorphism. Without loss of generality we may assume that $S = \{0, \ldots, i-1\}$ so that $\mathbb{RP}^S \cong \mathbb{RP}^{i-1}$ is the $(i-1)$-skeleton in the cell structure on $\mathbb{RP}^n$. We then have a map of pairs $(\mathbb{RP}^i, \mathbb{RP}^{i-1}) \to (\mathbb{RP}^n, \mathbb{RP}^{i-1})$, which gives a commutative square

$$\begin{array}{ccc} H^i(\mathbb{RP}^n, \mathbb{RP}^{i-1}; \mathbb{F}_2) & \longrightarrow & H^i(\mathbb{RP}^n; \mathbb{F}_2) \\ \downarrow & & \downarrow \\ H^i(\mathbb{RP}^i, \mathbb{RP}^{i-1}; \mathbb{F}_2) & \longrightarrow & H^i(\mathbb{RP}^i; \mathbb{F}_2). \end{array}$$

Since $\mathbb{RP}^i$ is the i-skeleton of $\mathbb{RP}^n$ and the cellular cochain complex has zero differential, we see that the right vertical and bottom horizontal maps are isomorphisms. We can then show that the left vertical map is an isomorphism by applying the 5-Lemma to the map of long exact sequences induced by the map of pairs $(\mathbb{RP}^i, \mathbb{RP}^{i-1}) \to (\mathbb{RP}^n, \mathbb{RP}^{i-1})$. It follows that the top horizontal map is also an isomorphism, as required. $\square$

Proof of Theorem 10.4.4. We will prove the case of $\mathbb{RP}^n$; the proof for $\mathbb{CP}^n$ is the same. The inclusion $\mathbb{RP}^i \hookrightarrow \mathbb{RP}^n$ of the i-skeleton gives a ring homomorphism $H^*(\mathbb{RP}^n; \mathbb{F}_2) \to H^*(\mathbb{RP}^i; \mathbb{F}_2)$ and this is an isomorphism for $* \leq i$ (using cellular cohomology). Thus the cup products that land in degrees $< n$ are determined by the cup products in $\mathbb{RP}^{n-1}$; we can therefore proceed by induction, and are left with proving that the top-degree cup product maps

$$H^i(\mathbb{RP}^n; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^{n-i}(\mathbb{RP}^n; \mathbb{F}_2) \to H^n(\mathbb{RP}^n; \mathbb{F}_2)$$

are isomorphisms. But by naturality we have a commutative square

$$\begin{array}{ccc} H^i(\mathbb{RP}^n, U_{\{0,\ldots,i-1\}}; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^{n-i}(\mathbb{RP}^n, U_{\{i+1,\ldots,n\}}; \mathbb{F}_2) & \xrightarrow{\smile} & H^n(\mathbb{RP}^n, U_{\mathbf{n}_+ \setminus \{i\}}; \mathbb{F}_2) \\ \downarrow & & \downarrow \\ H^i(\mathbb{RP}^n; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^{n-i}(\mathbb{RP}^n; \mathbb{F}_2) & \xrightarrow{\smile} & H^n(\mathbb{RP}^n; \mathbb{F}_2) \end{array}$$

where the top horizontal map is an isomorphism by Corollary 10.4.12 and the vertical maps are isomorphisms by Lemma 10.4.13.

Since $\mathbb{Z}/2 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ the relative tensor products over $\mathbb{F}_2$ here are isomorphic to tensor products of abelian groups.

For $\mathbb{RP}^\infty$ we observe that the inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty$ gives a ring homomorphism $H^*(\mathbb{RP}^\infty; \mathbb{F}_2) \to H^*(\mathbb{RP}^n; \mathbb{F}_2)$ that is an isomorphism in degrees $* \leq n$, so the cup products in $\mathbb{RP}^\infty$ in degrees $\leq n$ are determined by those in $\mathbb{RP}^n$, which gives the extension to $\mathbb{RP}^\infty$. □

Example 10.4.14. The topological spaces $S^2 \vee S^4$ and $\mathbb{CP}^2$ have isomorphic (co)homology groups: in both cases we have $\mathbb{Z}$ in dimensions $0, 2, 4$ and 0 elsewhere. However, their cup product structures are different: if x denotes a generator of H^2, then Theorem 10.4.4 implies that for $\mathbb{CP}^2$ the cup square x^2 is a generator of H^4; on the other hand, in Exercise 10.6 you will show that $x^2 = 0$ in $S^2 \vee S^4$. Thus the cohomology ring of a space is a more powerful invariant than the cohomology groups alone.

Exercise 10.8 ($*$)**.** Show that $H^*(\mathbb{CP}^n \times \mathbb{CP}^m)$ is a truncated graded polynomial ring in two variables,

$$H^*(\mathbb{CP}^n \times \mathbb{CP}^m) \cong \mathbb{Z}[x, y]/(x^{n+1}, y^{m+1}),$$

where both generators are in degree 2. [Hint: First check that there is an isomorphism of (ungraded) polynomial rings $\mathbb{Z}[x] \otimes \mathbb{Z}[y] \cong \mathbb{Z}[x, y]$. Use Exercise 10.2.]

11
Manifolds and Poincaré Duality

In §11.1 we introduce *manifolds*, which are spaces that locally look like $\mathbb{R}^n$, and prove some first results about their homology. Then we define *orientations* of manifolds in §11.2 and show that every oriented compact n-manifolds M has a *fundamental class* $[M] \in H_n(M)$. Poincaré duality for compact oriented manifolds is an isomorphism

$$H^k(M) \xrightarrow{\sim} H_{n-k}(M)$$

where the map is given by applying a general construction, the *cap product*, to the fundamental class; we introduce cap products in §11.3. The cap product is closely related to the cup product in cohomology, and in §11.4 we apply Poincaré duality to better understand cup products on manifolds; in particular, we give a simple proof of the cup product structure for real and complex projective spaces. In order to prove Poincaré duality we want to work locally on a manifold, which means we have to drop the compactness assumption; since Poincaré duality is trivially false for non-compact manifolds (such as $\mathbb{R}^n$) we must introduce a new variant of cohomology, namely *cohomology with compact support*, in §11.5. Using this we can then prove Poincaré duality in §11.6.

11.1 Manifolds

Definition 11.1.1. A *(topological) manifold* of dimension n (or just n-*manifold*) is a second-countable Hausdorff space M such that every point of M has a neighbourhood that is homeomorphic to $\mathbb{R}^n$.

Remark 11.1.2. A topological space X is *second-countable* if there exists a countable set of open sets $U_i \subseteq X$ ($i = 1, 2, \ldots$) such that every open set in X can be written as a union of U_i's. (For example, $\mathbb{R}^n$ is second-countable because we can cover any subset by open balls whose centers have rational coordinates and whose radii are rational.) This condition will not really play a role here, though it is commonly part of the definition of manifolds and will allow us to prove some things by induction that would otherwise require using transfinite induction.

Remark 11.1.3. Manifolds in this sense that are compact are often called *closed manifolds*, to distinguish them from compact manifolds with boundary (an important variant of the definition, though we probably won't discuss it).

Examples 11.1.4. Many of the spaces we have looked at in the course so far are manifolds:

- the spheres S^n
- the torus $S^1 \times S^1$
- the Klein bottle
- the orientable surface of genus g
- the projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$ (but not the infinite versions $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$)

Moreover, if M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.

Example 11.1.5 (3-manifolds from knots)**.** Suppose $K \subseteq S^3$ is a knot (a smooth embedding of S^1). Then we can choose N to be a little tube around K, which is an embedding of the solid torus $D^1 \times S^1$. If we remove this, then $S^3 \setminus N$ has a boundary $S^1 \times S^1$ and we can glue in the solid torus $S^1 \times D^1$ to get a new 3-manifold. The Lickorish–Wallace theorem from the 1960s states that all closed orientable connected smooth 3-manifolds can be obtained by variations of this procedure, which is called *Dehn surgery*.

Lemma 11.1.6. *If M is an n-manifold then for every point $x \in M$ we have*

$$H_*(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & * = n, \\ 0, & * \neq n. \end{cases}$$

Proof. Let U be a neighbourhood of x such that there is a homeomorphism $\phi\colon U \xrightarrow{\sim} \mathbb{R}^n$. Then we have isomorphisms

$$H_*(M, M \setminus \{x\}) \cong H_*(U, U \setminus \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{\phi(x)\})$$

by excising $M \setminus U$ and applying ϕ. But there is a homotopy equivalence between $(\mathbb{R}^n, \mathbb{R}^n \setminus \{\phi(x)\})$ and (D^n, S^{n-1}) so that

$$H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{\phi(x)\}) \cong H_*(D^n, S^{n-1}) \cong \tilde{H}_*(S^n),$$

which gives the required result. □

Remark 11.1.7. This shows in particular that if M is an n-manifold then it is not an m-manifold where $m \neq n$.

We are going to talk a lot about homology groups of the form $H_*(M, M \setminus K; R)$ where K is a compact subset of M, so it is convenient to introduce some short-hand notation for these:

We can think of $H_*(M|K;R)$ as the "homology of M in a small neighbourhood of K".

Notation 11.1.8. We write $H_*(M|K;R) := H_*(M, M \setminus K; R)$. If $K \subseteq L$ then we denote the homomorphism

$$H_*(M|L;R) \to H_*(M|K;R)$$

coming from the inclusion of pairs $(M, M \setminus L) \hookrightarrow (M, M \setminus K)$ by ρ^L_K, or just ρ_K if L is understood. If $K = \{x\}$ consists of a single point, we also write $H_*(M|x;R) := H_*(M|\{x\};R)$ and $\rho_x := \rho_{\{x\}}$.

Here is our first result on the homology of manifolds:

Proposition 11.1.9. *Let M be an n-manifold and K a compact subset of M.*

(i) $H_*(M|K;R) = 0$ *for* $* > n$.

(ii) A class $\alpha \in H_n(M|K;R)$ *is zero if and only if* $\rho_x\alpha = 0$ *in* $H_n(M|x;R)$ *for all* $x \in K$.

Remark 11.1.10. Part (ii) of the proposition is equivalent to: the homomorphism

$$H_n(M|K;R) \to \prod_{x\in K} H_n(M|x;R) \cong \prod_{x\in K} R$$

is injective.

For the proof we need a variant of the Mayer–Vietoris sequence:

Lemma 11.1.11. *Suppose $A, B \subseteq X$ are a reasonable pair of subspaces, and label the inclusions of subspace pairs as*

$$\begin{array}{ccc} (X, A\cap B) & \xrightarrow{i} & (X,A) \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle j} \\ (X,B) & \xrightarrow{j'} & (X, A\cup B). \end{array}$$

Then there is a natural long exact sequence

$$\cdots \to H_n(X, A\cap B) \xrightarrow{(i_*, i'_*)} H_n(X,A) \oplus H_n(X,B) \xrightarrow{j_* - j'_*} H_n(X, A\cup B) \to H_{n-1}(X, A\cap B) \to \cdots$$

Proof. Whenever we have two subgroups N, N' of an abelian group M, we have a commutative square

$$\begin{array}{ccc} M/(N\cap N') & \xrightarrow{p} & M/N \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle q} \\ M/N' & \xrightarrow{q'} & M/(N+N') \end{array}$$

where the maps give a natural short exact sequence

$$0 \to M/(N\cap N') \xrightarrow{(p,p')} M/N \oplus M/N' \xrightarrow{q-q'} M/(N+N') \to 0.$$

For any pair of subspaces A, B we therefore have a short exact sequence of chain complexes

$$0 \to S_\bullet(X, A\cap B) \to S_\bullet(X,A) \oplus S_\bullet(X,B) \to S_\bullet(X, A+B) \to 0.$$

If A, B are a reasonable pair of subspaces then this gives a long exact sequence of the required form by Proposition 9.5.9. ☐

Remark 11.1.12. If M is an n-manifold and $K, L \subseteq M$ are compact subsets, then this long exact sequence for the open subsets $M \setminus K$, $M \setminus L$ can be written as

$$\cdots \to H_k(M|K\cup L) \xrightarrow{(\rho_K^{K\cup L}, \rho_L^{K\cup L})} H_k(M|K) \oplus H_k(M|L) \xrightarrow{\rho^K_{K\cap L} - \rho^L_{K\cap L}} H_k(M|K\cap L) \to H_{k-1}(M|K\cup L) \to \cdots.$$

Proof of Proposition 11.1.9. We first prove the case $M = \mathbb{R}^n$ in several steps: First suppose K is a compact convex subset of $\mathbb{R}^n$ and x is a point of K. By Lemma 11.1.6 it is enough to show that the inclusion of pairs $(\mathbb{R}^n, \mathbb{R}^n \setminus K) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ gives an isomorphism in homology. Using the map of long exact sequences for the two pairs and the 5-Lemma we see that it suffices to show that the inclusion $\mathbb{R}^n \setminus K \hookrightarrow \mathbb{R}^n \setminus \{x\}$ gives isomorphisms in homology. To prove this we choose a large sphere S centred at x such that K is contained in the open ball bounded by S. Then the inclusion $S \hookrightarrow \mathbb{R}^n \setminus K$ is a deformation retract: we define a retraction $\rho\colon \mathbb{R}^n \setminus K \to S$ by taking a point y to the point where the line from x to y intersects S; the homotopy between ρ and the identity on $\mathbb{R}^n \setminus K$ is given by moving along the line from y to $\rho(y)$ — since K is convex this line segment necessarily lies in $\mathbb{R}^n \setminus K$. The inclusion of S into $\mathbb{R}^n \setminus \{x\}$ is a deformation retract in the same way, so the inclusion $\mathbb{R}^n \setminus K \hookrightarrow \mathbb{R}^n \setminus \{x\}$ must also give isomorphisms in homology by the 2-of-3 property for isomorphisms.

Next suppose $K \subseteq \mathbb{R}^n$ is a finite union $K_1 \cup \cdots \cup K_r$ where the K_i are compact convex subsets. We just proved the case $r = 1$ so we induct on r and set $K' := K_1 \cup \cdots \cup K_{r-1}$. Note that $K' \cap K_r$ is the union of the $r-1$ compact convex sets $K_1 \cap K_r, \ldots, K_{r-1} \cap K_r$. We use the exact sequence of Lemma 11.1.11 as in Remark 11.1.12 to get a long exact sequence

$$\cdots \to H_{k+1}(\mathbb{R}^n|K' \cap K_r) \to H_k(\mathbb{R}^n|K) \to H_k(\mathbb{R}^n|K') \oplus H_k(\mathbb{R}^n|K_r) \to \cdots$$

If $k > n$ the terms around $H_k(\mathbb{R}^n|K)$ are 0 by the inductive hypothesis, hence $H_k(\mathbb{R}^n|K) = 0$, while if $k = n$ we know that $H_{n+1}(\mathbb{R}^n|K' \cap K_r) = 0$ so that the map

$$H_n(\mathbb{R}^n|K) \to H_n(\mathbb{R}^n|K') \oplus H_n(\mathbb{R}^n|K_r)$$

is injective. This means a class $\alpha \in H_n(\mathbb{R}^n|K)$ is zero if and only if $\rho_{K'}\alpha = 0$ and $\rho_{K_r}\alpha = 0$. Applying the inductive hypothesis to these two classes it follows that $\alpha = 0$ if and only if $\rho_x\alpha = 0$ for all $x \in K$.

Now let K be an arbitrary compact subset of $\mathbb{R}^n$. Given $\alpha \in H_i(\mathbb{R}^n|K)$ we can lift α to a chain $\gamma \in S_i(\mathbb{R}^n)$ whose image in $S_i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$ is a cycle that represents α. The boundary $\partial\gamma$ is a linear combination of simplices whose image is contained in a compact set L disjoint from K. Choose a compact neighbourhood N of K (meaning N contains an open neighbourhood of every point of K) such that $N \cap L = \emptyset$. Then the image of γ in $S_i(\mathbb{R}^n, \mathbb{R}^n \setminus N)$ is also a cycle, representing a homology class $\alpha' \in H_i(\mathbb{R}^n|N)$ such that $\alpha = \rho_K\alpha'$. We can cover K by finitely many closed balls $B_1, \ldots, B_r$ such that $B_i \subseteq N$ and $B_i \cap K \neq \emptyset$. Set $B := B_1 \cup \cdots \cup B_r$. Then if $i > n$ we must have $\rho_B\alpha' = 0$ by the previous case, hence $\alpha = \rho_K^N\alpha' = \rho_K^B\rho_B^N\alpha' = 0$, which implies that $H_i(\mathbb{R}^n, \mathbb{R}^n|K) = 0$ for $i > n$. If $i = n$ and $\rho_x\alpha = 0$ for $x \in K$, then $\rho_{B_i}\alpha' = 0$ for all i since the first case implies $\rho_b^{B_i}$ is an isomorphism for any $b \in B_i$. Hence $\rho_x\alpha' = 0$ for all $x \in B$ and so $\rho_B^N\alpha' = 0$ which implies $\alpha = 0$, as required.

Now we consider a general n-manifold M. First suppose $K \subseteq M$ is a compact subset such that there exists an open subset U such that $U \cong \mathbb{R}^n$ and $K \subseteq U$. Then $H_*(M|K) \cong H_*(U|K)$ by excision, and the assertion follows from the case of $\mathbb{R}^n$. For a general K, we can write K as a finite union $K_1 \cup \cdots \cup K_r$ where the previous case applies to each K_i. We can then induct on r using the exact sequence of Lemma 11.1.11 as in the case of a finite union of compact convex subsets above. □

Corollary 11.1.13. *If M is a compact n-manifold, then $H_*(M; R) = 0$ for $* > n$.* □

11.2 Orientations and Fundamental Classes

We now want to define *orientations* of (compact) manifolds M, and prove that an orientation determines a unique generator of $H_n(M)$, the *fundamental class* of M. We can formulate the main result we need for this as a description of the image of the injection from Proposition 11.1.9:

Definition 11.2.1. For M an n-manifold and $T \subseteq M$ any subset, let $\Gamma(M|T; R)$ denote the subgroup of $\prod_{x \in T} H_n(M|x; R)$ consisting of elements $(\alpha_x)_{x \in T}$ such that for every $x \in T$ there exists a compact neighbourhood $N \subseteq T$ of x and a class $\alpha_N \in H_n(M|N; R)$ such that for every $y \in N$ we have $\rho_y \alpha_N = \alpha_y$.

Remark 11.2.2. By a compact neighbourhood of x in T we mean a compact subset N with $x \in N$ such that N contains an open neighbourhood of x in T.

Proposition 11.2.3. *For M an n-manifold and $K \subseteq M$ compact, the restrictions $\rho_x \colon H_n(M|K; R) \to H_n(M|x; R)$ for $x \in K$ induce an isomorphism*

$$H_n(M|K; R) \xrightarrow{\sim} \Gamma(M|K; R).$$

Proof. It is clear from the definition that the homomorphism

$$H_n(M|K; R) \to \prod_{x \in K} H_n(M|x; R)$$

factors through $\Gamma(M|K; R)$ (since we can take the neighbourhood K at each point), and this is moreover injective by Proposition 11.1.9. It therefore remains only to show that this homomorphism is surjective. Let $(\alpha_x)_{x \in K}$ be an element of $\Gamma(M|K; R)$. By assumption, for any $x \in K$ there exists a compact neighbourhood N and a class $\alpha_N \in H_n(M|N)$ such that $\alpha_y = \rho_y \alpha_N$ for all $y \in N$. Since K is compact we can write K as a finite union $K_1 \cup \cdots \cup K_r$ where each K_i is a compact set where such a class α_{K_i} exists. Set $K'_i := K_1 \cup \cdots \cup K_i$, then we will prove by induction on i that there exists a class $\alpha_{K'_i} \in H_n(M|K'_i; R)$ such that $\rho_x \alpha_{K'_i} = \alpha_x$ for all $x \in K'_i$.

If we know this for K'_{i-1} we can use the exact sequence of Lemma 11.1.11 as in Remark 11.1.12 to get a long exact sequence

$$\cdots \to H_{n+1}(M|K'_{i-1} \cap K_i) \to H_n(M|K'_i) \to H_n(M|K'_{i-1}) \oplus H_n(M|K_i) \to H_n(M|K'_{i-1} \cap K_i) \to \cdots$$

Here $H_{n+1}(M|K'_{i-1} \cap K_i) = 0$ by Proposition 11.1.9(i), while Proposition 11.1.9(ii) implies that $\rho_{K'_{i-1}\cap K_i}\alpha_{K'_{i-1}} - \rho_{K'_{i-1}\cap K_i}\alpha_{K_r} = 0$ since it restricts to 0 at all $x \in K' \cap K_r$ by construction. By exactness this implies that there is a class $\alpha_{K'_i} \in H_n(M|K'_i)$ such that $\rho_{K'_{i-1}}\alpha_{K'_i} = \alpha_{K'_{i-1}}$ and $\rho_{K_i}\alpha_{K'_i} = \alpha_{K_i}$. Then $\rho_x\alpha'_{K'_i} = \alpha_x$ for all $x \in K'_i$, as required. Since $K = K'_r$, by induction we get α_K as $\alpha_{K'_r}$. □

Definition 11.2.4. Let M be an n-manifold (not necessarily compact) and R a commutative ring. An *R-orientation* of M is an element $(\alpha_x)_{x\in M}$ of $\Gamma(M|M;R)$ such that α_x is a generator of $H_n(M|x;R) \cong R$ for every $x \in M$. We say that M is *R-orientable* if there exists an R-orientation, while M is *R-oriented* if we equip it with a choice of orientation. If $R = \mathbb{Z}$ we just say that M is *orientable*.

Remark 11.2.5. An n-manifold is orientable in this sense if and only if it is orientable in more geometric terms, but proving this is a little beyond the scope of this course as it requires covering spaces.

Remark 11.2.6. Not every n-manifold is $\mathbb{Z}$-orientable: if we start with a generator $u \in H_n(M|x)$ we can always extend this uniquely to a neighbourhood of x: if $U \subseteq M$ is an open neighbourhood of x that is homeomorphic to $\mathbb{R}^n$ we can let $B \subseteq U$ be the image of a closed ball around the image of x in $\mathbb{R}^n$; then $\rho^B_x \colon H_n(M|B) \xrightarrow{\sim} H_n(M|x)$ is an isomorphism, and we can assign the generator $\rho^B_y(\rho^B_x)^{-1}u$ to $y \in B$. The issue is that we may not be able to consistently extend this to all of M: although an extension to a neighbourhood of a point always exists, it might be that if we make a sequence of such extensions around a closed loop that starts at x, then the generator we get at the end is $-u$.

Remark 11.2.7. If we take $R = \mathbb{F}_2$, however, this issue does not arise, since there is no *choice* of generator involved: any isomorphism of free rank-1 $\mathbb{F}_2$-modules must take the unique non-zero element to itself. Thus *every* compact manifold is $\mathbb{F}_2$-orientable (and the orientation is unique).

Definition 11.2.8. If M is an R-oriented compact n-manifold, then there exists a unique homology class $[M] \in H_n(M)$ that corresponds to the orientation under the equivalence of Proposition 11.2.3 (in the case $K = M$). The class $[M]$ is called the *fundamental class* of M.

Remark 11.2.9. We should think of the class $[M]$ as representing "the whole manifold M". For example, if we can describe M as a Δ-set of dimension n then we can represent $[M]$ as a simplicial chain by adding up the top-dimensional simplices, but with appropriate signs: every $(n-1)$-simplex appears as a face of exactly 2 n-simplices, and we need these to appear with opposite signs in the boundary to get a cycle. If M is oriented it can be shown that the orientation can be used to determine such signs — alternatively, if we work over $\mathbb{F}_2$ we always get a cycle since the two copies of each face cancel.

Proposition 11.2.10. *If M is a compact connected n-manifold, then*

$$\rho_x \colon H_n(M;R) \to H_n(M|x;R) \cong R$$

is injective, and if M is R-orientable then this is an isomorphism.

Proof. By Proposition 11.2.3 to prove the first statement it suffices to show that the projection $\Gamma(M|M;R) \to H_n(M|x;R)$ is injective. For $(\alpha_x)_{x\in M} \in \Gamma(M|M;R)$, let $V \subseteq M$ be the subset of points x where $\alpha_x = 0$. We will show that V is both open and closed. For any point x in M, by assumption there exists a compact neighbourhood N of x and a class $\alpha_N \in H_n(M|N;R)$ such that $\alpha_y = \rho_y^N \alpha_N$ for $y \in N$. We can then choose an open neighbourhood U of x such that $U \cong \mathbb{R}^n$ and a compact neighbourhood $B \subseteq U \cap N$ corresponding to a closed ball around x under this homeomorphism. Then we know from the proof of Proposition 11.1.9 that $\rho_y^B \colon H_n(M|B) \to H_n(M|y)$ is an isomorphism for all $y \in B$. Thus if $x \in V$ we have that $\alpha_y = \rho_y^B(\rho_x^B)^{-1}\alpha_x = 0$ for all $y \in B$. On the other hand, if $x \notin V$ then $\alpha_y \neq 0$ for $y \in B$ by the same argument. In particular, both V and $M \setminus V$ contain open neighbourhoods of each of their points, and hence are open. Since M is connected, it follows that we must have $V = \emptyset$ or $V = M$, which implies injectivity.

Finally, if M is orientable, then we know the image of $H_n(M)$ in $H_n(M|x;R)$ contains a generator, and hence ρ_x^M must also be surjective. □

Remark 11.2.11. With some more work (which is again a bit beyond the scope of the course as it involves covering spaces) it can be shown that if M is *not* R-orientable then the image of ρ_x is the 2-torsion subgroup of R. In particular, for $R = \mathbb{Z}$ it follows that for a compact n-manifold M the group $H_n(M)$ is either $\mathbb{Z}$, if M is orientable, or 0, if M is not orientable. For example, we can immediately read off from the homology of $\mathbb{RP}^n$ that $\mathbb{RP}^n$ is orientable if n is odd, and non-orientable if n is even.

11.3 Cap Products

Our next goal is to state the Poincaré duality theorem, which says that for a compact oriented manifold M there is an isomorphism between homology and cohomology groups,

$$H^k(M) \cong H_{n-k}(M).$$

The map that gives this isomorphism arises by applying a general construction, the *cap product*, to the fundamental class of M.

Definition 11.3.1. For any chain complex $C_\bullet$ and abelian group M, there is a natural *evaluation pairing*

$$\mathrm{ev}\colon \mathrm{Hom}(C,M)_\bullet \otimes C_\bullet \to M[0],$$

given in degree 0 on the generator $\phi \otimes x$ for $x \in C_n$, $\phi \in \mathrm{Hom}(C,M)_{-n} = \mathrm{Hom}(C_n, M)$ by

$$\mathrm{ev}(\phi \otimes x) = (-1)^{\lambda(n)}\phi(x)$$

where

$$\lambda(n) = \begin{cases} 0, & n \equiv 0, 3 \pmod 4, \\ 1, & n \equiv 1, 2 \pmod 4. \end{cases}$$

In particular, for any topological space X this gives a natural chain map

$$\mathrm{ev}\colon S^\bullet(X; M) \otimes S_\bullet(X) \to M[0].$$

If R is a ring, we can combine this with multiplication in R to get a pairing

$$\mathrm{ev}_R\colon S^\bullet(X;R) \otimes S_\bullet(X;R) \cong S^\bullet(X;R) \otimes S_\bullet(X) \otimes R[0] \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} R[0] \otimes R[0] \to R[0].$$

This descends on homology to a pairing

$$\kappa_R\colon H^*(X;R) \otimes H_*(X;R) \to R[0],$$

called the *Kronecker pairing*.

In fact, this pairing is R-bilinear, and so corresponds to a graded R-module homomorphism $H^*(X;R) \otimes_R H_*(X;R) \to R[0]$

Remark 11.3.2. The sign is necessary to get a chain map with our sign convention for $\mathrm{Hom}(C, M)$, since for $x \in C_{n+1}, \phi \in \mathrm{Hom}(C_n, M)$ we need

$$\begin{aligned} \mathrm{ev}(d(\phi \otimes x)) &= \mathrm{ev}(d\phi \otimes x + (-1)^n \phi \otimes dx) \\ &= (-1)^{\lambda(n+1)} \phi(dx) + (-1)^n (-1)^{\lambda(n)} \phi(dx) \\ &= \left((-1)^{\lambda(n+1)} + (-1)^{n+\lambda(n)}\right) \phi(dx) \end{aligned}$$

so that the parity of $\lambda(n+1)$ must be the opposite of that of $n + \lambda(n)$ for all n. The evaluation pairing corresponds to the identity map of $\mathrm{Hom}(C, M)_\bullet$ under the equivalence of Exercise 9.4.

Definition 11.3.3. For any topological space X and ring R, combining the diagonal and the Eilenberg–Zilber map gives a natural chain map

$$S_\bullet(X;R) \xrightarrow{\Delta_*} S_\bullet(X \times X; R) \to S_\bullet(X) \otimes S_\bullet(X;R).$$

Combining this with the evaluation pairing, we get a natural chain map

$$S^\bullet(X;R) \otimes S_\bullet(X;R) \to S^\bullet(X;R) \otimes S_\bullet(X) \otimes S_\bullet(X;R) \to R[0] \otimes S_\bullet(X;R) \to S_\bullet(X;R),$$

where the last map uses the multiplication in R. which we call the (chain-level) *cap product*. We denote the cap product of $\phi \in S^n(X;R)$, $c \in S_m(X;R)$ by $\phi \frown c \in S_{m-n}(X;R)$. The cap product descends to homology as a map

$$H^*(X;R) \otimes H_*(X;R) \to H_*(X;R),$$

which we also call the cap product and denote in the same way; note that this is independent of the choice of Eilenberg–Zilber maps.

Remark 11.3.4. The fact that the cap product is a chain map amounts to the relation

$$\partial(\phi \frown c) = \delta\phi \frown c + (-1)^n \phi \frown \partial c$$

in $S_{m-n}(X;R)$ for $\phi \in S^n(X;R)$, $c \in S_m(X;R)$.

Variant 11.3.5. If A and B are a reasonable pair of subspaces of X, then we can use the chain map

$$S_\bullet(X, A \cup B) \xrightarrow{\Delta^*} S_\bullet(X \times X, A \times X \cup X \times B)$$

and the chain homotopy equivalence between $S_\bullet(X \times X, A \times X \cup X \times B)$ and $S_\bullet(X, A) \otimes S_\bullet(X, B)$ to get a *relative cap product* as the composite

$$S^\bullet(X, A; R) \otimes S_\bullet(X, A \cup B; R) \to S^\bullet(X, A; R) \otimes S_\bullet(X, A) \otimes S_\bullet(X, B; R) \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} R[0] \otimes S_\bullet(X, B; R) \to S_\bullet(X, B; R)$$

where the last map uses the multiplication in R. In particular, for any subspace A we have a relative cap product

$$S^\bullet(X, A; R) \otimes S_\bullet(X, A; R) \to S_\bullet(X; R).$$

Lemma 11.3.6. *The cap product is related to the Kronecker pairing and cup product by the following formulae for $\phi, \psi \in H^*(X; R)$, $\alpha \in H_*(X; R)$:*

(i) $\kappa_R(\phi, \psi \frown \alpha) = \kappa_R(\phi \smile \psi, \alpha)$

(ii) $(\phi \smile \psi) \frown \alpha = \phi \frown (\psi \frown \alpha)$,

(iii) $1 \frown \alpha = \alpha$.

Proof. We will prove the case $R = \mathbb{Z}$, to keep the diagrams a bit smaller, but the same strategy works for a general ring R. We use some naturality properties of the evaluation pairings, which we leave to the reader to verify: First, for any chain map $\phi\colon C_\bullet \to D_\bullet$, we have a commutative square

$$\begin{array}{ccc}
\mathrm{Hom}(D, M)_\bullet \otimes C_\bullet & \xrightarrow{\mathrm{id}\otimes\phi} & \mathrm{Hom}(D, M)_\bullet \otimes D_\bullet \\
\Big\downarrow{\scriptstyle \phi^*\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{ev}_D} \\
\mathrm{Hom}(C, M)_\bullet \otimes C_\bullet & \xrightarrow{\mathrm{ev}_C} & M[0].
\end{array}$$

Second, for chain complexes $C_\bullet, D_\bullet$, we have a commutative triangle

$$\begin{array}{ccc}
\mathrm{Hom}(C, M)_\bullet \otimes C_\bullet \otimes \mathrm{Hom}(D, N)_\bullet \otimes D_\bullet & & \\
& \searrow{\scriptstyle \mathrm{ev}_C\otimes\mathrm{ev}_D} & \\
\Big\downarrow & & M[0] \otimes N[0] \\
& \nearrow{\scriptstyle \mathrm{ev}_{C\otimes D}} & \\
\mathrm{Hom}(C \otimes D, M \otimes N)_\bullet \otimes C_\bullet \otimes D_\bullet. & &
\end{array}$$

To prove (i), consider the diagram

$$\begin{array}{ccccc}
S^\bullet(X) \otimes S^\bullet(X) \otimes S_\bullet(X) & \longrightarrow & \mathrm{Hom}(S_\bullet(X) \otimes S_\bullet(X), \mathbb{Z}) \otimes S_\bullet(X) & \longrightarrow & S^\bullet(X) \otimes S_\bullet(X) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{ev}} \\
S^\bullet(X) \otimes S^\bullet(X) \otimes S_\bullet(X) \otimes S_\bullet(X) & \longrightarrow & \mathrm{Hom}(S_\bullet(X) \otimes S_\bullet(X), \mathbb{Z}) \otimes S_\bullet(X) \otimes S_\bullet(X) & \xrightarrow{\mathrm{ev}} & \mathbb{Z}[0] \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{ev}\otimes\mathrm{id}} & & & \nearrow{\scriptstyle \mathrm{ev}} & \\
S^\bullet(X) \otimes S_\bullet(X). & & & &
\end{array}$$

Here the top left square commutes by naturality, the top right square commutes by the first naturality property of ev (applied to the composite $S_\bullet(X) \xrightarrow{\Delta_*} S_\bullet(X \times X) \to S_\bullet(X) \otimes S_\bullet(X)$) and the bottom triangle by the second. The path along the top row and right column gives $\kappa_{\mathbb{Z}}(\phi \smile \psi, \alpha)$ in homology, while the path along the left column and bottom gives $\kappa_{\mathbb{Z}}(\phi, \psi \frown \alpha)$.

To prove (ii), consider the diagram

$$\begin{array}{ccccc}
S^\bullet(X) \otimes S^\bullet(X) \otimes S_\bullet(X) & \longrightarrow & \mathrm{Hom}(S_\bullet(X) \otimes S_\bullet(X), \mathbb{Z}) \otimes S_\bullet(X) & \longrightarrow & S^\bullet(X) \otimes S_\bullet(X) \\
\downarrow & & \downarrow & & \downarrow \\
S^\bullet(X) \otimes S^\bullet(X) \otimes S_\bullet(X) \otimes S_\bullet(X) & \longrightarrow & \mathrm{Hom}(S_\bullet(X) \otimes S_\bullet(X), \mathbb{Z}) \otimes S_\bullet(X) \otimes S_\bullet(X) & \longrightarrow & S^\bullet(X) \otimes S_\bullet(X) \otimes S_\bullet(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev} \otimes \mathrm{id}} \\
S^\bullet(X) \otimes S^\bullet(X) \otimes S_\bullet(X) \otimes S_\bullet(X) \otimes S_\bullet(X) & \longrightarrow & \mathrm{Hom}(S_\bullet(X) \otimes S_\bullet(X), \mathbb{Z}) \otimes S_\bullet(X) \otimes S_\bullet(X) \otimes S_\bullet(X) & \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} & S_\bullet(X) \\
\downarrow & & & \nearrow{\scriptstyle \mathrm{ev} \otimes \mathrm{id}} & \\
S^\bullet(X) \otimes S_\bullet(X) \otimes S_\bullet(X). & & & &
\end{array}$$

Here the three squares in the top row and the leftmost column commute by naturality, and the last square and bottom triangle arise from the naturality properties of ev by tensoring with $S_\bullet(X)$.

Finally, to prove (iii) we consider the diagram

$$\begin{array}{ccccc}
\mathbb{Z} \otimes S_\bullet(X) & \longrightarrow & S^\bullet(*) \otimes S_\bullet(X) & \longrightarrow & S^\bullet(X) \otimes S_\bullet(X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} \otimes S_\bullet(X) \otimes S_\bullet(X) & \longrightarrow & S^\bullet(*) \otimes S_\bullet(X) \otimes S_\bullet(X) & \longrightarrow & S^\bullet(X) \otimes S_\bullet(X) \otimes S_\bullet(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev} \otimes \mathrm{id}} \\
\mathbb{Z} \otimes S_\bullet(*) \otimes S_\bullet(X) & \longrightarrow & S^\bullet(*) \otimes S_\bullet(*) \otimes S_\bullet(X) & \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} & S_\bullet(X) \\
\downarrow & & & \nearrow{\scriptstyle \mathrm{ev} \otimes \mathrm{id}} & \\
\mathbb{Z} \otimes \mathbb{Z} \otimes S_\bullet(X). & & & &
\end{array}$$

Here the three squares in the top row and left-most column commute by naturality, and the lower right and (deformed) bottom squares commute by the first naturality property of ev applied to the maps $S_\bullet(X) \to S_\bullet(*)$ and $S_\bullet(*) \to \mathbb{Z}[0]$. The composite along the left column and bottom gives the identity in homology because the evaluation pairing for $\mathbb{Z}$ is the canonical isomorphism, while the composite $S_\bullet(X) \to S_\bullet(*) \otimes S_\bullet(X) \to \mathbb{Z} \otimes S_\bullet(X)$ is chain homotopic to the canonical isomorphism by Proposition 10.3.1(ii). □

The following exercise gives a more concrete proof of these identities:

Exercise 11.1. Use the explicit formula for the Alexander–Whitney map to get a formula for the chain-level cap product, and use this to prove the identities relating the cap product to the cup product and Kronecker pairing.

Lemma 11.3.7. *For $f\colon X \to Y$ a continuous map, $\phi \in H^n(Y; R)$ and $\alpha \in H_m(X; R)$ we have*

$$f_*(f^*\phi \frown \alpha) = \phi \frown f_*\alpha$$

in $H_{m-n}(Y; R)$, while if $n = m$ we also have

$$\kappa_R(\phi, f_*\alpha) = \kappa_R(f^*\phi, \alpha).$$

Proof. We prove the first statement; the second is proved similarly by a slightly simpler diagram. Consider the diagram

$$
\begin{tikzcd}
 & S^\bullet(Y)\otimes S_\bullet(X) \arrow[rr, "\mathrm{id}\otimes f_*"] \arrow[dl, "f^*\otimes\mathrm{id}"'] \arrow[d] & & S^\bullet(Y)\otimes S_\bullet(Y) \arrow[d] \\
S^\bullet(X)\otimes S_\bullet(X) \arrow[dr] & S^\bullet(Y)\otimes S_\bullet(X)\otimes S_\bullet(X) \arrow[r, "\mathrm{id}\otimes f_*\otimes\mathrm{id}"] \arrow[d, "f^*\otimes\mathrm{id}\otimes\mathrm{id}"] & S^\bullet(Y)\otimes S_\bullet(Y)\otimes S_\bullet(X) \arrow[r, "\mathrm{id}\otimes\mathrm{id}\otimes f_*"] \arrow[d, "\mathrm{ev}\otimes\mathrm{id}"] & S^\bullet(Y)\otimes S_\bullet(Y)\otimes S_\bullet(Y) \arrow[d, "\mathrm{ev}\otimes\mathrm{id}"] \\
 & S^\bullet(X)\otimes S_\bullet(X)\otimes S_\bullet(X) \arrow[r, "\mathrm{ev}\otimes\mathrm{id}"] & S_\bullet(X) \arrow[r, "f_*"] & S_\bullet(Y).
\end{tikzcd}
$$

The middle square in the bottom row commutes by the first naturality property of ev from the proof of Lemma 11.3.6, while the three other squares commute by naturality. □

We can now make a precise statement of the version of Poincaré duality we want to prove:

Theorem 11.3.8 (Compact Poincaré duality)**.** *Let M be an R-oriented compact n-manifold. Then the cap product with the fundamental class gives isomorphisms*

$$- \frown [M] \colon H^k(M;R) \xrightarrow{\sim} H_{n-k}(M;R).$$

Remark 11.3.9. If k is a field then the universal coefficient theorem gives an isomorphism $H^i(M;k) \cong \mathrm{Hom}_k(H_i(M;k),k) = H_i(M;k)^\vee$. Assuming these k-vector spaces are finite-dimensional (which is true at least if $n \neq 4$ or the manifold is smooth, since then it is a finite cell complex) Poincaré duality implies that there are non-canonical isomorphisms

$$H_i(M;k) \cong H_{n-i}(M;k).$$

Example 11.3.10. $S^1 \vee S^3$ is not homotopy equivalent to a compact manifold: We have

$$H_*(S^1 \vee S^3; \mathbb{F}_2) \cong \begin{cases} \mathbb{Z}/2, & * = 0,1,3, \\ 0, & \text{otherwise.} \end{cases}$$

Thus such a manifold would have to be 3-dimensional, as the top non-zero homology group is in degree 3, but then Poincaré duality would imply

$$H_1(S^1 \vee S^3; \mathbb{F}_2) \cong H_{3-1}(S^1 \vee S^3; \mathbb{F}_2),$$

which is false.

Exercise 11.2. Let M be a compact n-manifold whose homology groups are all finitely generated. (This is in fact true for all compact smooth manifolds.)

(i) Show that if M is orientable, then it is R-orientable for every commutative ring R. [Use the universal coefficient theorem.]

(ii) Show that if M is orientable, then $H_{n-1}(M)$ contains no torsion. [Apply Poincaré duality with $\mathbb{Z}/p$-coefficients and the universal coefficient theorem for every prime p.]

(iii) Suppose M is non-orientable and assume this implies M is also not $\mathbb{Z}/p$-orientable for any odd prime p, and that $H_n(M) = H_n(M;\mathbb{Z}/p) = 0$. Show that the torsion subgroup of $H_{n-1}(M)$ is $\mathbb{Z}/2$.

11.4 Application to Cup Products

Before we turn to the proof of Theorem 11.3.8, we will discuss some consequences for cup products. This will lead to a simple proof of the cup product structure on projective spaces.

Definition 11.4.1. Let R be a commutative ring. By a generalization of Exercise 8.4, a homomorphism of R-modules $\phi\colon M \otimes_R N \to P$ corresponds to a homomorphism $M \to \mathrm{Hom}_R(N, P)$ given by $m \mapsto \phi(m \otimes -)$. We will refer to this as the *adjoint homomorphism* in the second variable (with the adjoint in the first variable defined similarly, or using the natural isomorphism $M \otimes_R N \cong N \otimes_R M$).

Warning 11.4.2. The adjoint of the Kronecker pairing is a natural map

$$\kappa'_R\colon H^*(X; R) \to \mathrm{Hom}_R(H_*(X; R), R)$$

Note that for $R = \mathbb{Z}$ this is not *quite* the same as the map we used earlier in the course to prove the universal coefficient theorm, as it differs from it by the sign $(-1)^{\lambda(n)}$ in degree n. However, the statement of the universal coefficient theorem is equally true for this map.

Remark 11.4.3. The relation from Lemma 11.3.6(i) can be interpreted as saying that for any topological space X and $\phi \in H^k(X; R)$, there are commutative squares

$$\begin{array}{ccc} H^l(X;R) & \xrightarrow{\kappa'_R} & \mathrm{Hom}_R(H_l(X;R), R) \\ \downarrow{\scriptstyle \phi\smile -} & & \downarrow{\scriptstyle (\phi\frown -)^*} \\ H^{k+l}(X;R) & \xrightarrow{\kappa'_R} & \mathrm{Hom}_R(H_{k+l}(X;R), R). \end{array}$$

If R is a field or $R = \mathbb{Z}$ and the homology groups of X are free then the universal coefficient theorem for cohomology implies (as the Ext term vanishes) that the horizontal maps here are isomorphisms. Thus in this case the cup products are determined by cap products, and vice versa.

Definition 11.4.4. For R-modules M, N, a *perfect pairing* is an R-module homomorphism

$$\mu\colon M \otimes_R N \to R$$

such that the adjoint homomorphisms in each variable are isomorphisms,

$$M \xrightarrow{\sim} \mathrm{Hom}_R(N, R), \quad N \xrightarrow{\sim} \mathrm{Hom}_R(M, R).$$

Proposition 11.4.5. *Let M be a compact R-oriented n-manifold. Then the cup product pairing*

$$H^{n-k}(M; R) \otimes_R H^k(M; R) \xrightarrow{\smile} H^n(M; R) \xrightarrow{\kappa_R(-,[M])} R$$

is perfect if either R is a field or $R = \mathbb{Z}$ and the homology of M is free.

Proof. By Remark 11.4.3 we can equivalently define this pairing as the composite

$$H^{n-k}(M;R) \otimes_R H^k(M;R) \xrightarrow{\mathrm{id} \otimes (-\frown[M])} H^{n-k}(M;R) \otimes H_{n-k}(M;R) \xrightarrow{\kappa_R} R.$$

The adjoint in the second variable is therefore the composite

$$H^{n-k}(M;R) \xrightarrow{\kappa_R'} \mathrm{Hom}_R(H_{n-k}(M;R),R) \xrightarrow{(-\frown[M])^*} \mathrm{Hom}_R(H^k(M;R),R).$$

Here our assumptions guarantee that the first map is an isomorphism by the universal coefficient theorem for cohomology, while the second is an isomorphism by Poincaré duality. The proof for the first variable is the same. □

Corollary 11.4.6. *Let M be a compact connected R-oriented n-manifold.*

(i) If $R = \mathbb{Z}$ and the homology of M is free and $\alpha \in H^k(M)$ generates a summand $\mathbb{Z}\alpha$ of $H^k(M)$, then there exists a class $\beta \in H^{n-k}(M)$ such that $\alpha \smile \beta$ is a generator of $H^n(M) \cong \mathbb{Z}$.

(ii) If R is a field then for any $\alpha \in H^k(M;R)$ there exists an element $\beta \in H^{n-k}(M;R)$ such that $\alpha \smile \beta \neq 0$ in $H^n(M;R) \cong R$.

Proof. In (i), projection to the summand generated by α defines a homomorphism $\pi\colon H^k(M) \to \mathbb{Z}$ with $\pi(\alpha) = 1$. By Proposition 11.4.5 there exists a unique class $\beta \in H^{n-k}(M)$ such that for any $\xi \in H^k(M)$ we have

$$\pi(\xi) = \kappa_{\mathbb{Z}}(\xi \smile \beta, [M]).$$

Thus in particular $\kappa_{\mathbb{Z}}(\alpha \smile \beta, [M]) = \pi(\alpha) = 1$, which implies that $\kappa_{\mathbb{Z}}(1, (\alpha \smile \beta) \frown [M]) = 1$. But

$$\kappa_{\mathbb{Z}}(1,-)\colon H_0(M) \to \mathbb{Z}$$

is an isomorphism since M is connected, so this implies that the class $(\alpha \smile \beta) \frown [M]$ is a generator of $H_0(M)$, and hence $\alpha \smile \beta$ is a generator of $H^n(M)$ by Poincaré duality. The proof of (ii) is the same. □

Example 11.4.7. This implies that the space $S^2 \vee S^4$ is not homotopy equivalent to a compact manifold: Since its top-dimension homology group with $\mathbb{F}_2$-coefficients is in degree 4, it would have to be a 4-manifold, but if x is a generator of $H^2(S^2 \vee S^4; \mathbb{F}_2) \cong \mathbb{Z}/2$ then we know $x^2 = 0$, and so the cup product pairing is not perfect.

This result also gives an easy description of the cup product for real and complex projective space:

Corollary 11.4.8. *There are isomorphisms of graded rings*

$$H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$$

where x is a generator in degree 1,

$$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$$

where x is a generator in degree 2.

Proof. The space $\mathbb{RP}^n$ is a compact connected n-manifold (necessarily $\mathbb{F}_2$-oriented), while $\mathbb{CP}^n$ is a compact connected $\mathbb{Z}$-orientable $2n$-manifold. As in the previous proof of Theorem 10.4.4 we can induct on n, and so to get the ring structure it is enough to show that $x^{n-1} \smile x$ is a generator of $H^n(\mathbb{RP}^n; \mathbb{F}_2)$ or $H^{2n}(\mathbb{CP}^n)$. By Corollary 11.4.6 there exists some integer m such that $(mx^{n-1}) \smile x = m(x^{n-1} \smile x)$ is a generator, which is clearly impossible unless $m = \pm 1$. □

Remark 11.4.9. Suppose M is a compact connected oriented n-manifold and $N, K \subseteq M$ are oriented submanifolds of dimensions p, q, respectively. If N and K intersect transversely (a somewhat technical condition that can always be achieved by perturbing the submanifolds) then $N \cap K$ is a submanifold of dimension $p + q - n$ (empty if $p + q < n$), with a canonical orientation inherited from those of N, K, M. Let us write $[N] \in H_p(M)$ for the image of $[N] \in H_p(N)$ via the inclusion $N \hookrightarrow M$, and similarly for $[K]$ and $[N \cap K]$, and denote by $[N]^\vee \in H^{n-p}(M)$ etc. the Poincaré dual cohomology classes, then it can be shown that we have

$$[N]^\vee \smile [K]^\vee = [N \cap K]^\vee$$

in $H^{2n-p-q}(M)$, giving a geometric interpretation of this cup product in terms of intersections. For $p + q = n$ the intersection $N \cap K$ consists of a finite set of points, and so under the isomorphism $H^n(M) \cong \mathbb{Z}$ we get

$$[N]^\vee \smile [K]^\vee = \sum_{x \in N \cap K} \pm 1,$$

with the signs determined by the orientations. In particular, this signed sum over intersection points is invariant under deformations of the submanifolds N and K. (If we work instead with $\mathbb{F}_2$-coefficients, we see that (without orientations) the parity of the number of intersection points is invariant under deformations.)

Exercise 11.3. Use Poincaré duality to show that $S^n \vee S^m$ is not homotopy equivalent to a compact manifold for $n, m > 0$. [In the case $m = 2n$ you need to use the cup product, which you computed in Exercise 10.6.]

Exercise 11.4. Use the Künneth theorem to compute the integral (co)homology of the n-torus

$$T^n := (S^1)^{\times n}.$$

Apply Poincaré duality to deduce the binomial coefficient identity

$$\binom{n}{k} = \binom{n}{n-k}.$$

What is the ring structure?

Exercise 11.5. For which n does there exist a compact connected oriented $2n$-manifold M such that $H_n(M) \cong \mathbb{Z}$?

11.5 (⋆) Cohomology with Compact Support

Although our goal is to prove the Poincaré duality isomorphism for compact manifolds, to do so we want to work locally on the manifold.

But since an open subset of a compact manifold is typically no longer compact, this means we need to consider a version of Poincaré duality for non-compact manifolds. If we look at the non-compact manifold $\mathbb{R}^n$ then we know that

$$H_*(\mathbb{R}^n) \cong H^*(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0, \end{cases}$$

so we clearly *don't* have Poincaré duality in the form of Theorem 11.3.8. To fix this we need to introduce a new variant of cohomology, namely *cohomology with compact support*:

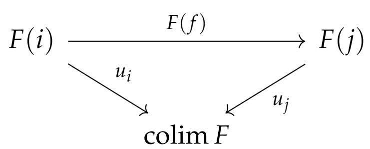
Definition 11.5.1. Let X be a topological space. A cochain $\phi \in S^k(X;R) \cong R^{\mathrm{Sing}_k(X)}$ has *compact support* if there exists a compact subset $K \subseteq X$ such that for every singular simplex $\sigma\colon \Delta^k \to X$ we have $\phi(\sigma) = 0$ if $\sigma(\Delta^k) \subseteq X \setminus K$. The cochains with compact support form a subgroup of $S^k(X;R)$, which we denote $S^k_c(X;R)$. If ϕ has compact support, then so does $\delta\phi$, so the cochains with compact support form a subcomplex $S^\bullet_c(X;R)$. We denote its homology by $H^*_c(X;R)$ and call this the *cohomology with compact support* of X (with coefficients in R).

Remark 11.5.2. If X is itself compact then every cochain has compact support, so in this case $H^*_c(X;R) \cong H^*(X;R)$.

Exercise 11.6. Show that if X is path-connected and non-compact, then $H^0_c(X) = 0$. [Hint: Use the definition of $S^\bullet_c(X)$ as a subcomplex of $S^\bullet(X)$.]

It is convenient to reformulate this definition using the general categorical notion of *colimits*:

Definition 11.5.3. Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor. The *colimit* of F, if it exists, is an object $\operatorname{colim} F$ together with maps $u_i\colon F(i) \to \operatorname{colim} F$ such that for every morphism $f\colon i \to j$ the triangle



commutes, with the universal property that given an object $x \in \mathcal{C}$ and morphisms $\phi_i\colon F(i) \to x$ such that $\phi_i = \phi_j F(f)$ for every morphism $f\colon i \to j$ in $\mathcal{C}$, then there exists a unique morphism $\phi\colon \operatorname{colim} F \to x$ such that $\phi_i = \phi u_i$ for all i.

Example 11.5.4. Coproducts, pushouts, and sequential colimits are special cases of colimits we have encountered earlier in the course.

Remark 11.5.5. Given a functor $F\colon \mathcal{I} \to \mathsf{Ab}$, its colimit exists and fits in a short exact sequence

$$0 \to \bigoplus_{f\colon i \to j} F(i) \xrightarrow{\alpha - \beta} \bigoplus_{i \in \mathcal{I}} F(i) \to \operatorname{colim}_{\mathcal{I}} F \to 0,$$

where α takes $x \in F(i)$ in the component indexed by $f\colon i \to j$ to x in the component $F(i)$ indexed by i and β takes it to $F(f)(x)$ in the component indexed by j. Moreover, colimits in Ch can be computed degreewise, and we get a similar short exact sequence of chain complexes for a functor $F\colon \mathcal{I} \to \mathsf{Ch}$.

Notation 11.5.6. As we did for homology, let us also write

$$H^*(M|K;R) := H^*(M, M \setminus K; R)$$

for M a manifold and $K \subseteq M$, and similarly for the corresponding singular cochains.

Lemma 11.5.7. *If M is a manifold, let $\mathrm{Cpt}(M)$ denote the partially ordered set of compact subsets of M, ordered by inclusion, viewed as a category. Then $S^\bullet_c(M;R)$ is the colimit of the functor $\mathrm{Cpt}(M) \to \mathsf{Ch}$ that takes K to $S^\bullet(M|K;R)$ and an inclusion $K \hookrightarrow L$ to the chain map $S^\bullet(M|K;R) \to S^\bullet(M|L;R)$ induced by the inclusion of pairs $(M, M \setminus L) \to (M, M \setminus K)$.*

We leave the proof as an exercise for the reader.

Definition 11.5.8. A category $\mathcal{I}$ is *filtered* if given any pair of objects i, i' there exists a third object i'' and maps $i \to i''$, $i' \to i''$, and if for every pair of parallel morphisms $f, g\colon i \to j$ there exists a morphism $h\colon j \to k$ such that $hf = hg$. (Note that if $\mathcal{I}$ is a partially ordered set, the second condition is vacuous, and the first says that for any pair of objects i, i' there exists a third object that is bigger than both i and i'.)

Example 11.5.9. If M is a manifold, then the partially ordered set $\mathrm{Cpt}(M)$ is filtered, since a finite union of compact sets is compact.

Fact 11.5.10 (Homology commutes with filtered colimits)**.** *If $\mathcal{I}$ is a filtered category then for any functor $F\colon \mathcal{I} \to \mathsf{Ch}$ the natural map*

$$\mathrm{colim}_{\mathcal{I}} H_*(F) \to H_*(\mathrm{colim}_{\mathcal{I}} F)$$

is an isomorphism.

Corollary 11.5.11. *If M is a manifold then*

$$H^*_c(M;R) \cong \mathrm{colim}_{K \in \mathrm{Cpt}(M)} H^*(M|K;R).$$

Remark 11.5.12. If $\mathcal{I}$ is a partially ordered set, then a subset $\mathcal{J} \subseteq \mathcal{I}$ is called *cofinal* if for every $i \in \mathcal{I}$ there exists $j \in \mathcal{J}$ such that $i \leq j$. Given a functor $F\colon \mathcal{I} \to \mathcal{C}$, then for a cofinal subset $\mathcal{J}$ the canonical map

$$\mathrm{colim}_{\mathcal{J}} F|_{\mathcal{J}} \to \mathrm{colim}_{\mathcal{I}} F$$

is an isomorphism.

Example 11.5.13. In $\mathbb{R}^n$, the subset of $\mathrm{Cpt}(\mathbb{R}^n)$ consisting of closed balls centred at 0 is cofinal. Therefore $H^*_c(\mathbb{R}^n)$ is the colimit of $H^*(\mathbb{R}^n|B)$ over all such balls. But $H^*(\mathbb{R}^n|0) \to H^*(\mathbb{R}^n|B)$ is an isomorphism for every such B, and since $\mathrm{Cpt}(M)$ is filtered it can be shown that this implies that $H^*(\mathbb{R}^n|0) \cong H^*_c(\mathbb{R}^n)$. We know that $H^*(\mathbb{R}^n|0) \cong \tilde{H}^*(S^n)$ and so

$$H^*_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}, & * = n, \\ 0, & * \neq 0. \end{cases}$$

Thus $H^*_c(\mathbb{R}^n) \cong H_{n-*}(\mathbb{R}^n)$ as needed for Poincaré duality to hold here.

Remark 11.5.14. If the one-point compactification M^+ of a manifold M is reasonable, then there is an isomorphism $H_c^*(M) \cong \tilde{H}^*(M^+)$. Since the one-point compactification of $\mathbb{R}^n$ is S^n, this result generalizes the preceding computation.

Definition 11.5.15. Let M be an R-oriented n-manifold. Then for every compact subset K, Proposition 11.2.3 implies that there exists a unique class $\mu_K^M \in H_n(M|K;R)$ (or just μ_K if M is clear) such that $\rho_x\mu_K$ agrees with the orientation at x for every $x \in K$. In particular, for $K \subseteq L$ we must have $\mu_K = \rho_K^L \mu_L$. We have relative cap products

$$H^i(M|K;R) \otimes H_j(M|K;R) \to H_{j-i}(M;R),$$

and so in particular a homomorphism

$$- \frown \mu_K \colon H^i(M|K;R) \to H_{n-i}(M;R).$$

The naturality properties of the cap product then imply that for $K \subseteq L$ the triangle

$$\begin{array}{ccc} H^i(M|K;R) & \longrightarrow & H^i(M|L;R) \\ & {\scriptstyle \frown\mu_K} \searrow \quad \swarrow {\scriptstyle \frown\mu_L} & \\ & H_{n-i}(M;R) & \end{array}$$

commutes. By the universal property of the colimit, this means there exists a unique homomorphism

$$D_M \colon H_c^i(M;R) \to H_{n-i}(M;R)$$

such that the triangle

$$\begin{array}{ccc} H^i(M|K;R) & \longrightarrow & H_c^i(M;R) \\ & {\scriptstyle \frown\mu_K} \searrow \quad \swarrow {\scriptstyle D_M} & \\ & H_{n-i}(M;R) & \end{array}$$

commutes for every $K \subseteq M$ compact. We call this D_M the *duality map*.

Remark 11.5.16. If M is a compact n-manifold then the duality map D_M is just $- \frown [M]$, since this has the property that uniquely characterizes D_M.

Remark 11.5.17. If we apply the Kronecker pairing to the classes μ_K then we similarly get a homomorphism

$$\int_M \colon H_c^n(M;R) \to R$$

such that the triangle

$$\begin{array}{ccc} H^i(M|K;R) & \longrightarrow & H_c^i(M;R) \\ & {\scriptstyle \kappa_R(-,\mu_K)} \searrow \quad \swarrow {\scriptstyle \int_M} & \\ & R & \end{array}$$

commutes. We can informally think of this map as "integrating" a compactly supported cochain over the manifold M.

We then have the following generalization of Poincaré duality for manifolds that are not necessarily compact:

Theorem 11.5.18 (Poincaré duality)**.** *If M is an R-oriented n-manifold, then the duality map*

$$D_M\colon H^k_c(M;R) \to H_{n-k}(M;R)$$

is an isomorphism.

11.6 (⋆) Proof of Duality

Let us start by proving Theorem 11.5.18 when M is $\mathbb{R}^n$:

Lemma 11.6.1. *The duality map*

$$D_{\mathbb{R}^n}\colon H^k_c(\mathbb{R}^n;R) \to H_{n-k}(\mathbb{R}^n;R)$$

is an isomorphism.

Proof. Let $B \subseteq \mathbb{R}^n$ be a closed ball. Then we know $H_*(\mathbb{R}^n|B) \cong \tilde{H}_*(S^n)$ with μ_B as the generator for $* = n$; in particular, this is free. By the universal coefficient theorem for cohomology, the map

Strictly speaking given the version of this that we have proved we should assume here that R is a principal ideal domain, which is certainly true in the interesting cases $R = \mathbb{Z}, \mathbb{F}_2$.

$$\kappa'_R\colon H^n(\mathbb{R}^n|B;R) \to \mathrm{Hom}_R(H_n(\mathbb{R}^n|B;R),R)$$

is an isomorphism, so there exists a generator $\gamma \in H^n(\mathbb{R}^n|B;R)$ such that $\kappa_R(\gamma,\mu_B) = 1$. The identity

$$\kappa_R(\gamma,\mu_B) = \kappa_R(1,\gamma \frown \mu_B),$$

which follows from a relative variant of Lemma 11.3.6, implies that $\gamma \frown \mu_B$ is a generator of $H_0(\mathbb{R}^n;R) \cong R$. Thus $\frown \mu_B$ is an isomorphism

$$H^*(\mathbb{R}^n|B;R) \xrightarrow{\sim} H_{n-*}(\mathbb{R}^n;R).$$

As in Example 11.5.13 the cohomology with compact support of $\mathbb{R}^n$ is given by taking the colimit over such closed balls B, so (as the diagram is filtered) we obtain in the colimit that the duality map

$$D_{\mathbb{R}^n}\colon H^*_c(\mathbb{R}^n;R) \xrightarrow{\sim} H_{n-*}(\mathbb{R}^n;R),$$

is an isomorphism. □

We want to prove Theorem 11.5.18 by reducing it to the case where M is $\mathbb{R}^n$. To do so we need to set up a new long exact sequence for cohomology with compact support:

Lemma 11.6.2. *Suppose M is an n-manifold and U, V are open subsets of M such that $M = U \cup V$. Then there is a long exact sequence*

$$\cdots \to H^i_c(U \cap V;R) \to H^i_c(U;R) \oplus H^i_c(V;R) \to H^i_c(M;R) \to H^{i+1}_c(U \cap V;R) \to \cdots$$

Remark 11.6.3. Note that the maps here go the "wrong" way compared to what we normally have in cohomology: for the inclusion

$U \subseteq M$ we have a covariant map $H^*_c(U;R) \to H^*_c(M;R)$. This is defined as follows: for $K \subseteq U$ compact the map

$$H^*(M, M \setminus K; R) \to H^*(U, U \setminus K; R)$$

is an isomorphism by excision. Taking the colimit over K, we get an isomorphism

$$\mathrm{colim}_{K \in \mathrm{Cpt}(U)} H^*(M, M \setminus K; R) \xrightarrow{\sim} H^*_c(U;R).$$

On the other hand, the inclusion $\mathrm{Cpt}(U) \subseteq \mathrm{Cpt}(M)$ gives a natural map on colimits

$$\mathrm{colim}_{K \in \mathrm{Cpt}(U)} H^*(M, M \setminus K; R) \to \mathrm{colim}_{K \in \mathrm{Cpt}(M)} H^*(M, M \setminus K; R) \cong H^*_c(M;R).$$

Combining the two we get a map $H^*_c(U;R) \to H^*_c(M;R)$, as required.

Proof of Lemma 11.6.2. Given $K \subseteq U$ and $L \subseteq V$ compact, we have a diagram

$$\begin{array}{ccccc} S^\bullet(M, (M \setminus K) + (M \setminus L); R) & \longrightarrow & S^\bullet(M|K;R) \oplus S^\bullet(M|L;R) & \longrightarrow & S^\bullet(M|K \cup L; R) \\ \downarrow & & \downarrow & & \| \\ S^\bullet(U \cap V, (U \cap V \setminus K \cap V) + (U \cap V \setminus U \cap L); R) & & S^\bullet(U, U \setminus K; R) \oplus S^\bullet(V, V \setminus L; R) & & S^\bullet(M, M \setminus K \cup L; R) \end{array}$$

where the top row is a short exact sequence (it is $\mathrm{Hom}(-,R)$ applied to the short exact sequence from the proof of Lemma 11.1.11) and the vertical maps give isomorphisms in homology by excision. In homology we therefore get a long exact sequence

$$\cdots \to H^i(U \cap V|K \cap L; R) \to H^i(U|K;R) \oplus H^i(V|L;R) \to H^i(M|K \cup L; R) \to H^{i-1}(U \cap V|K \cap L; R) \to \cdots.$$

The long exact sequence we want is then obtained by taking the colimit of these long exact sequences over $K \subseteq U, L \subseteq V$ compact (using cofinality arguments to identify the colimit of $H^i(M|K \cup L; R)$ with $H^i_c(M;R)$ and that of $H^i(U \cap V|K \cap L; R)$ with $H^i_c(U \cap V)$. □

We also need to know the duality morphisms relate this new long exact sequence to a Mayer–Vietoris sequence in homology:

Proposition 11.6.4. *Suppose M is an R-oriented n-manifold and U, V are open subsets of M such that $M = U \cup V$. Then there is a commutative diagram*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^i_c(U \cap V;R) & \longrightarrow & H^i_c(U;R) \oplus H^i_c(V;R) & \longrightarrow & H^i_c(M;R) & \longrightarrow & H^{i+1}_c(U \cap V;R) & \longrightarrow \cdots \\ & \downarrow D_{U \cap V} & & \downarrow (D_U, D_V) & & \downarrow D_M & & \downarrow D_{U \cap V} & \\ \cdots \longrightarrow & H_{n-i}(U \cap V;R) & \longrightarrow & H_{n-i}(U;R) \oplus H_{n-i}(V;R) & \longrightarrow & H_{n-i}(M;R) & \longrightarrow & H_{n-i-1}(U \cap V;R) & \longrightarrow \cdots, \end{array}$$

where the bottom row is a Mayer–Vietoris sequence, and the top row is the long exact sequence from Lemma 11.6.2.

Proof. To obtain the required compatibility it is convenient to derive the Mayer–Vietoris sequence as the homology long exact sequence corresponding to the short exact sequence of chain complexes

$$0 \to S_\bullet(U \cap V; R) \to S_\bullet(U;R) \oplus S_\bullet(V;R) \to S_\bullet(U + V; R) \to 0,$$

which has the right form in homology by the locality theorem for the cover of M by U and V. We will prove that given $K \subseteq U$ and $L \subseteq V$ compact, we can choose representatives of the classes $\mu^M_{K\cup L}, \mu^U_K, \mu^V_L, \mu^{U\cap V}_{K\cap L}$ (which we denote by $\bar{\mu}$ with the same decorations, such that we have a commutative diagram of chain complexes

$$\begin{array}{ccccc}
S^\bullet(M|K\cap L;R) & \longrightarrow & S^\bullet(M|K;R)\oplus S^\bullet(M|L;R) & \longrightarrow & S^\bullet(M|K\cup L;R) \\
\downarrow & & \downarrow & & \\
S^\bullet(U\cap V|K\cap L;R) & & S^\bullet(U|K;R)\oplus S^\bullet(V|L;R) & & \downarrow{\scriptstyle \frown\bar{\mu}^M_{K\cup L}} \\
\downarrow{\scriptstyle \frown\bar{\mu}^{U\cap V}_{K\cap L}} & & \downarrow{\scriptstyle (\frown\bar{\mu}^U_K, \frown\bar{\mu}^V_L)} & & \\
S_\bullet(U\cap V;R)[n] & \longrightarrow & S_\bullet(U;R)[n]\oplus S_\bullet(V;R)[n] & \longrightarrow & S_\bullet(M;R)[n],
\end{array}$$

where $C_\bullet[n] := C_\bullet \otimes \mathbb{Z}[n]$ denotes the shift of a chain complex $C_\bullet$ by n.

The rows are not short exact sequences, but they are naturally chain homotopy equivalent to the short exact sequences of chain complexes that we used to define the two long exact sequences; we therefore get (by choosing appropriate chain homotopy inverses) a commutative diagram relating these short exact sequences, and hence a morphism between the associated long exact sequences in homology. Taking the colimit of these over K and L as in the proof of Lemma 11.6.2, we obtain the required commutative diagram of long exact sequences.

Since the open sets $U \setminus L, V \setminus K, U \cap V$ cover M, by locality we can represent the homology class $\mu^M_{U\cup V}$ by a sum

$$\bar{\mu}^M_{U\cup V} = \alpha_{U\setminus L} + \alpha_{V\setminus K} + \alpha_{U\cap V}$$

where $\alpha_{U\setminus L}$ is a chain in $U \setminus L$, $\alpha_{V\setminus K}$ is a chain in $V \setminus K$, and $\alpha_{U\cap V}$ is a chain in $U \cap V$. Then the chain $\bar{\mu}_{K\cap L} := \alpha_{U\cap V}$ represents $\mu^{U\cap V}_{K\cap L}$ since the other two chains lie in the complement of $K \cap L$ and hence vanish in $H_n(M|K \cap L) \cong H_n(U \cap V|K \cap L)$. Similarly, $\bar{\mu}_K := \alpha_{U\setminus L} + \alpha_{U\cap V}$ represents μ^U_K and $\bar{\mu}_L := \alpha_{V\setminus K} + \alpha_{U\cap V}$ represents μ^V_L. To prove commutativity in the left square we must show that for ξ a cochain in $S^\bullet(M|K \cap L)$, we have $\xi \frown \bar{\mu}_{K\cap L} = \xi \frown \bar{\mu}_K$ in $S_\bullet(U;R)$. This is true because by definition ξ vanishes on chains outside $K \cap L$. Similarly, we have $\xi \frown \bar{\mu}_{K\cap L} = \xi \frown \bar{\mu}_L$ in $S_\bullet(V;R)$.

To prove the right square commutes, we need to check that for $(\phi, \psi) \in S^\bullet(M|K;R) \oplus S^\bullet(M|L;R)$, we have

$$(\phi - \psi) \frown \bar{\mu}_{K\cup L} = \phi \frown \bar{\mu}_K - \psi \frown \bar{\mu}_L.$$

This is true because

$$\phi \frown \bar{\mu}_{K\cup L} = \phi \frown (\bar{\mu}_K + \alpha_{V\setminus K}) = \phi \frown \bar{\mu}_K$$

since ϕ vanishes on chains outside K, such as $\alpha_{V\setminus K}$, and similarly for ψ. □

Corollary 11.6.5. *Let M be an R-oriented n-manifold. If M is the union of open subsets U and V, and if $D_U, D_V, D_{U\cap V}$ are isomorphisms, then so is D_M.*

Proof. This is immediate from Proposition 11.6.4 and the 5-Lemma. □

Corollary 11.6.6. *Let M be an R-oriented n-manifold. If M is the union of a sequence of open sets $U_1 \subseteq U_2 \subseteq \cdots$ and D_{U_i} is an isomorphism for each $i = 1, 2, \ldots$, then D_M is an isomorphism.*

For the proof we need the following observation:

Lemma 11.6.7. *Suppose a topological space X is the union of subspaces X_i, $i \in \mathcal{I}$, where $\mathcal{I}$ is a filtered partially ordered set, such that for every compact set in X there is some X_i that contains it. Then the natural map*

$$\operatorname{colim}_{i\in\mathcal{I}} H_*(X_i; M) \to H_*(X; M)$$

is an isomorphism for every M.

Proof. To see the map is surjective, represent a class $[\alpha] \in H_m(X; M)$ by $\alpha \in S_m(X; M)$; the images of the simplices in the linear combination α give a compact subset of X, hence α is in the image of $S_m(X_i; M)$ for some $i \in \mathcal{I}$. Similarly, if a cycle γ in some X_i is a boundary in X, then the bounding chain must lie in some $X_{i'}$ and since $\mathcal{I}$ is filtered there is some i'' such that $i \leq i'', i' \leq i''$ and hence the image of γ is a boundary in $S_m(X_{i''}; M)$, which means γ represents 0 in $\operatorname{colim}_{i\in\mathcal{I}} H_*(X_i; M)$. This shows the map is injective. □

Proof of Corollary 11.6.6. First we note that by excision we can regard $H^*_c(U_i; R)$ as the colimit of $H^*(M|K; R)$ as K ranges over $\operatorname{Cpt}(U_i) \subseteq \operatorname{Cpt}(M)$. Since $\operatorname{Cpt}(U_i) \subseteq \operatorname{Cpt}(U_{i+1})$ there is a natural map $H^*_c(U_i; R) \to H^*_c(U_{i+1}; R)$. The naturality of (relative) cap products also gives commutative squares

$$\begin{array}{ccc} H^*_c(U_i; R) & \longrightarrow & H^*_c(U_{i+1}; R) \\ \downarrow{\scriptstyle D_{U_i}} & & \downarrow{\scriptstyle D_{U_{i+1}}} \\ H_{n-*}(U_i; R) & \longrightarrow & H_{n-*}(U_{i+1}; R). \end{array}$$

Every compact subset of M lies in some U_i so if we take the colimit along these maps we get

$$\operatorname{colim}_i H^*_c(U_i; R) \cong H^*_c(M; R).$$

Moreover, through the isomorphism $H_*(M; R) \cong \operatorname{colim}_i H_*(U_i; R)$ of Lemma 11.6.7 and the naturality of cap products the duality map D_M is identified with the map on colimits

$$\operatorname{colim}_i D_{U_i} \colon \operatorname{colim}_i H^*_c(U_i; R) \to \operatorname{colim}_i H_{n-*}(U_i; R),$$

which is therefore an isomorphism. □

Now we can complete the proof the theorem:

Proof of Theorem 11.5.18. We first consider subsets U of $\mathbb{R}^n$ of the form $U_1 \cup \cdots \cup U_r$ where each U_i is a convex open subset and proceed by induction on r. If $r = 1$ then U_1 is homeomorphic to $\mathbb{R}^n$ and we're done by Lemma 11.6.1. For $r > 1$ set $U' = U_1 \cup \cdots \cup U_{r-1}$; then $U' \cap U_r$ is the union of the $r-1$ convex open subsets $U_i \cap U_r$, $i < r$, and so we can apply Corollary 11.6.5 to U' and U_r to conclude that D_U is an isomorphism.

Next consider an arbitrary open subset $U \subseteq \mathbb{R}^n$. Then we can write U as a countable union of convex open subsets U_i, $i = 1, 2, \ldots$. Set $V_i = \bigcup_{j \leq i} U_i$ then we know D_{V_i} is an isomorphism for every i, and so Corollary 11.6.6 implies that D_U is an isomorphism since U is the union of the V_i's.

Now consider a manifold M that can be written as a finite union of open subsets $U_i, i = 1 \ldots, r$ with U_i homeomorphic to an open subset of $\mathbb{R}^n$, and induct on r. Set $U' = U_1 \cup \cdots \cup U_{r-1}$; then $U' \cap U_r$ is the union of $r-1$ open subsets homeomorphic to open subsets of $\mathbb{R}^n$, and we can apply Corollary 11.6.5 to U' and U_r to conclude that D_M is an isomorphism.

Finally, we consider an arbitrary manifold M. Because of our assumption that a manifold is second-countable, we can write M as a countable union of open subsets $U_1, U_2, \ldots$ such that each U_i is homeomorphic to an open subset of $\mathbb{R}^n$. If we set $V_i = \bigcup_{j \leq i} U_i$ then we know D_{V_i} is an isomorphism for every i, so we can apply Corollary 11.6.6 to conclude that D_M is an isomorphism. □

The assumption that M is second-countable is not really necessary, but if we drop it we need to make use of transfinite induction here (or equivalently Zorn's lemma).